

The parameterization of all plants stabilized by a Proportional-Derivative controller

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ABSTRACT

In this paper, we examine the parameterization of all plants stabilized by a proportional-derivative (PD) controller. A PD controller is a kind of Proportional-Integral-Derivative (PID) controllers. PID controller structure is the most widely used one in industrial applications. Recently, if stabilizing PID controllers for the plant exist, the parameterization of all stabilizing PID controllers has been considered. However, no paper examines the parameterization of all plants stabilized by a PID controller. In this paper, we clarify the parameterization of all plants stabilized by a PD controller. In addition, we present the parameterization of all stabilizing PD controllers for the plant stabilized by a PD controller.

Keywords: PID Control, Parameterization, Stabilizability, Admissible Set

1. INTRODUCTION

Proportional-Integral-Derivative (PID) controller is the most widely used controller structure in industrial applications. Its structural simplicity and sufficient ability of solving many practical control problems have contributed to this wide acceptance [1–3].

Several papers on tuning methods for PID parameters have been considered [4–14]. However, methods in [4–14] do not guarantee the stability of the closed-loop system. If admissible sets of PID parameters to guarantee the stability of the closed-loop system are obtained, we can easily design stabilizing PID controllers to meet control specifications.

The problem to obtain admissible sets of PID parameters to guarantee the stability of the closed-loop system is known as a parameterization problem [15–19]. Recently, if there exists a stabilizing PID controller, the parameterization of all stabilizing PID controllers is considered in [3, 20, 21]. However, methods in [3, 20, 21] remain a difficulty. Using methods by [3, 20, 21], we cannot design a stabilizing PID controller for a certain class of plants. Because, for a cer-

tain class of plants, for example, a plant with fourth or more number of poles in the right half plane, it is difficult to stabilize by three parameters in a PID controller. In addition, the class of plants those can be stabilized by a PID controller is not clarified. If the class of plants those can be stabilized by a PID controller is obtained, we have possibility as follows:

1. We can easily find whether or not, the plant can be stabilized by a PID controller.
2. We can easily design stabilizing PID controllers for the plant, which can be stabilized, by a PID controller.

Hagiwara et al. tackled this problem and clarified the parameterization of all plants stabilized by a proportional controller [22]. In addition, the parameterization of all stabilizing proportional controllers for the plant that can be stabilized by a proportional controller was clarified. However, since the result by Hagiwara et al. clarified only the parameterization of all plants stabilized by a proportional controller, the result by Hagiwara et al. cannot apply for a derivative controller and a proportional-derivative controller. Since the parameterization is useful as shown in [15–19] and proportional-derivative controllers are powerful ones to control practical plants as shown in [23–26], problems to obtain the parameterization of all plants stabilized by a proportional-derivative controller and to obtain the parameterization of all stabilizing proportional-derivative controllers are important ones to solve.

In this paper, we clarify the parameterization of all plants stabilized by a proportional-derivative controller. In addition, we present the parameterization of all stabilizing proportional-derivative controllers for the plant that can be stabilized by a proportional-derivative controller. This paper is organized as follows: In Section 2., we introduce a derivative controller and a proportional-derivative controller and describe the problem considered in this paper. In Section 3., we clarify the parameterization of all plants stabilized by a derivative controller. In addition, in Section 3., when the plant is given, a procedure to check whether or not, the plant can be stabilized by a derivative controller is presented. In Section 4., we present the parameterization of all stabilizing derivative controllers for the plant that can be stabilized by a derivative controller. In Section 5., we expand the result in Section 3. and propose the parameterization

Manuscript received on July 25, 2011 ; revised on September 27, 2011.

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of all plants stabilized by a proportional-derivative controller. In Section 6., we present the parameterization of all stabilizing proportional-derivative controllers for the plant that can be stabilized by a proportional-derivative controller. In Section 7., we illustrate a numerical example to show the effectiveness of proposed results. Section 8. gives concluding remarks.

Notation

R	the set of real numbers.
$R(s)$	the set of real rational functions with s .
RH_∞	the set of stable proper real rational functions.

2. PROBLEM FORMULATION

Consider the closed-loop system written by

$$\begin{cases} y = G(s)u \\ u = C(s)(r - y) \end{cases}, \quad (1)$$

where $G(s) \in R(s)$ is the single-input/single-output strictly proper plant, $C(s) \in R(s)$ is the controller, $u \in R$ is the control input, $y \in R$ is the output and $r \in R$ is the reference input.

When the controller $C(s)$ has the form written by

$$C(s) = \frac{a_D s}{\tau_D s + 1}, \quad (2)$$

then the controller $C(s)$ is called the derivative controller (D controller) [1, 3, 20, 21], where $\tau_D \in R$ is small positive number and $a_D \in R$. In addition, when the controller $C(s)$ has the form written by

$$C(s) = a_P + \frac{a_D s}{\tau_D s + 1}, \quad (3)$$

then the controller $C(s)$ is called the proportional-derivative controller (PD controller) [1, 3, 20, 21], where $a_P \in R$, $\tau_D \in R$ is small positive number and $a_D \in R$. Transfer functions from the reference input r to the output y in (1) using the derivative controller $C(s)$ in (2) and using the proportional-derivative controller $C(s)$ in (3) are written by

$$y = \frac{G(s) \frac{a_D s}{\tau_D s + 1}}{1 + G(s) \frac{a_D s}{\tau_D s + 1}} r \quad (4)$$

and

$$y = \frac{G(s) \left(a_P + \frac{a_D s}{\tau_D s + 1} \right)}{1 + G(s) \left(a_P + \frac{a_D s}{\tau_D s + 1} \right)} r. \quad (5)$$

It is obvious that when a_P and a_D are settled at random, the stability of the closed-loop system in (1) is not guaranteed. In addition, there exist plants $G(s)$ those cannot be stabilized by the derivative controller

$C(s)$ in (2) and the proportional-derivative controller $C(s)$ in (3).

The purpose of this paper is to propose parameterizations of all plants stabilized by a derivative controller and of all plants stabilized by a proportional-derivative controller.

3. THE PARAMETERIZATION OF ALL PLANTS STABILIZED BY A DERIVATIVE CONTROLLER

In this section, we propose the parameterization of all plants $G(s)$ stabilized by a derivative controller.

The parameterization of all plants $G(s)$ stabilized by a derivative controller is summarized in the following theorem.

Theorem 1: The parameterization of all plants $G(s)$ stabilized by a derivative controller in (2) is written by

$$G(s) = \frac{Q(s)}{1 - \frac{P_D s}{\tau_D s + 1} Q(s)}, \quad (6)$$

where $P_D \in R$ is any real number and $Q(s) \in RH_\infty$ is any function.

Proof: First, the necessity is shown. That is, we show that if a derivative controller $C(s)$ in (2) stabilizes the plant $G(s)$, then the plant $G(s)$ is written by (6). From the assumption that a derivative controller $C(s)$ in (2) stabilizes the plant $G(s)$, transfer functions $C(s)G(s)/(1+C(s)G(s))$, $C(s)/(1+C(s)G(s))$, $G(s)/(1+C(s)G(s))$ and $1/(1+C(s)G(s))$ are stable. From simple manipulations, we have

$$\frac{C(s)G(s)}{1+C(s)G(s)} = \frac{\frac{a_D s}{\tau_D s + 1} G(s)}{1 + \frac{a_D s}{\tau_D s + 1} G(s)}, \quad (7)$$

$$\frac{C(s)}{1+C(s)G(s)} = \frac{\frac{a_D s}{\tau_D s + 1}}{1 + \frac{a_D s}{\tau_D s + 1} G(s)}, \quad (8)$$

$$\frac{G(s)}{1+C(s)G(s)} = \frac{G(s)}{1 + \frac{a_D s}{\tau_D s + 1} G(s)} \quad (9)$$

and

$$\frac{1}{1+C(s)G(s)} = \frac{1}{1 + \frac{a_D s}{\tau_D s + 1} G(s)}. \quad (10)$$

Since the transfer function in (9) is stable, when $Q(s)$ is settled by

$$Q(s) = \frac{G(s)}{1 + \frac{a_D s}{\tau_D s + 1} G(s)}, \quad (11)$$

then $Q(s)$ in (11) satisfies $Q(s) \in RH_\infty$. Equation (11) is rewritten by

$$G(s) = \frac{Q(s)}{1 - \frac{a_D s}{\tau_D s + 1} Q(s)}. \quad (12)$$

Let $P_D = a_D$. Equation (12) corresponds to (6). We have shown the necessity.

Next, we show the sufficiency. That is, we show that if the plant $G(s)$ is written by (6), then there exists a stabilizing derivative controller $C(s)$ in (2). Let

$$C(s) = \frac{P_D s}{\tau_D s + 1}. \quad (13)$$

Transfer functions $C(s)G(s)/(1+C(s)G(s))$, $C(s)/(1+C(s)G(s))$, $G(s)/(1+C(s)G(s))$ and $1/(1+C(s)G(s))$ are written by

$$\frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{P_D s}{\tau_D s + 1} Q(s), \quad (14)$$

$$\frac{C(s)}{1 + C(s)G(s)} = \frac{P_D s}{\tau_D s + 1} \left(1 - \frac{P_D s}{\tau_D s + 1} Q(s) \right), \quad (15)$$

$$\frac{G(s)}{1 + C(s)G(s)} = Q(s) \quad (16)$$

and

$$\frac{1}{1 + C(s)G(s)} = 1 - \frac{P_D s}{\tau_D s + 1} Q(s). \quad (17)$$

Since $Q(s) \in RH_\infty$, transfer functions in (14) ~ (17) are stable. We have shown the sufficiency.

We have thus proved Theorem 1. ■

From Theorem 1, we have following corollary. If there exists a stabilizing derivative controller in (2) for the plant $G(s)$, $G(s)$ has a zero in $(-1/\tau_D, 0)$.

Proof: From the assumption that there exists a stabilizing derivative controller in (2) for the plant $G(s)$, $G(s)$ is written by (6). When $Q(s)$ in (6) is denoted by

$$Q(s) = \frac{q_n(s)}{q_d(s)}, \quad (18)$$

where $q_n(s)$ and $q_d(s)$ are polynomials, $G(s)$ in (6) is rewritten by

$$G(s) = \frac{(\tau_D s + 1) q_n(s)}{(\tau_D s + 1) q_d(s) - P_D s q_n(s)}. \quad (19)$$

From this equation, we have

$$\begin{aligned} G\left(-\frac{1}{\tau_D}\right) &= \frac{\left\{ \tau_D \left(-\frac{1}{\tau_D}\right) + 1 \right\} q_n\left(-\frac{1}{\tau_D}\right)}{-P_D \left(-\frac{1}{\tau_D}\right) q_n\left(-\frac{1}{\tau_D}\right)} \\ &= \frac{\left\{ \tau_D \left(-\frac{1}{\tau_D}\right) + 1 \right\}}{-P_D \left(-\frac{1}{\tau_D}\right)} \\ &= 0. \end{aligned} \quad (20)$$

This implies that $G(s)$ has a zero in $(-1/\tau_D, 0)$.

We have thus proved Corollary 3. ■

Next, when the plant $G(s)$ is given, a procedure to check whether or not, $G(s)$ can be stabilized by a derivative controller is presented. A procedure is summarized as follows:

1. $G(s)$ is assumed to be written by

$$G(s) = \frac{n(s)}{d(s)}, \quad (21)$$

where $n(s)$ and $d(s)$ are polynomials. From Corollary 3, $n(s)$ is written by

$$n(s) = (\tau_D s + 1) \bar{n}(s), \quad (22)$$

where $\bar{n}(s)$ is a polynomial.

2. Let $H(s)$ be

$$H(s) = P_D s \bar{n}(s) + d(s). \quad (23)$$

3. Using the Routh-Hurwitz stability method [27], find $P_D \in R$ to make $H(s) = 0$ have no root in the closed right half plane. If $P_D \in R$ to make $H(s) = 0$ have no root in the closed right half plane does not exist, then there exists no stabilizing derivative controller. Conversely, if $P_D \in R$ to make $H(s) = 0$ have no root in the closed right half plane exists, then there exists a stabilizing derivative controller. This fact is confirmed as follows: Using $P_D \in R$ to make $H(s) = 0$ have no root in the closed right half plane and $H(s)$ in (23), $G(s)$ in (21) is rewritten by

$$\begin{aligned} G(s) &= \frac{n(s)}{d(s)} \\ &= \frac{(\tau_D s + 1) \bar{n}(s)}{H(s) - P_D s \bar{n}(s)} \\ &= \frac{(\tau_D s + 1) \bar{n}(s)}{H(s)} \\ &= \frac{H(s)}{1 - \frac{P_D s}{\tau_D s + 1} \frac{(\tau_D s + 1) \bar{n}(s)}{H(s)}}. \end{aligned} \quad (24)$$

Since P_D is settled to make $H(s) = 0$ have no root in the closed right half plane and $\tau_D > 0$, $(\tau_D s + 1) \bar{n}(s)/H(s) \in RH_\infty$. This fact and (24) imply that $G(s)$ in (21) is written by the form in (6). That is, there exists a stabilizing derivative controller for $G(s)$ in (6).

4. THE PARAMETERIZATION OF ALL STABILIZING DERIVATIVE CONTROLLERS

In this section, we present the parameterization of all stabilizing derivative controllers for the plant $G(s)$ written by the form in (6).

The parameterization of all stabilizing derivative controllers for the plant $G(s)$ written by the form in (6) is summarized as follows:

Theorem 2: The parameterization of all stabilizing derivative controllers $C(s)$ for the plant $G(s)$ written by the form in (6) is written by

$$C(s) = \frac{\frac{P_D s}{\tau_D s + 1} + \left(1 - \frac{P_D s}{\tau_D s + 1} Q(s)\right) \tilde{Q}(s)}{1 - Q(s)\tilde{Q}(s)}, \quad (25)$$

where $\tilde{Q}(s) \in RH_\infty$ is written by

$$\tilde{Q}(s) = \frac{\frac{(\alpha - P_D) s}{\tau_D s + 1}}{1 + \frac{(\alpha - P_D) s}{\tau_D s + 1} Q(s)} \quad (26)$$

and $\alpha \in R$ is any number to make $\tilde{Q}(s)$ stable. Proof of this theorem requires following lemma.

Lemma 1: The closed-loop system in (1) is internally stable if and only if $C(s)$ is written by

$$C(s) = \frac{X(s) + D(s)\tilde{Q}(s)}{Y(s) - N(s)\tilde{Q}(s)}, \quad (27)$$

where $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G(s)$ on RH_∞ satisfying

$$G(s) = \frac{N(s)}{D(s)}, \quad (28)$$

$X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are functions satisfying

$$X(s)N(s) + Y(s)D(s) = 1 \quad (29)$$

and $\tilde{Q}(s) \in RH_\infty$ is any function [19].

Using Lemma 1, we shall show the proof of Theorem 2.

Proof: Coprime factors $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ for $G(s)$ in (6) are derived by

$$N(s) = Q(s) \quad (30)$$

and

$$D(s) = 1 - \frac{P_D s}{\tau_D s + 1} Q(s). \quad (31)$$

Then $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ satisfying (29) are given by

$$X(s) = \frac{P_D s}{\tau_D s + 1} \quad (32)$$

and

$$Y(s) = 1. \quad (33)$$

From Lemma 1, the parameterization of all stabilizing controllers for the plant $G(s)$ in (6) is written by

$$C(s) = \frac{\frac{P_D s}{\tau_D s + 1} + \left\{1 - \frac{P_D s}{\tau_D s + 1} Q(s)\right\} \tilde{Q}(s)}{1 - Q(s)\tilde{Q}(s)}, \quad (34)$$

where $\tilde{Q}(s) \in RH_\infty$ is any function.

Next, we show that if $C(s)$ in (34) works as a derivative controller, that is, $C(s)$ in (34) is written by

$$C(s) = \frac{\alpha s}{\tau_D s + 1}, \quad (35)$$

then $\tilde{Q}(s)$ is written by (26) and α makes $\tilde{Q}(s)$ stable. From the assumption that $C(s)$ in (34) is written by (35), substitution of (35) for (34) gives (26). From Lemma 1, $\tilde{Q}(s)$ in (26) satisfies $\tilde{Q}(s) \in RH_\infty$. Therefore, α makes $\tilde{Q}(s)$ stable.

Conversely, we show that if α makes $\tilde{Q}(s)$ stable and $\tilde{Q}(s)$ is set by (26), then $C(s)$ in (25) works as a derivative controller and makes the closed-loop system in (1) stable. Substitution of (26) for (25) gives us

$$C(s) = \frac{\alpha s}{\tau_D s + 1}. \quad (36)$$

Thus we have shown that if $\tilde{Q}(s)$ is set by (26), then $C(s)$ in (25) is constant. In addition, since α makes $\tilde{Q}(s)$ stable, $\tilde{Q}(s)$ in (26) satisfies $\tilde{Q}(s) \in RH_\infty$. Therefore, $C(s)$ in (25) makes the closed-loop system in (1) stable.

We have thus proved Theorem 2. \blacksquare

Next, we present a method to obtain admissible set of derivative controllers, which is the set of stabilizing derivative controllers, using graphical method. Admissible set of derivative controllers, which is the set of derivative controllers to make the closed-loop system in (1) stable, is easily obtained using the Nyquist theorem. Because $Q(s)$ in (26) satisfies $Q(s) \in RH_\infty$. That is, admissible set of derivative controllers is obtained using the gain margin of $sQ(s)/(\tau_D s + 1)$. When the gain margin of $sQ(s)/(\tau_D s + 1)$ and that of $-sQ(s)/(\tau_D s + 1)$ are g_{q1} and g_{q2} , elements α of admissible set of derivative controllers are satisfying

$$P_D \leq \alpha < P_D + 10^{\frac{g_{q1}}{20}} \quad (37)$$

and

$$P_D - 10^{\frac{g_{q2}}{20}} < \alpha \leq P_D. \quad (38)$$

The fact that admissible set of derivative controllers satisfies (37) or (38) is confirmed as follows. From

Theorem 2, $C(s) = \alpha s / (\tau_D s + 1)$ is an element of admissible set if and only if $\{(\alpha - P_D)s / (\tau_D s + 1)\} / [1 + \{(\alpha - P_D)sQ(s) / (\tau_D s + 1)\}] \in RH_\infty$. From $Q(s) \in RH_\infty$, $\tau_D > 0$ and the Nyquist theorem, the necessary and sufficient condition that α is an element of admissible set is that the Nyquist plot of $(\alpha - P_D)sQ(s) / (\tau_D s + 1)$ does not encircle $(-1, 0)$. This implies that when we illustrate the Nyquist plot of $sQ(s) / (\tau_D s + 1)$ in Fig. 1 and that of $-sQ(s) / (\tau_D s + 1)$ in Fig. 2, the condition is equivalent to satisfy

$$0 \leq (\alpha - P_D) \left| \overrightarrow{OP_1} \right| < 1 \quad (39)$$

and

$$0 \leq -(\alpha - P_D) \left| \overrightarrow{OP_2} \right| < 1. \quad (40)$$

From the definition of the gain margin, the gain mar-

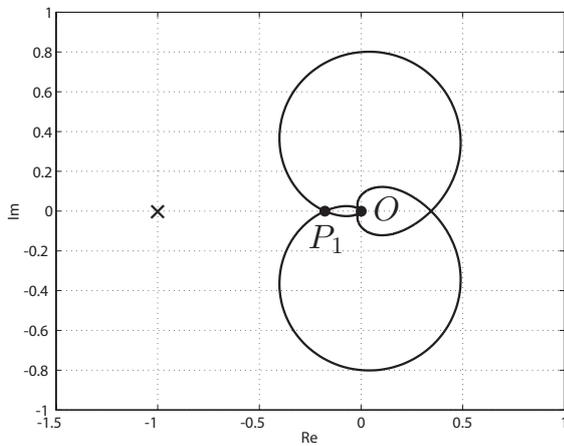


Fig.1: The Nyquist plot of $\frac{sQ(s)}{\tau_D s + 1}$

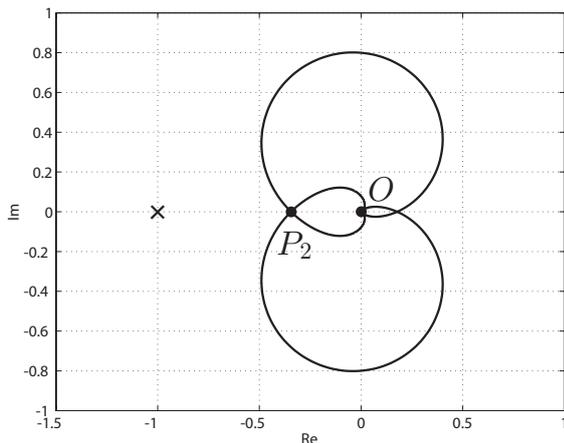


Fig.2: The Nyquist plot of $-\frac{sQ(s)}{\tau_D s + 1}$

gin g_{q1} and g_{q2} are written by

$$g_{q1} = 20 \log \frac{1}{\left| \overrightarrow{OP_1} \right|} \quad (41)$$

and

$$g_{q2} = 20 \log \frac{1}{\left| \overrightarrow{OP_2} \right|}, \quad (42)$$

respectively. Substituting (41) and (42) into (39) and (40), we have (37) and (38).

5. THE PARAMETERIZATION OF ALL PLANTS STABILIZED BY A PROPORTIONAL-DERIVATIVE CONTROLLER

In this section, we propose the parameterization of all plants $G(s)$ stabilized by a proportional-derivative controller.

The parameterization of all plants $G(s)$ stabilized by a proportional-derivative controller is summarized in the following theorem.

Theorem 3: The parameterization of all plants $G(s)$ stabilized by a proportional-derivative controller is written by

$$G(s) = \frac{Q(s)}{1 - \frac{(P_P \tau_D + P_D)s + P_P}{\tau_D s + 1} Q(s)}, \quad (43)$$

where $P_P \in R$ and $P_D \in R$ are any real numbers and $Q(s) \in RH_\infty$ is any function.

Proof: First, the necessity is shown. That is, we show that if a proportional-derivative controller $C(s)$ in (3) stabilizes the plant $G(s)$, then the plant $G(s)$ is written by (43). From the assumption that a proportional-derivative controller $C(s)$ in (3) stabilizes the plant $G(s)$, transfer functions $C(s)G(s) / (1 + C(s)G(s))$, $C(s) / (1 + C(s)G(s))$, $G(s) / (1 + C(s)G(s))$ and $1 / (1 + C(s)G(s))$ are stable. From simple manipulations, we have

$$\frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1} G(s)}{1 + \frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1} G(s)}, \quad (44)$$

$$\frac{C(s)}{1 + C(s)G(s)} = \frac{\frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1}}{1 + \frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1} G(s)}, \quad (45)$$

$$\frac{G(s)}{1 + C(s)G(s)} = \frac{G(s)}{1 + \frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1} G(s)} \quad (46)$$

and

$$\frac{1}{1 + C(s)G(s)} = \frac{1}{1 + \frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1} G(s)}. \quad (47)$$

Since the transfer function in (46) is stable, when $Q(s)$ is settled by

$$Q(s) = \frac{G(s)}{1 + \frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1} G(s)}, \quad (48)$$

then $Q(s)$ in (48) satisfies $Q(s) \in RH_\infty$. Equation (48) is rewritten by

$$G(s) = \frac{Q(s)}{1 - \frac{(a_P \tau_D + a_D)s + a_P}{\tau_D s + 1} Q(s)}. \quad (49)$$

Let $P_P = a_P$ and $P_D = a_D$. Equation (49) corresponds to (43). We have shown the necessity.

Next, we show the sufficiency. That is, we show that if the plant $G(s)$ is written by (43), then there exists a stabilizing proportional-derivative controller $C(s)$ in (3). Let

$$C(s) = P_P + \frac{P_D s}{\tau_D s + 1}. \quad (50)$$

Transfer functions $C(s)G(s)/(1+C(s)G(s))$, $C(s)/(1+C(s)G(s))$, $G(s)/(1+C(s)G(s))$ and $1/(1+C(s)G(s))$ are written by

$$\frac{C(s)G(s)}{1+C(s)G(s)} = \frac{(P_P \tau_D + P_D)s + P_P}{\tau_D s + 1} Q(s), \quad (51)$$

$$\begin{aligned} & \frac{C(s)}{1+C(s)G(s)} \\ &= \frac{(P_P \tau_D + P_D)s + P_P}{\tau_D s + 1} \\ & \left\{ 1 - \frac{(P_P \tau_D + P_D)s + P_P}{\tau_D s + 1} Q(s) \right\}, \quad (52) \end{aligned}$$

$$\frac{G(s)}{1+C(s)G(s)} = Q(s) \quad (53)$$

and

$$\frac{1}{1+C(s)G(s)} = 1 - \frac{(P_P \tau_D + P_D)s + P_P}{\tau_D s + 1} Q(s). \quad (54)$$

Since $Q(s) \in RH_\infty$, transfer functions in (51) ~ (54) are stable. We have shown the sufficiency.

We have thus proved Theorem 3. \blacksquare

From Theorem 3, we have following corollary. If there exists a stabilizing proportional-derivative controller in (3) for the plant $G(s)$, $G(s)$ has a zero in $(-1/\tau_D, 0)$.

Proof: From the assumption that there exists a stabilizing proportional-derivative controller in (3) for the plant $G(s)$, $G(s)$ is written by (43). When $Q(s)$ in (43) is denoted by

$$Q(s) = \frac{q_n(s)}{q_d(s)}, \quad (55)$$

where $q_n(s)$ and $q_d(s)$ are polynomials, $G(s)$ in (43) is rewritten by

$$G(s) = \frac{(\tau_D s + 1) q_n(s)}{(\tau_D s + 1) q_d(s) - \{(P_P \tau_D + P_D)s + P_P\} q_n(s)}. \quad (56)$$

From this equation, we have

$$G\left(-\frac{1}{\tau_D}\right) = 0. \quad (57)$$

This implies that $G(s)$ has a zero in $(-1/\tau_D, 0)$.

We have thus proved Corollary 5. \blacksquare

Next, when the plant $G(s)$ is given, a procedure to check whether or not, $G(s)$ can be stabilized by a proportional-derivative controller is presented. A procedure is summarized as follows:

1. $G(s)$ is assumed to be written by (21), where $n(s)$ and $d(s)$ are polynomials. From Corollary 5, $n(s)$ is written by

$$n(s) = (\tau_D s + 1) \bar{n}(s), \quad (58)$$

where $\bar{n}(s)$ is a polynomial.

2. Let $H_2(s)$ be

$$H_2(s) = \{(P_P \tau_D + P_D)s + P_P\} \bar{n}(s) + d(s). \quad (59)$$

3. Using the Routh-Hurwitz stability method [27], find $P_P \in R$ and $P_D \in R$ to make $H_2(s) = 0$ have no root in the closed right half plane. If $P_P \in R$ and $P_D \in R$ to make $H_2(s) = 0$ have no root in the closed right half plane does not exist, then there exists no stabilizing proportional-derivative controller. Conversely, if $P_P \in R$ and $P_D \in R$ to make $H_2(s) = 0$ have no root in the closed right half plane exists, then there exists a stabilizing proportional-derivative controller. This fact is confirmed as follows: Using $P_P \in R$ and $P_D \in R$ to make $H_2(s) = 0$ have no root in the closed right half plane and $H_2(s)$ in (59), $G(s)$ in (21) is rewritten by

$$\begin{aligned} G(s) &= \frac{n(s)}{d(s)} \\ &= \frac{(\tau_D s + 1) \bar{n}(s)}{H_2(s) - \{(P_P \tau_D + P_D)s + P_P\} \bar{n}(s)} \\ &= \frac{(\tau_D s + 1) \bar{n}(s)}{H_2(s)} \\ &= \frac{(\tau_D s + 1) \bar{n}(s)}{1 - \frac{(P_P \tau_D + P_D)s + P_P}{\tau_D s + 1} \frac{(\tau_D s + 1) \bar{n}(s)}{H_2(s)}}. \quad (60) \end{aligned}$$

Since P_P and P_D are settled to make $H_2(s) = 0$ in (59) have no root in the closed right half plane and $\tau_D > 0$, $(\tau_D s + 1) \bar{n}(s)/H_2(s) \in RH_\infty$. This fact and (60) imply that $G(s)$ in (21) is written by the form in (43). That is, there exists a stabilizing proportional-derivative controller for $G(s)$ in (43).

6. THE PARAMETERIZATION OF ALL STABILIZING PROPORTIONAL-DERIVATIVE CONTROLLERS

In this section, we present the parameterization of all stabilizing proportional-derivative controllers for the plant $G(s)$ written by the form in (43).

The parameterization of all stabilizing proportional-derivative controllers for the plant $G(s)$ written by the form in (43) is summarized as follows:

Theorem 4: The parameterization of all stabilizing proportional-derivative controllers $C(s)$ for the plant $G(s)$ written by the form in (43) is written by

$$C(s) = \frac{C_n(s)}{C_d(s)}, \quad (61)$$

where

$$\begin{aligned} C_n(s) &= \frac{(P_P\tau_D + P_D)s + P_P}{\tau_D s + 1} \\ &+ \left\{ 1 - \frac{(P_P\tau_D + P_D)s + P_P}{\tau_D s + 1} Q(s) \right\} \tilde{Q}(s), \end{aligned} \quad (62)$$

$$C_d(s) = 1 - Q(s)\tilde{Q}(s), \quad (63)$$

$\tilde{Q}(s) \in RH_\infty$ is written by

$$\tilde{Q}(s) = \frac{\tilde{Q}_n(s)}{\tilde{Q}_d(s)}, \quad (64)$$

$$\begin{aligned} \tilde{Q}_n(s) &= \frac{\{(\alpha_P - P_P)\tau_D + (\alpha_D - P_D)\}s + (\alpha_P - P_P)}{\tau_D s + 1}, \end{aligned} \quad (65)$$

$$\begin{aligned} \tilde{Q}_d(s) &= 1 + \frac{\{(\alpha_P - P_P)\tau_D + (\alpha_D - P_D)\}s + (\alpha_P - P_P)}{\tau_D s + 1} \\ &\cdot Q(s), \end{aligned} \quad (66)$$

$\alpha_P \in R$ and $\alpha_D \in R$ are any numbers to make $\tilde{Q}(s)$ stable.

Proof: Coprime factors $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ for $G(s)$ in (43) are derived by

$$N(s) = Q(s) \quad (67)$$

and

$$D(s) = 1 - \frac{(P_P\tau_D + P_D)s + P_P}{\tau_D s + 1} Q(s). \quad (68)$$

Then $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ satisfying (29) are given by

$$X(s) = \frac{(P_P\tau_D + P_D)s + P_P}{\tau_D s + 1} \quad (69)$$

and

$$Y(s) = 1. \quad (70)$$

From Lemma 1, the parameterization of all stabilizing controllers for the plant $G(s)$ in (43) is written by where $C_n(s)$ and $C_d(s)$ are written by (62) and (63), respectively, and $\tilde{Q}(s) \in RH_\infty$ is any function.

Next, we show that if $C(s)$ in (61) works as a proportional-derivative controller, that is, $C(s)$ in (61) is written by

$$C(s) = \alpha_P + \frac{\alpha_D s}{\tau_D s + 1}, \quad (71)$$

then $\tilde{Q}(s)$ is written by (64) and α_P and α_D make $\tilde{Q}(s)$ stable. From the assumption that $C(s)$ in (61) is written by (71), substitution of (71) for (61) gives (64). From Lemma 1, $\tilde{Q}(s)$ in (64) satisfies $\tilde{Q}(s) \in RH_\infty$. Therefore, α_P and α_D make $\tilde{Q}(s)$ stable.

Conversely, we show that if α_P and α_D make $\tilde{Q}(s)$ stable and $\tilde{Q}(s)$ is set by (64), then $C(s)$ in (61) works as a proportional-derivative controller and makes the closed-loop system in (1) stable. Substitution of (64) for (61) gives us

$$C(s) = \alpha_P + \frac{\alpha_D s}{\tau_D s + 1}. \quad (72)$$

Thus we have shown that if $\tilde{Q}(s)$ is set by (64), then $C(s)$ in (61) is constant. In addition, since α_P and α_D make $\tilde{Q}(s)$ stable, $\tilde{Q}(s)$ in (64) satisfies $\tilde{Q}(s) \in RH_\infty$. Therefore, $C(s)$ in (61) makes the closed-loop system in (1) stable.

We have thus proved Theorem 4. \blacksquare

7. NUMERICAL EXAMPLE

In this section, a numerical example is illustrated to show the effectiveness of the proposed method.

Consider the problem to design a stabilizing derivative controller $C(s)$ in (2) for the plant $G(s)$ written by

$$G(s) = \frac{5s + 5}{s^4 + 8s^3 + 21s^2 + 12s + 8}. \quad (73)$$

Since the plant $G(s)$ in (73) is rewritten by (6), where P_D , $Q(s)$ and τ_D are written by

$$P_D = 2, \quad (74)$$

$$Q(s) = \frac{5}{s^3 + 7s^2 + 14s + 8} \in RH_\infty \quad (75)$$

and

$$\tau_D = 1. \quad (76)$$

Next, using the result in Section 4, we obtain admissible set of derivative controller

$$C(s) = \frac{\alpha s}{\tau_D s + 1}. \quad (77)$$

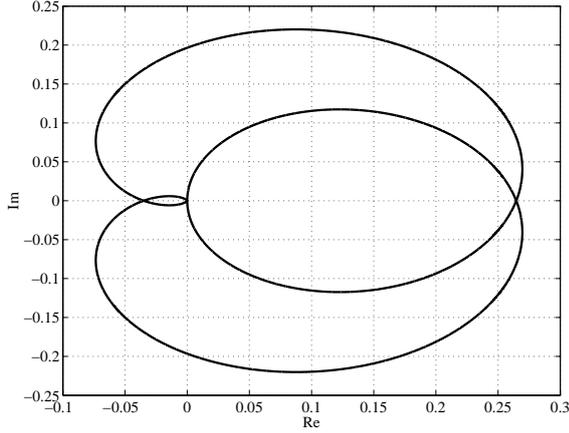


Fig.3: The Nyquist plot of $\frac{sQ(s)}{\tau_D s + 1}$ with $Q(s)$ in (75)

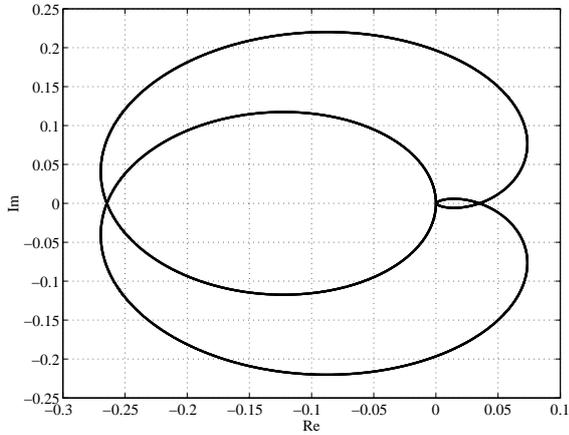


Fig.4: The Nyquist plot of $-\frac{sQ(s)}{\tau_D s + 1}$ with $Q(s)$ in (75)

Nyquist plots of $sQ(s)/(\tau_D s + 1)$ and $-sQ(s)/(\tau_D s + 1)$ with $Q(s)$ in (75) are shown in Fig. 3 and Fig. 4, respectively. From Fig. 3 and Fig. 4, the gain margin g_{q1} of $sQ(s)/(\tau_D s + 1)$ and the gain margin g_{q2} of $-sQ(s)/(\tau_D s + 1)$ are written by

$$g_{q1} = 20 \log \frac{1}{0.035} = 20 \log 28.57 \quad (78)$$

and

$$g_{q2} = 20 \log \frac{1}{0.265} = 20 \log 3.77, \quad (79)$$

respectively. From (37) and (38), elements α in (77) of admissible set of derivative controllers are written by

$$2 \leq \alpha < 2 + 28.57 = 30.57 \quad (80)$$

or

$$2 - 3.77 = -1.77 < \alpha \leq 2, \quad (81)$$

that is

$$-1.77 < \alpha < 30.57. \quad (82)$$

Using above-mentioned parameters, we have a stabilizing derivative controller $C(s)$. Selecting an element $\alpha = 20$, the controller $C(s)$ is an element of admissible set, since

$$C(s) = \frac{\alpha s}{\tau_D s + 1} = \frac{20s}{s + 1} \quad (83)$$

satisfies (39). Using designed derivative controller, the response of the output $y(t)$ of the closed-loop system in (1) for the step reference input $r(t) = 1$ is shown in Fig. 5. Figure 5 shows that the designed

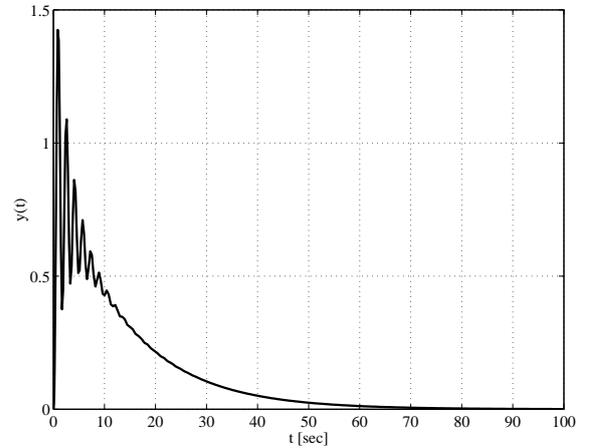


Fig.5: Response of the output $y(t)$ of the closed-loop system in (1) for the step reference input $r(t) = 1$ using a stabilizing derivative controller $C(s)$ in (83)

derivative controller $C(s)$ in (83) for the plant $G(s)$ makes the closed-loop system stable.

Next, when the element α does not satisfy (39), that is α is not included in admissible set, we show that the closed-loop system is unstable. When $C(s)$ is settled by

$$C(s) = \frac{\alpha s}{\tau_D s + 1} = \frac{30.59s}{s + 1}, \quad (84)$$

which is not an element of admissible set, the response of the output $y(t)$ of the closed-loop system in (1) for the step reference input $r(t) = 1$ is shown in Fig. 6. Figure 6 shows that the closed-loop system in (1) is unstable.

In this way, it is shown that if the plant $G(s)$ is written by the form (6), we can easily design a stabilizing derivative controller $C(s)$ and obtain admissible set of stabilizing derivative controllers.

8. CONCLUSIONS

In this paper, we clarified parameterizations of all plants stabilized by a derivative controller and of

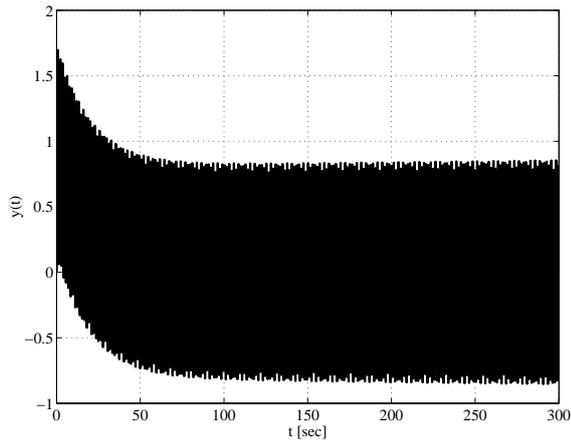


Fig.6: Response of the output $y(t)$ of the closed-loop system in (1) for the step reference input $r(t) = 1$ using a derivative controller $C(s)$ in (84)

all plants stabilized by a proportional-derivative controller. In addition, we presented parameterizations of all stabilizing derivative controllers for the plant that can be stabilized by a derivative controller and of all stabilizing proportional-derivative controllers for the plant that can be stabilized by a proportional-derivative controller. Admissible set of derivative parameters to make the closed-loop system stable was clarified. A numerical example was illustrated to show the effectiveness of the proposed method.

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