A Design Method for Robust Stabilizing Simple Multi-Period Repetitive Controllers for Multiple-Input/Multiple-Output Plants

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ABSTRACT

The multi-period repetitive control system is a type of servomechanism for a periodic reference input. Even if a plant does not include time-delays, using multi-period repetitive controllers, the transfer function from the periodic reference input to the output and that from the disturbance to the output of the multi-period repetitive control system generally have infinite numbers of poles. To specify the input-output characteristic and the disturbance attenuation characteristic easily, Yamada and Takenaga proposed the concept of simple multi-period repetitive control systems, such that the controller works as a stabilizing multi-period repetitive controller and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles. In addition, the parameterization of all stabilizing simple multi-period repetitive controllers was clarified. Recently, the parameterization of all robust stabilizing simple multi-period repetitive controllers for the plant with uncertainty was clarified by Yamada et al. However, they did not clarify the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plants. In this paper, we propose the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plants.

Keywords: Robust Stability, Parameterization, Multi-Period Repetitive Controller, Finite Number of Poles, Multiple-Input/Multiple-Output Plant

1. INTRODUCTION

A modified repetitive control system is a type of servomechanism for a periodic reference input, i.e., it follows a periodic reference input without steady state error, even when there exists a periodic disturbance or an uncertainty of a plant [1–10]. However, the modified repetitive control system has a bad effect on the disturbance attenuation characteristic [11], in that at certain frequencies, the sensitivity to disturbances of a control system with a conventional repetitive controller becomes twice as worse as that of a control system without a repetitive controller. Gotou et al. overcame this problem by proposing a multi-period repetitive control system [11]. However, the phase angle of the low-pass filter in a multi-period repetitive controller has a bad effect on the disturbance attenuation characteristics [12, 13]. Yamada et al. overcame this problem and proposed a design method for multi-period repetitive controllers to attenuate disturbances effectively [14, 15] using the time advance compensation described in [12, 13, 16]. Using this multi-period repetitive control structure, Steinbuch proposed a design method for repetitive control systems with uncertain period time [17].

On the other hand, there is an important control problem of finding all stabilizing controllers, named the parameterization problem [18–22]. The parameterization of all stabilizing multi-period repetitive controllers was solved in [23, 24]. However, when we design stabilizing multi-period repetitive controllers using the parameterization in [23, 24], the input-output frequency characteristics of the control system cannot be determined easily. From a practical point of view, the input-output frequency characteristics of a control system must be determined easily. Satoh et al. proposed the parameterization of all stabilizing multi-period repetitive controllers with specified input-output frequency characteristics [25, 26]. However, multi-period repetitive controllers in [25, 26] cannot guarantee the stability of control system for plants with uncertainties. Almost all real plants include uncertainties. In some cases, the uncertainties in the plant make the control system unstable. Yamada et al. proposed the parameterization of all robust stabilizing multi-period repetitive controllers for plants with uncertainties [27].

Using the multi-period repetitive controllers in [11, 14, 15, 23–26], even if the plant does not include time delays, the transfer function from the periodic reference input to the output and that from the disturbance to the output have infinite numbers of poles. In this situation, it is difficult to specify the input-

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output characteristic and the disturbance attenuation characteristic. From a practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easy to determine. To do this, transfer functions from the periodic reference input to the output and from the disturbance to the output should have finite numbers of poles. To overcome this problem, Yamada and Takenaga proposed simple multi-period repetitive control systems such that the controller works as a multi-period repetitive controller and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles [29]. In addition, Yamada and Takenaga clarified the parameterization of all stabilizing simple multi-period repetitive controllers. However, simple multi-period repetitive controllers in [29] cannot guarantee the stability of control system for plants with uncertainties. Yamada et al. proposed the parameterization of all robust stabilizing simple multi-period repetitive controllers for plants with uncertainties [30]. However, the parameterization in [30] cannot be applied to multiple-input/multiple-output plants, because this parameterization is obtained using the characteristics of single-input/single-output systems. Many real plants include multiple-input and multiple-output. In addition, the parameterization is useful to design stabilizing controllers [18–22]. Therefore, the problem of obtaining the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plants is important.

In this paper, we propose the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plants such that the controller works as a robust stabilizing multi-period repetitive controller and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles, when the uncertainty does not exist. This paper is organized as follows. In Section 2., the concept of the robust stabilizing simple multi-period repetitive controller is presented. In addition, in Section 2., the problem considered in this paper is described. In Section 3., the parameterization of all robust stabilizing simple multi-period repetitive controllers is clarified. In Section 4., control characteristics of a robust stabilizing simple multi-period repetitive control system are described. In Section 5., we present a design procedure of robust stabilizing simple multi-period repetitive control system. In Section 6., we show a numerical example to illustrate the effectiveness of the proposed method. Section 7. gives concluding remarks.

Notation

R	the set of real numbers.
R_+	$R \cup \{\infty\}$.
$R(s)$	the set of real rational functions with s.
RH_∞	the set of stable proper real rational functions.
H_∞	the set of stable causal functions.
$D^\perp$	orthogonal complement of D, i.e., $\begin{bmatrix} D & D^\perp \end{bmatrix}$ or $\begin{bmatrix} D \\ D^\perp \end{bmatrix}$ is unitary.
A^T	transpose of A.
$A^\dagger$	pseudo inverse of A.
$\rho(\{\cdot\})$	spectral radius of $\{\cdot\}$.
$\bar{\sigma}(\{\cdot\})$	largest singular value of $\{\cdot\}$.
$\\|\{\cdot\}\\|_\infty$	H_∞ norm of $\{\cdot\}$.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	represents the state space description $C(sI-A)^{-1}B+D$.

2. ROBUST STABILIZING SIMPLE MULTI-PERIOD REPETITIVE CONTROL SYSTEMS AND PROBLEM FORMULATION

Consider the unity feedback control system in

$$\begin{cases} y &= G(s)u + d \\ u &= C(s)(r-y) \end{cases}, \tag{1}$$

where $G(s) \in R^{m\times p}(s)$ is the plant, $C(s)$ is the controller, $u \in R^p$ is the control input, $d \in R^m$ is the disturbance, $y \in R^m$ is the output and $r \in R^m$ is the periodic reference input with period T satisfying

$$r(t+T) = r(t) \quad (\forall t \geq 0). \tag{2}$$

It is assumed that $m \leq p$. The nominal plant of $G(s)$ is denoted by $G_m(s) \in R^{m\times p}(s)$. Both $G(s)$ and $G_m(s)$ are assumed to have no zero or pole on the imaginary axis. In addition, it is assumed that the number of poles of $G(s)$ in the closed right half plane is equal to that of $G_m(s)$. The relation between the plant $G(s)$ and the nominal plant $G_m(s)$ is written as

$$G(s) = (I + \Delta(s))G_m(s), \tag{3}$$

where $\Delta(s)$ is an uncertainty. The set of $\Delta(s)$ is all rational functions satisfying

$$\bar{\sigma}\{\Delta(j\omega)\} < |W_T(j\omega)| \quad (\forall \omega \in R_+), \tag{4}$$

where $W_T(s) \in R(s)$ is a stable rational function.

The robust stability condition for the plant $G(s)$ with uncertainty $\Delta(s)$ satisfying (4) is given by

$$\|T(s)W_T(s)\|_\infty < 1, \tag{5}$$

where $T(s)$ is the complementary sensitivity function written by

$$T(s) = (I + G_m(s)C(s))^{-1} G_m(s)C(s). \tag{6}$$

According to [11, 14, 15, 23–26], in order for the output y to follow the periodic reference input r with period T in (1) with small steady state error, the multi-period repetitive controller $C(s)$ is written as

$$C(s) = C_0(s) + \sum_{i=1}^{N} C_i(s)e^{-sT_i} \left(I - \sum_{i=1}^{N} q_i(s)e^{-sT_i}\right)^{-1}, \quad (7)$$

where N is an arbitrary positive integer, $T_i > 0 \in R$ $(i = 1, \ldots, N)$, $C_0(s) \in R^{p\times m}(s)$, $C_i(s) \in R^{p\times m}(s)(i = 1, \ldots, N)$ satisfying rank $C_i(s) = m(i = 1, \ldots, N)$, $q_i(s) \in R^{m\times m}(s)(i = 1, \ldots, N)$ are low-pass filters satisfying $\sum_{i=1}^{N} q_i(0) = I$ and rank $q_i(s) = m(i = 1, \ldots, N)$. In the following, $e^{-sT_i}(I - \sum_{i=1}^{N} q_i(s)e^{-sT_i})^{-1}$ defines the internal model for the periodic signal with period T. According to [11, 14, 15, 23–26], if the low-pass filters $q_i(s)(i = 1, \ldots, N)$ satisfy

$$\bar{\sigma}\left\{I - \sum_{i=1}^{N} q_i(j\omega_k)\right\} \simeq 0 \quad (k = 0, \ldots, N_{max}), \quad (8)$$

where $\omega_k(k = 0, \ldots, N_{max})$ are frequency components of the periodic reference input r written by

$$\omega_k = \frac{2\pi}{T}k \quad (k = 0, \ldots, N_{max}), \quad (9)$$

and $\omega_{N_{max}}$ is the maximum frequency component of the periodic reference input r, then the output y in (1) follows the periodic reference input r with a small steady state error. The controller written by (7) is called the multi-period repetitive controller [11, 14, 15, 23–26].

Using the multi-period repetitive controller $C(s)$ in (7), transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y in (1) are written as

$$\begin{aligned} y &= (I + G(s)C(s))^{-1} G(s)C(s)r \\ &= (I + \Delta(s))G_m(s)\left\{C_0(s) + \sum_{i=1}^{N} (C_i(s) -C_0(s)q_i(s))\, e^{-sT_i}\right\} \\ &\quad \left[I + (I + \Delta(s))\, G_m(s)C_0(s) - \sum_{i=1}^{N} [\{I+ (I + \Delta(s))G_m(s)C_0(s)\}\, q_i(s) - (I + \Delta(s)) G_m(s)C_i(s)]\, e^{-sT_i}\right]^{-1} r \end{aligned} \quad (10)$$

and

$$\begin{aligned} y &= (I + G(s)C(s))^{-1} d \\ &= \left(I - \sum_{i=1}^{N} q_i(s)e^{-sT_i}\right) \\ &\quad \left[I + (I + \Delta(s))\, G_m(s)C_0(s) - \sum_{i=1}^{N} [\{I+ (I + \Delta(s))G_m(s)C_0(s)\}\, q_i(s) - (I + \Delta(s)) G_m(s)C_i(s)]\, e^{-sT_i}\right]^{-1} d, \end{aligned} \quad (11)$$

respectively.

Generally, transfer functions from the periodic reference input r to the output y in (10) and from the disturbance d to the output y in (11) have infinite numbers of poles, even if $\Delta(s) = 0$. When transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y have infinite numbers of poles, it is difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From the practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easily specified. In order to specify the input-output characteristic and the disturbance attenuation characteristic easily, transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y are desirable to have finite numbers of poles.

From above practical requirement, we define a robust stabilizing simple multi-period repetitive controller as Definition 1 and clarify the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plants.

Definition 1: (robust stabilizing simple multi-period repetitive controller for multiple-input/multiple-output plants)
We call the controller $C(s)$ a "robust stabilizing simple multi-period repetitive controller for multiple-input/multiple-output plants", if following expressions hold true:

1. The controller $C(s)$ works as a multi-period repetitive controller. That is, the controller $C(s)$ is written by (7), where $C_0(s) \in R^{p\times m}(s)$, $C_i(s) \in R^{p\times m}(s)(i = 1, \ldots, N)$ satisfies rank $C_i(s) = m(i = 1, \ldots, N)$ and $q_i(s) \in R^{m\times m}(s)$ $(i = 1, \ldots, N)$ satisfy $\sum_{i=1}^{N} q_i(0) = I$ and rank $q_i(s) = m(i = 1, \ldots, N)$.
2. When $\Delta(s) = 0$, the controller $C(s)$ makes transfer functions from the periodic reference input r to the output y in (1) and from the disturbance d to the output y in (1) have finite numbers of poles.
3. The controller $C(s)$ satisfies the robust stability condition in (5).

3. THE PARAMETERIZATION OF ALL ROBUST STABILIZING SIMPLE MULTI-PERIOD REPETITIVE CONTROLLERS FOR MULTIPLE-INPUT/MULTIPLE -OUTPUT PLANTS

In this section, we clarify the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plants defined in Definition 1.

In order to obtain the parameterization of all robust stabilizing simple multi-period repetitive controllers, we must see that controllers $C(s)$ satisfying (5). The problem of obtaining the controller $C(s)$, which is not necessarily a simple multi-period repetitive controller, satisfying (5) is equivalent to the following H_∞ control problem. In order to obtain the controller $C(s)$ satisfying (5), we consider the control system shown in Fig. 1. $P(s)$ is selected such that

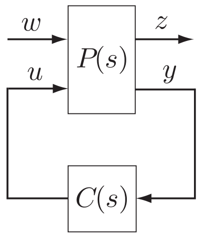

Fig.1: *Block diagram of H_∞ control problem*

the transfer function from w to z in Fig. 1 is equal to $T(s)W_T(s)$. The state space description of $P(s)$ is, in general,

$$\left\{\begin{array}{rcllll} \dot{x}(t) &=& Ax(t) & +B_1w(t) & +B_2u(t) & \\ z(t) &=& C_1x(t) & & +D_{12}u(t) & , \\ y(t) &=& C_2x(t) & +D_{21}w(t) & & \end{array}\right. \quad (12)$$

where $A \in R^{n\times n}$, $B_1 \in R^{n\times m}$, $B_2 \in R^{n\times p}$, $C_1 \in R^{m\times n}$, $C_2 \in R^{m\times n}$, $D_{12} \in R^{m\times p}$, $D_{21} \in R^{m\times m}$, $x(t) \in R^n$, $w(t) \in R^m$, $z(t) \in R^m$, $u(t) \in R^p$ and $y(t) \in R^m$. $P(s)$ is called the generalized plant. $P(s)$ is assumed to satisfy the following assumptions [31]:

1. (C_2, A) is detectable, (A, B_2) is stabilizable.
2. D_{12} has full column rank, and D_{21} has full row rank.
3. rank $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + p \quad (\forall\omega \in R_+)$,

rank $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + m \quad (\forall\omega \in R_+)$.

Under these assumptions, according to [31], following lemma holds true.

Lemma 1: If controllers satisfying (5) exist, both

$$\begin{aligned} & X\left(A - B_2D_{12}^\dagger C_1\right) + \left(A - B_2D_{12}^\dagger C_1\right)^T X \\ & +X\left\{B_1B_1^T - B_2\left(D_{12}^TD_{12}\right)^{-1}B_2^T\right\}X \\ & +\left(D_{12}^\perp C_1\right)^T D_{12}^\perp C_1 = 0 \end{aligned} \quad (13)$$

and

$$\begin{aligned} & Y\left(A - B_1D_{21}^\dagger C_2\right)^T + \left(A - B_1D_{21}^\dagger C_2\right)Y \\ & +Y\left\{C_1^TC_1 - C_2^T\left(D_{21}D_{21}^T\right)^{-1}C_2\right\}Y \\ & +B_1D_{21}^\perp\left(B_1D_{21}^\perp\right)^T = 0 \end{aligned} \quad (14)$$

have solutions $X \geq 0$ and $Y \geq 0$ such that

$$\rho(XY) < 1 \quad (15)$$

and both $A-B_2D_{12}^\dagger C_1+\{B_1B_1^T-B_2(D_{12}^TD_{12})^{-1}B_2^T\}X$ and $A - B_1D_{21}^\dagger C_2 + Y\{C_1^TC_1 - C_2^T(D_{21}D_{21}^T)^{-1}C_2\}$ have no eigenvalue in the closed right half plane. Using X and Y, the parameterization of all controllers satisfying (5) is given by

$$\begin{aligned} & C(s) \\ & \quad = C_{11}(s) + C_{12}(s)Q(s)(I - C_{22}(s)Q(s))^{-1}C_{21}(s), \end{aligned} \quad (16)$$

where

$$\begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{array}\right], \quad (17)$$

$$\begin{aligned} A_c &= A + B_1B_1^TX - B_2\left(D_{12}^\dagger C_1 + E_{12}^{-1}B_2^TX\right) \\ &\quad -(I - YX)^{-1}\left(B_1D_{21}^\dagger + YC_2^TE_{21}^{-1}\right) \\ &\quad \left(C_2 + D_{21}B_1^TX\right), \end{aligned}$$

$$\begin{aligned} B_{c1} &= (I - YX)^{-1}\left(B_1D_{21}^\dagger + YC_2^TE_{21}^{-1}\right), \\ B_{c2} &= (I - YX)^{-1}\left(B_2 + YC_1^TD_{12}\right)E_{12}^{-1/2}, \end{aligned}$$

$$\begin{aligned} C_{c1} &= -D_{12}^\dagger C_1 - E_{12}^{-1}B_2^TX, \\ C_{c2} &= -E_{21}^{-1/2}\left(C_2 + D_{21}B_1^TX\right), \end{aligned}$$

$$D_{c11} = 0, \quad D_{c12} = E_{12}^{-1/2}, \quad D_{c21} = E_{21}^{-1/2}, \quad D_{c22} = 0,$$

$$E_{12} = D_{12}^TD_{12}, \quad E_{21} = D_{21}D_{21}^T$$

and $Q(s) \in H_\infty$ is any function satisfying $\|Q(s)\|_\infty < 1$.

$C(s)$ in (16) is written using Linear Fractional Transformation(LFT). Using homogeneous transformation, (16) is rewritten by

$$\begin{aligned} C(s) &= (Z_{11}(s)Q(s) + Z_{12}(s)) \\ &\quad (Z_{21}(s)Q(s) + Z_{22}(s))^{-1} \\ &= \left(Q(s)\tilde{Z}_{21}(s) + \tilde{Z}_{22}(s)\right)^{-1} \\ &\quad \left(Q(s)\tilde{Z}_{11}(s) + \tilde{Z}_{12}(s)\right), \end{aligned} \quad (18)$$

where $Z_{ij}(s)(i = 1,2; j = 1,2)$ and $\tilde{Z}_{ij}(s)(i = 1,2; j = 1,2)$ are denoted by

$$\begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix} = \begin{bmatrix} C_{12}(s)-C_{11}(s)C_{21}^{-1}(s)C_{22}(s) & C_{11}(s)C_{21}^{-1}(s) \\ -C_{21}^{-1}(s)C_{22}(s) & C_{21}^{-1}(s) \end{bmatrix} \quad (19)$$

and

$$\begin{bmatrix} \tilde{Z}_{11}(s) & \tilde{Z}_{12}(s) \\ \tilde{Z}_{21}(s) & \tilde{Z}_{22}(s) \end{bmatrix} = \begin{bmatrix} C_{21}(s)-C_{22}(s)C_{12}^{-1}(s)C_{11}(s) & C_{12}^{-1}(s)C_{11}(s) \\ -C_{22}(s)C_{12}^{-1}(s) & C_{12}^{-1}(s) \end{bmatrix} \quad (20)$$

and satisfy

$$\begin{bmatrix} \tilde{Z}_{22}(s) & \tilde{Z}_{12}(s) \\ \tilde{Z}_{21}(s) & \tilde{Z}_{11}(s) \end{bmatrix} \begin{bmatrix} Z_{11}(s) & -Z_{12}(s) \\ -Z_{21}(s) & Z_{22}(s) \end{bmatrix} = I = \begin{bmatrix} Z_{11}(s) & -Z_{12}(s) \\ -Z_{21}(s) & Z_{22}(s) \end{bmatrix} \begin{bmatrix} \tilde{Z}_{22}(s) & \tilde{Z}_{12}(s) \\ \tilde{Z}_{21}(s) & \tilde{Z}_{11}(s) \end{bmatrix}. \quad (21)$$

Using Lemma 1, the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/ multiple-output plants is given by following theorem.

Theorem 1: If simple multi-period repetitive controllers satisfying (5) exist, both (13) and (14) have solutions $X \geq 0$ and $Y \geq 0$ such that (15) and both $A - B_2D_{12}^{\dagger}C_1 + \{B_1B_1^T - B_2(D_{12}^TD_{12})^{-1}B_2^T\}X$ and $A - B_1D_{21}^{\dagger}C_2 + Y\{C_1^TC_1 - C_2^T(D_{21}D_{21}^T)^{-1}C_2\}$ have no eigenvalue in the closed right half plane. Using X and Y, the parameterization of all robust stabilizing simple multi-period repetitive controllers satisfying (5) is given by

$$\begin{aligned} C(s) &= (Z_{11}(s)Q(s) + Z_{12}(s)) (Z_{21}(s)Q(s) + Z_{22}(s))^{-1} \\ &= \left(Q(s)\tilde{Z}_{21}(s) + \tilde{Z}_{22}(s)\right)^{-1} \left(Q(s)\tilde{Z}_{11}(s) + \tilde{Z}_{12}(s)\right), \end{aligned} \quad (22)$$

where $Z_{ij}(s)(i = 1,2; j = 1,2)$ and $\tilde{Z}_{ij}(s)(i = 1,2; j = 1,2)$ are written by (19) and (20), and satisfy (21), $C_{ij}(s)(i = 1,2; j = 1,2)$ are given by (17) and $Q(s) \in H_\infty^{p\times m}$ is any function satisfying $\|Q(s)\|_\infty < 1$ and written by

$$Q(s) = \left(Q_{n0}(s) + \sum_{i=1}^{N} Q_{ni}(s)e^{-sT_i}\right) \left(Q_{d0}(s) + \sum_{i=1}^{N} Q_{di}(s)e^{-sT_i}\right)^{-1}, \quad (23)$$

$$Q_{ni}(s) = G_{2d}(s)\bar{Q}_i(s) \in RH_\infty^{p\times m} \quad (i = 1, \ldots, N) \quad (24)$$

and

$$Q_{di}(s) = -G_{1d}(s)G_{2n}(s)\bar{Q}_i(s) \in RH_\infty^{m\times m} \quad (i = 1, \ldots, N). \quad (25)$$

Here, $G_{1n}(s) \in RH_\infty^{m\times m}$, $G_{1d}(s) \in RH_\infty^{m\times m}$, $G_{2n}(s) \in RH_\infty^{m\times p}$ and $G_{2d}(s) \in RH_\infty^{p\times p}$ are coprime factors satisfying

$$Z_{22}(s) + G_m(s)Z_{12}(s) = G_{1n}(s)G_{1d}^{-1}(s) \quad (26)$$

and

$$G_{1n}^{-1}(s)\left(Z_{21}(s) + G_m(s)Z_{11}(s)\right) = G_{2n}(s)G_{2d}^{-1}(s). \quad (27)$$

$Q_{n0}(s) \in RH_\infty^{p\times m}$, $Q_{d0}(s) \in RH_\infty^{m\times m}$ and $\bar{Q}_i(s) \in RH_\infty^{p\times m}$ $(i = 1, \ldots, N)$ are any functions satisfying

$$\bar{\sigma}\left\{Z_{22}(0)\left(Q_{d0}(0) + \sum_{i=1}^{N} Q_{di}(0)\right) + Z_{21}(0)\left(Q_{n0}(0) + \sum_{i=1}^{N} Q_{ni}(0)\right)\right\} = 0, \quad (28)$$

$$\text{rank}\ \left(Q_{ni}(s) - Q_{n0}(s)Q_{d0}^{-1}(s)Q_{di}(s)\right) = m \quad (i = 1, \ldots, N) \quad (29)$$

and rank $\bar{Q}_i(s) = m$ $(i = 1, \ldots, N)$.

Proof: First, the necessity is shown. That is, we show that if the multi-period repetitive controller written by (7) stabilizes the control system in (1) robustly and makes transfer functions from the periodic reference input r to the output y in (10) and from the disturbance d to the output y in (11) have finite numbers of poles, when $\Delta(s) = 0$, then $C(s)$ and $Q(s)$ are written by (22) and (23), respectively. From Lemma 1, the parameterization of all robust stabilizing controllers $C(s)$ for $G(s)$ is written by (22), where $\|Q(s)\|_\infty < 1$. In addition, $Q(s) \in H_\infty^{p\times m}$ is any function. In order to prove the necessity, we will show that if the controller $C(s)$ written by (22) works as a multi-period repetitive controller, then $Q(s) \in H_\infty^{p\times m}$ in (22) is written by (23). Substituting $C(s)$ in (7) into (22), we have (23), where

$$Q_{n0}(s) = -Z_{0n}(s)Z_d(s)\tilde{Z}_{0d}(s)\tilde{Z}_d(s)q_d(s), \quad (30)$$

$$\begin{aligned} Q_{ni}(s) &= -Z_{in}(s)\tilde{Z}_{0d}(s)\tilde{Z}_d(s)q_d(s) \\ &\quad + Z_{0n}(s)Z_d(s)\tilde{Z}_{0d}(s)\tilde{Z}_d(s)q_{in}(s) \\ &\quad (i = 1, \ldots, N), \end{aligned} \quad (31)$$

$$Q_{d0}(s) = \tilde{Z}_{0n}(s)\tilde{Z}_d(s)q_d(s) \quad (32)$$

and

$$Q_{di}(s) = \tilde{Z}_{in}(s)q_d(s) - \tilde{Z}_{0n}(s)\tilde{Z}_d(s)q_{in}(s) \quad (i = 1, \ldots, N). \tag{33}$$

Here, $C_{0n}(s) \in RH_\infty^{p\times m}$, $C_{0d}(s) \in RH_\infty^{m\times m}$, $C_{in}(s) \in RH_\infty^{p\times m}$, $C_d(s) \in RH_\infty^{m\times m}$, $q_{in}(s) \in RH_\infty^{m\times m}$, $q_d(s) \in RH_\infty^{m\times m}$, $Z_{0n}(s) \in RH_\infty^{p\times m}$, $Z_{0d}(s) \in RH_\infty^{m\times m}$ $Z_{in}(s) \in RH_\infty^{p\times m}$, $Z_d(s) \in RH_\infty^{m\times m}$ $\tilde{Z}_{0n}(s) \in RH_\infty^{m\times m}$, $\tilde{Z}_{0d}(s) \in RH_\infty^{m\times m}$ $\tilde{Z}_{in}(s) \in RH_\infty^{m\times m}$ and $\tilde{Z}_d(s) \in RH_\infty^{m\times m}$ are coprime factors satisfying

$$C_0(s) = C_{0n}(s)C_{0d}^{-1}(s), \tag{34}$$

$$C_i(s)C_{0d}(s) = C_{in}(s)C_d^{-1}(s) \quad (i = 1, \ldots, N), \tag{35}$$

$$q_i(s) = q_{in}(s)q_d^{-1}(s) \quad (i = 1, \ldots, N), \tag{36}$$

$$\left(\tilde{Z}_{22}(s)C_{0n}(s) - \tilde{Z}_{12}(s)C_{0d}(s)\right) C_d(s) = Z_{0n}(s)Z_{0d}^{-1}(s), \tag{37}$$

$$\tilde{Z}_{22}(s)C_{in}(s)Z_{0d}(s) = Z_{in}(s)Z_d^{-1}(s) \quad (i = 1, \ldots, N), \tag{38}$$

$$\left(\tilde{Z}_{21}(s)C_{0n}(s) - \tilde{Z}_{11}(s)C_{0d}(s)\right) C_d(s)Z_{0d}(s)Z_d(s) = \tilde{Z}_{0n}(s)\tilde{Z}_{0d}^{-1}(s) \quad (i = 1, \ldots, N) \tag{39}$$

and

$$\tilde{Z}_{21}(s)C_{in}(s)Z_{0d}(s)Z_d(s)\tilde{Z}_{0d}(s) = \tilde{Z}_{in}(s)\tilde{Z}_d^{-1}(s) \quad (i = 1, \ldots, N). \tag{40}$$

From (30)∼(33), all of $Q_{n0}(s)$, $Q_{ni}(s)$ $(i = 1, \ldots, N)$, $Q_{d0}(s)$ and $Q_{di}(s)$ $(i = 1, \ldots, N)$ are included in RH_∞. Thus, we have shown that if $C(s)$ written by (7) stabilize the control system in (1) robustly, $Q(s)$ in (22) is written by (23). Since $\sum_{i=1}^N q_i(0) = I$, from (30)∼(33) and (21), (28) holds true. In addition, from the assumption of rank $C_i(s) = m$ $(i = 1, \ldots, N)$ and from (36), (38) and (40),

$$\text{rank } Z_{in}(s) = m \quad (i = 1, \ldots, N) \tag{41}$$

and

$$\text{rank } \tilde{Z}_{in}(s) = m \quad (i = 1, \ldots, N) \tag{42}$$

hold true. From (41), (42), (30), (31), (32) and (33), (29) is satisfied. The rest to prove the necessity is to show that when $\Delta(s) = 0$, if $C(s)$ in (7) makes transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y have finite numbers of poles, then $Q_{ni}(s)$ and $Q_{di}(s)$ are written by (24) and (25), respectively. From (23), when $\Delta(s) = 0$, transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y are written by

$$y = G_{ryn}(s)G_{ryd}^{-1}(s)r \tag{43}$$

and

$$y = G_{dyn}(s)G_{dyd}^{-1}(s)d, \tag{44}$$

respectively, where

$$G_{ryn}(s) = G_m(s)\left\{Z_{12}(s)Q_{d0}(s) + Z_{11}(s)Q_{n0}(s) + \sum_{i=1}^N \left(Z_{12}(s)Q_{di}(s) + Z_{11}(s)Q_{ni}(s)\right) e^{-sT_i}\right\}, \tag{45}$$

$$\begin{aligned} G_{ryd}(s) = {} & \left(Z_{22}(s) + G_m(s)Z_{12}(s)\right) Q_{d0}(s) \\ & + \left(Z_{21}(s) + G_m(s)Z_{11}(s)\right) Q_{n0}(s) \\ & + \sum_{i=1}^N \left\{\left(Z_{22}(s) + G_m(s)Z_{12}(s)\right) Q_{di}(s) \right. \\ & \left. + \left(Z_{21}(s) + G_m(s)Z_{11}(s)\right) Q_{ni}(s)\right\} e^{-sT_i}, \end{aligned} \tag{46}$$

$$G_{dyn}(s) = Z_{22}(s)Q_{d0}(s) + Z_{21}(s)Q_{n0}(s) + \sum_{i=1}^N \left(Z_{22}(s)Q_{di}(s) + Z_{21}(s)Q_{ni}(s)\right) e^{-sT_i} \tag{47}$$

and

$$\begin{aligned} G_{dyd}(s) = {} & \left(Z_{22}(s) + G_m(s)Z_{12}(s)\right) Q_{d0}(s) \\ & + \left(Z_{21}(s) + G_m(s)Z_{11}(s)\right) Q_{n0}(s) \\ & + \sum_{i=1}^N \left\{\left(Z_{22}(s) + G_m(s)Z_{12}(s)\right) Q_{di}(s) \right. \\ & \left. + \left(Z_{21}(s) + G_m(s)Z_{11}(s)\right) Q_{ni}(s)\right\} e^{-sT_i}. \end{aligned} \tag{48}$$

From the assumption that transfer functions from the periodic reference input r to the output y in (43) and from the disturbance d to the output y in (44) have finite numbers of poles, (46) and (48),

$$\left(Z_{22}(s) + G_m(s)Z_{12}(s)\right) Q_{di}(s) + \left(Z_{21}(s) + G_m(s)Z_{11}(s)\right) Q_{ni}(s) = 0 \tag{49}$$

is satisfied. Using (26) and (27), this equation is rewritten by

$$Q_{di}(s) = -G_{1d}(s)G_{2n}(s)G_{2d}^{-1}(s)Q_{ni}(s). \quad (50)$$

Since $Q_{ni}(s) \in RH_\infty^{p\times m}$ and $Q_{di}(s) \in RH_\infty^{m\times m}$, $Q_{ni}(s)$ and $Q_{di}(s)$ are written by (24) and (25), respectively, where $\bar{Q}_i(s) \in RH_\infty^{p\times m}$ $(i = 1, \dots, N)$. From the assumption that rank $C_i(s) = m(i = 1, \dots, N)$ and from (31) and (33), rank $\bar{Q}_i(s) = m(i = 1, \dots, N)$ holds true. We have thus proved the necessity.

Next, the sufficiency is shown. That is, it is shown that if $C(s)$ and $Q(s) \in H_\infty^{p\times m}$ are settled by (22) and (23), respectively, then the controller $C(s)$ is written by the form in (7), $\sum_{i=1}^{N} q_i(0) = I$ holds true and transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y have finite numbers of poles. Substituting (23) into (22), we have (7), where $C_0(s)$, $C_i(s)(i = 1, \dots, N)$ and $q_i(s)(i = 1, \dots, N)$ are written by

$$C_0(s) = (Z_{11}(s)Q_{n0}(s) + Z_{12}(s)Q_{d0}(s))(Z_{21}(s)Q_{n0}(s) + Z_{22}(s)Q_{d0}(s))^{-1}, \quad (51)$$

$$C_i(s) = \left(Q_{n0}(s)Q_{d0}^{-1}(s)\tilde{Z}_{21}(s) + \tilde{Z}_{22}(s)\right)^{-1}\left(Q_{ni}(s) - Q_{n0}(s)Q_{d0}^{-1}(s)Q_{di}(s)\right)(Z_{21}(s)Q_{n0}(s) + Z_{22}(s)Q_{d0}(s))^{-1} \quad (i = 1, \dots, N) \quad (52)$$

and

$$q_i(s) = -(Z_{21}(s)Q_{ni}(s) + Z_{22}(s)Q_{di}(s))(Z_{21}(s)Q_{n0}(s) + Z_{22}(s)Q_{d0}(s))^{-1} \quad (i = 1, \dots, N). \quad (53)$$

We find that if $C(s)$ and $Q(s)$ are settled by (22) and (23), respectively, then the controller $C(s)$ is written by the form in (7). From rank $\bar{Q}_i(s) = m(i = 1, \dots, N)$ and (52), rank $C_i(s) = m(i = 1, \dots, N)$ hold true. Substituting (28) into (53), we have $\sum_{i=1}^{N} q_i(0) = I$. In addition, from (24) and (25) and easy manipulation, we can confirm that when $\Delta(s) = 0$, transfer functions from the periodic reference input r to the output y and from the disturbance d to the output y have finite numbers of poles.

We have thus proved Theorem 1. ■

4. CONTROL CHARACTERISTICS

In this section, we describe control characteristics of control system in (1) using the parameterization of all robust stabilizing simple multi-period repetitive controllers $C(s)$ in (22).

First, we mention the input-output characteristic. The transfer function $S(s)$ from the periodic reference input r to the error $e = r - y$ is written by

$$S(s) = (I + C(s)G(s))^{-1} = S_n(s)S_d^{-1}(s), \quad (54)$$

where

$$S_n(s) = C_{21}^{-1}(s)\left\{I + \sum_{i=1}^{N}(Q_{di}(s) - C_{22}(s)Q_{ni}(s))(Q_{d0}(s) - C_{22}(s)Q_{n0}(s))^{-1}e^{-sT_i}\right\}(Q_{d0}(s) - C_{22}(s)Q_{n0}(s)) \quad (55)$$

and

$$S_d(s) = (Z_{22}(s) + G(s)Z_{12}(s))Q_{d0}(s) + (Z_{21}(s) + G(s)Z_{11}(s))Q_{n0}(s) + \sum_{i=1}^{N}\{(Z_{22}(s) + G(s)Z_{12}(s))Q_{di}(s) + (Z_{21}(s) + G(s)Z_{11}(s))Q_{ni}(s)\}e^{-sT_i}. \quad (56)$$

From (54), for $\omega_k(k = 0, \dots, N_{max})$ in (9), which are the frequency components of the periodic reference input r, if

$$\bar{\sigma}\left\{I + \sum_{i=1}^{N}(Q_{di}(j\omega_k) - C_{22}(j\omega_k)Q_{ni}(j\omega_k))(Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k))^{-1}\right\} \simeq 0 \quad (k = 0, \dots, N_{max}), \quad (57)$$

then the output y in (1) follows the periodic reference input r with small steady state error.

Next, we mention the disturbance attenuation characteristic. The transfer function $S(s)$ from the disturbance d to the output y is written by (54), (55) and (56). From (54), for the disturbance d with same frequency components $\omega_k(k = 0, \dots, N_{max})$ in (9) of the periodic reference input r, if

$$\bar{\sigma}\left\{I + \sum_{i=1}^{N}(Q_{di}(j\omega_k) - C_{22}(j\omega_k)Q_{ni}(j\omega_k))(Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k))^{-1}\right\} \simeq 0 \quad (k = 0, \dots, N_{max}), \quad (58)$$

then the disturbance d is attenuated effectively. For the frequency component ω_d of the disturbance d that is different from that of the periodic reference input r, that is $\omega_d \neq \omega_k$, even if

$$\bar{\sigma}\left\{I + \sum_{i=1}^{N}(Q_{di}(j\omega_d) - C_{22}(j\omega_d)Q_{ni}(j\omega_d))(Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d))^{-1}\right\} \simeq 0, \quad (59)$$

the disturbance d cannot be attenuated, because

$$e^{-j\omega_d T_i} \neq 1 \tag{60}$$

and

$$\bar{\sigma}\left\{I + \sum_{i=1}^{N}\left(Q_{di}(j\omega_d) - C_{22}(j\omega_d)Q_{ni}(j\omega_d)\right)\left(Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d)\right)^{-1} e^{-j\omega_d T_i}\right\} \neq 0. \tag{61}$$

In order to attenuate the frequency component ω_d of the disturbance d that is different from that of the periodic reference input r, if

$$\bar{\sigma}\left\{Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d)\right\} \simeq 0, \tag{62}$$

then the disturbance d is attenuated effectively.

5. DESIGN PROCEDURE

In this section, a design procedure of robust stabilizing simple multi-period repetitive controllers C(s) for multiple-input/multiple-output plants is presented.

A design procedure of robust stabilizing simple multi-period repetitive controllers C(s) satisfying Theorem 1 is summarized as follows:

Procedure

Step 1) Obtain $C_{11}(s)$, $C_{12}(s)$, $C_{21}(s)$ and $C_{22}(s)$ by solving the robust stability problem using the Riccati equation based H_∞ control.

Step 2) $Q_{n0}(s) \in RH_\infty$ is settled so that for the frequency component ω_d of the disturbance d, $\bar{\sigma}(Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d))$ is effectively small. In order to design $Q_{n0}(s)$ to make $\bar{\sigma}(Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d))$ effectively small, $Q_{n0}(s)$ is settled by

$$Q_{n0}(s) = C_{22o}^{\dagger}(s)\bar{q}_d(s)Q_{d0}(s), \tag{63}$$

where $C_{22o}(s) \in RH_\infty$ is an outer function of $C_{22}(s)$ satisfying

$$C_{22}(s) = C_{22i}(s)C_{22o}(s), \tag{64}$$

$C_{22i}(s) \in RH_\infty$ is an inner function satisfying $C_{22i}(0) = I$ and $\bar{\sigma}\{C_{22i}(j\omega)\} = 1$ $(\forall\omega \in R_+)$, $\bar{q}_d(s)$ is a low-pass filter satisfying $\bar{q}_d(0) = I$, as

$$\bar{q}_d(s) = \text{diag}\left\{\frac{1}{(1+s\tau_{d1})^{\alpha_{d1}}}, \cdots, \frac{1}{(1+s\tau_{dp})^{\alpha_{dp}}}\right\} \tag{65}$$

is valid, α_{di} $(i = 1, \ldots, p)$ are arbitrary positive integers to make $C_{22o}^{\dagger}(s)\bar{q}_d(s)$ proper and $\tau_{di} \in R$ $(i = 1, \ldots, p)$ are any positive real numbers satisfying

$$\bar{\sigma}\left[I - C_{22i}(j\omega_d)\text{diag}\left\{\frac{1}{(1+j\omega_d\tau_{d1})^{\alpha_{d1}}}, \cdots, \frac{1}{(1+j\omega_d\tau_{dp})^{\alpha_{dp}}}\right\}\right] \simeq 0. \tag{66}$$

Step 3) $\bar{Q}_i(s) \in RH_\infty$ $(i = 1, \ldots, N)$ are settled so that for the frequency component $\omega_k (k = 0, \ldots, N_{max})$ of the periodic reference input r, $\bar{\sigma}\{I + \sum_{i=1}^{N}(Q_{di}(j\omega_k) - C_{22}(j\omega_k)Q_{ni}(j\omega_k))(Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k))^{-1}\} \simeq 0 (k = 0, \ldots, N_{max})$ is satisfied. In order to design $\bar{Q}_i(s)$ $(i = 1, \ldots, N)$ to hold

$$\begin{aligned}
&\bar{\sigma}\left\{I + \sum_{i=1}^{N}\left(Q_{di}(j\omega_k) - C_{22}(j\omega_k)Q_{ni}(j\omega_k)\right)\left(Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k)\right)^{-1}\right\} \\
&= \bar{\sigma}\left\{I - \sum_{i=1}^{N}\left(G_{1d}(j\omega_k)G_{2n}(j\omega_k) + C_{22}(j\omega_k)G_{2d}(j\omega_k)\right)\bar{Q}_i(j\omega_k)\left(Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k)\right)^{-1}\right\} \\
&\simeq 0 (k = 0, \ldots, N_{max}),
\end{aligned} \tag{67}$$

$\bar{Q}_i(s) \in RH_\infty (i = 1, \ldots, N)$ are settled by

$$\bar{Q}_i(s) = H_o^{\dagger}(s)\bar{q}_{ri}(s)\left(Q_{d0}(s) - C_{22}(s)Q_{n0}(s)\right) \quad (i = 1, \ldots, N), \tag{68}$$

where $H_o(s) \in RH_\infty$ is an outer function of

$$H(s) = G_{1d}(s)G_{2n}(s) + C_{22}(s)G_{2d}(s) \tag{69}$$

satisfying

$$H(s) = H_i(s)H_o(s), \tag{70}$$

$H_i(s) \in RH_\infty$ is an inner function satisfying $H_i(0) = I$ and $\bar{\sigma}\{H_i(j\omega)\} = 1$ $(\forall\omega \in R_+)$, $\bar{q}_{ri}(s)$ $(i = 1, \ldots, N)$ are low-pass filters satisfying $\sum_{i=1}^{N}\bar{q}_{ri}(0) = I$, as

$$\bar{q}_{ri}(s) = \frac{1}{N}\text{diag}\left\{\frac{1}{(1+s\tau_{r1})^{\alpha_{r1}}}, \cdots, \frac{1}{(1+s\tau_{rp})^{\alpha_{rp}}}\right\} \quad (i = 1, \ldots, N) \tag{71}$$

are valid, $\alpha_{ri} (i = 1, \ldots, p)$ are arbitrary positive integers to make $H_o^{\dagger}(s)\bar{q}_{ri}(s)$ proper and $\tau_{ri} \in R (i = 1, \ldots, p)$ are any positive real numbers satisfying

$$\bar{\sigma}\left[I - H_i(j\omega_k)\sum_{i=1}^{N}\frac{1}{N}\text{diag}\left\{\frac{1}{(1+j\omega_k\tau_{r1})^{\alpha_{r1}}}, \cdots, \frac{1}{(1+j\omega_k\tau_{rp})^{\alpha_{rp}}}\right\}\right] \simeq 0 \quad (k = 0, \ldots, N_{max}). \tag{72}$$

6. NUMERICAL EXAMPLE

In this section, a numerical example is illustrated to show the effectiveness of the proposed method.

Consider the problem to obtain the parameterization of all robust stabilizing simple multi-period repetitive controllers for the plant $G(s)$ in (3), where the nominal plant $G_m(s)$ and the upper bound $W_T(s)$ of the set of $\Delta(s)$ are given by

$$G_m(s) = \begin{bmatrix} \frac{s+3}{(s-2)(s+9)} & \frac{2}{(s-2)(s+9)} \\ \frac{s+3}{(s-2)(s+9)} & \frac{s+4}{(s-2)(s+9)} \end{bmatrix}, \quad (73)$$

and

$$W_T(s) = \frac{s+400}{550}. \quad (74)$$

The period T of the periodic reference input r in (2) is $T = 4$[sec]. Solving the robust stability problem using Riccati equation based H_∞ control as Theorem 1, the parameterization of all robust stabilizing simple multi-period repetitive controllers $C(s)$ is obtained as (22), where N is selected as $N = 3$ and T_i $(i = 1, 2, 3)$ are set as $T_i = T \cdot i$ $(i = 1, 2, 3)$.

In order for disturbances both

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \sin(\pi t) + \sin(2\pi t) + \sin(3\pi t) \\ 2\sin(\pi t) + 2\sin(2\pi t) + 2\sin(3\pi t) \end{bmatrix} \quad (75)$$

and

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \sin\left(\frac{\pi t}{8}\right) + \sin\left(\frac{\pi t}{4}\right) + \sin\left(\frac{3\pi t}{8}\right) \\ 2\sin\left(\frac{\pi t}{8}\right) + 2\sin\left(\frac{\pi t}{4}\right) + 2\sin\left(\frac{3\pi t}{8}\right) \end{bmatrix} \quad (76)$$

to be attenuated effectively and for the output $y = [y_1, y_2]^T$ to follow the periodic reference input

$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \sin\left(\frac{\pi t}{2}\right) + \sin(\pi t) + \sin\left(\frac{3\pi t}{2}\right) \\ 2\sin\left(\frac{\pi t}{2}\right) + 2\sin(\pi t) + 2\sin\left(\frac{3\pi t}{2}\right) \end{bmatrix} \quad (77)$$

with small steady state error, $Q_{n0}(s)$ and $\bar{Q}_i(s) (i = 1, 2, 3)$ are settled by (63) and (68), where

$$Q_{d0}(s) = I, \quad (78)$$

$$\bar{q}_d(s) = \begin{bmatrix} \frac{1}{0.002s+1} & 0 \\ 0 & \frac{1}{0.002s+1} \end{bmatrix}, \quad (79)$$

and

$$\bar{q}_{ri}(s) = \begin{bmatrix} \frac{1}{3(0.01s+1)} & 0 \\ 0 & \frac{1}{3(0.01s+1)} \end{bmatrix} \quad (i = 1, 2, 3). \quad (80)$$

When $Q_{d0}(s)$ and $\bar{Q}_i(s)$ $(i = 1, 2, 3)$ are set as (78) and (68), the fact that $Q(s) \in H_\infty$ in (23) is confirmed as follows: Since $Q_{n0}(s) \in RH_\infty$ and $Q_{ni}(s) \in RH_\infty$ $(i = 1, 2, 3)$, if the Nyquist plot of $\det\{Q_{d0}(s) + \sum_{i=1}^{3} Q_{di}(s)e^{-sT_i}\}$ does not encircle the origin, then $Q(s) \in H_\infty$ holds true. The Nyquist plot of $\det\{Q_{d0}(s) + \sum_{i=1}^{3} Q_{di}(s)e^{-sT_i}\}$ is shown in Fig. 2. From Fig. 2, since the Nyquist plot of

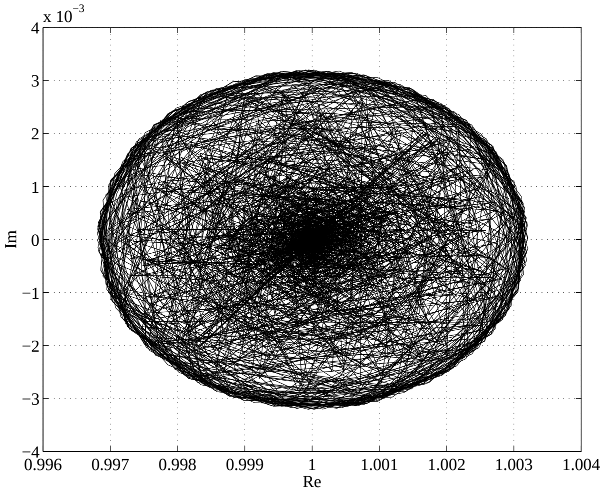

Fig.2: *The Nyquist plot of* $\det\{Q_{d0}(s) + \sum_{i=1}^{3} Q_{di}(s)e^{-sT_i}\}$

$\det\{Q_{d0}(s) + \sum_{i=1}^{3} Q_{di}(s)e^{-sT_i}\}$ does not encircle the origin, we find that $Q(s) \in H_\infty$ holds true.

The largest singular value plot of $Q(s)$ is shown in Fig. 3. Figure 3 shows that the designed $Q(s)$

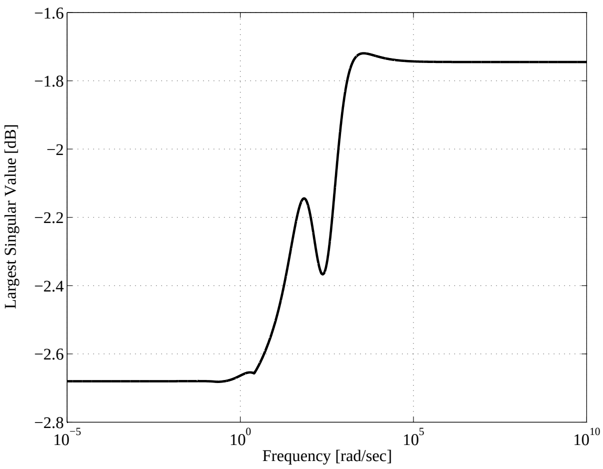

Fig.3: *The largest singular value plot of* $Q(s)$

satisfies $\|Q(s)\|_\infty < 1$.

When $\Delta(s)$ is given by

$$\Delta(s) = \begin{bmatrix} \frac{s-100}{s+500} & \frac{200}{s+600} \\ \frac{200}{s+500} & \frac{s-100}{s+600} \end{bmatrix}, \tag{81}$$

the largest singular value plot of $\Delta(s)$ and the gain plot of $W_T(s)$ are shown in Fig. 4. Here, the solid

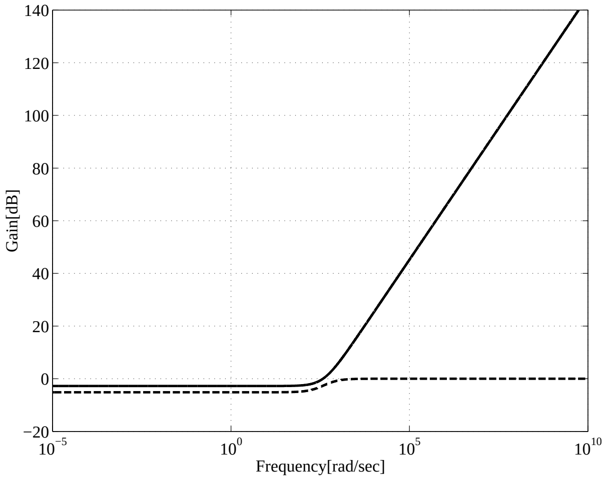

Fig.4: *The largest singular value plot of $\Delta(s)$ and the gain plot of $W_T(s)$*

line shows the gain plot of $W_T(s)$ and the dotted line shows the largest singular value plot of $\Delta(s)$. Figure 4 shows that the uncertainty $\Delta(s)$ satisfies (4).

Using above-mentioned parameters, we have a robust stabilizing simple multi-period repetitive controller. When the designed robust stabilizing simple multi-period repetitive controller $C(s)$ is used, the response of the error $e = r - y = [e_1, e_2]^T$ in (1) for the periodic reference input r in (77) is shown in Fig. 5. Here, the broken line shows the response of

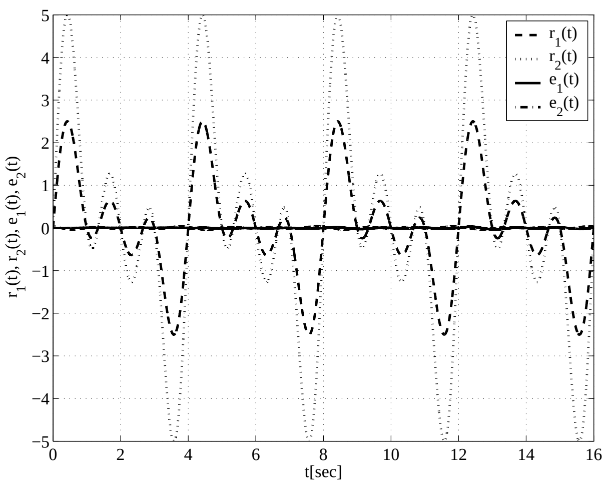

Fig.5: *The response of the error $e(t)$ for the periodic reference input $r(t)$ in (77)*

the periodic reference input r_1, the dotted line shows that of the periodic reference input r_2, the solid line shows that of the error e_1 and the dotted and broken line shows that of the error e_2. Figure 5 shows that the output y follows the periodic reference input r in (77) with small steady state error, even if the plant has uncertainty $\Delta(s)$.

Next, using the designed robust stabilizing simple multi-period repetitive controller $C(s)$, the disturbance attenuation characteristic is shown. The response of the output y for the disturbance d in (75) of which the frequency component is equivalent to that of the periodic reference input r is shown in Fig. 6. Here, the broken line shows the response of the

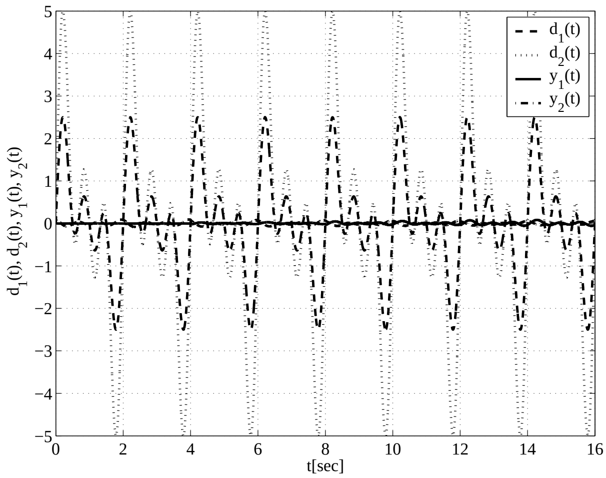

Fig.6: *The response of the output $y(t)$ for the disturbance $d(t)$ in (75)*

disturbance d_1, the dotted line shows that of the disturbance d_2, the solid line shows that of the output y_1 and the dotted and broken line shows that of the output y_2. Figure 6 shows that the disturbance d in (75) is attenuated effectively. Finally, the response of the output y for the disturbance d in (76) of which the frequency component is different from that of the periodic reference input r is shown in Fig. 7. Here, the broken line shows the response of the disturbance d_1, the dotted line shows that of the disturbance d_2, the solid line shows that of the output y_1 and the dotted and broken line shows that of the output y_2. Figure 7 shows that the disturbance d in (76) is attenuated effectively.

In this way, we find that we can easily design a robust stabilizing simple multi-period repetitive controller using Theorem 1.

7. CONCLUSIONS

In this paper, we proposed the parameterization of all robust stabilizing simple multi-period repetitive controllers for multiple-input/multiple-output plant with uncertainties. That is, we found out the parameterization of all robust stabilizing simple multi-period repetitive controllers $C(s)$ written as the form in (7)

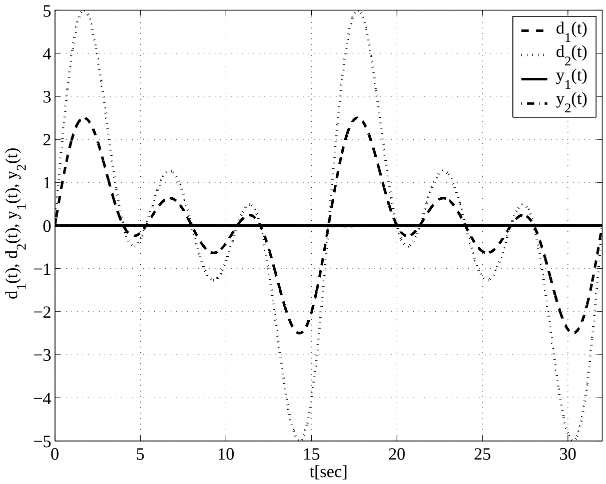

Fig.7: *The response of the output $y(t)$ for the disturbance $d(t)$ in (76)*

such that the control system in (1) is robustly stable, the output y follows the periodic reference input r with small steady state error even in the presence of uncertainty $\Delta(s)$ and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles when the uncertainty $\Delta(s) = 0$. Control characteristics of a robust stabilizing simple multi-period repetitive control system are presented, as well as a design procedure for a robust stabilizing simple multi-period repetitive controller. A numerical example was shown to illustrate the effectiveness of the proposed parameterization.

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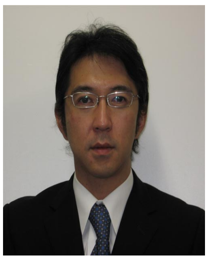

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