Criterion of Approximation for Designing 2×2 Feedback Systems with Inputs Satisfying Bounding Conditions

Tadchanon Chuman[*1] and **Suchin Arunsawatwong**[*2], Non-members

ABSTRACT

A common practice in designing a feedback system with a non-rational transfer matrix is to replace the non-rational matrix with an appropriate rational approximant during the design process so that reliable and efficient computational tools for rational systems can be utilized. Consequently, a criterion of approximation is required to ensure that the controller obtained from the approximant still provides satisfactory results for the original system. This paper derives such a criterion for the case of two-input two-output feedback systems in which the design objective is to ensure that the errors and the controller outputs always stay within prescribed bounds whenever the inputs satisfy certain bounding conditions. For a given rational approximant matrix, the criterion is expressed as a set of inequalities that can be solved in practice. It will be seen that the criterion generalizes a known result for single-input single-output systems. Finally, a controller for a binary distillation column is designed by using the criterion in conjunction with the method of inequalities.

Keywords: Feedback systems, Rational approximants, Two-input two-output systems, Non-rational systems, Peak outputs, Method of inequalities.

1. INTRODUCTION

A common practice in designing a feedback system in which the plant is described by non-rational transfer functions, is to replace the non-rational functions with rational approximants so that well-developed computational tools for rational systems can be utilized. To this end, a number of researchers have been prompted to investigate and/or develop methods for approximating a non-rational transfer function by a rational one. For example, Gibilaro and Lees [1] and Zakian [2] investigated methods for simplifying transfer functions using moment approximants, whereas Lam [3] reduced models of delay systems using Padé approximants for exponential functions. In addition, Gu et al. [4] used a method based on Fourier transform techniques. A number of references can be found in the literature concerning how to obtain rational approximants (see, for example, [1–4] and the references therein).

The approach of replacing non-rational transfer functions with rational approximants in the design process is useful especially when computational tools for non-rational transfer functions are not readily available. However, it may fail to give satisfactory results for the original system if the approximants are not sufficiently close to the original models. In order to ensure that the design carried out with the approximants is valid for the original system, a criterion of approximation needs to be explicitly taken into account in the formulation of the design problem.

Consider the two-input two-output feedback system described by

$$\left.\begin{aligned} u_i &= k_{i1} \star e_1 + k_{i2} \star e_2 \\ e_i &= f_i - g_{i1} \star u_1 - g_{i2} \star u_2 \end{aligned}\right\}, \quad i = 1, 2 \qquad (1)$$

where $\mathbf{f} \triangleq [f_1, f_2]^T$ is the input vector, $\mathbf{G}(s) \triangleq [G_{ij}(s)]_{2\times 2}$ is the plant transfer matrix and $\mathbf{K}(s, \mathbf{p}) \triangleq [K_{ij}(s, \mathbf{p})]_{2\times 2}$ is the controller transfer matrix characterized by a design parameter $\mathbf{p} \in \mathbb{R}^N$ (see Fig. 1). Let $\mathbf{e} \triangleq [e_1, e_2]^T$ and $\mathbf{u} \triangleq [u_1, u_2]^T$ be the error vector and the control vector of the system (1), respectively. Also, let $k_{ij} : [0, \infty) \to \mathbb{R}$ and $g_{ij} : [0, \infty) \to \mathbb{R}$ denote the inverse Laplace transforms of $K_{ij}(s, \mathbf{p})$ and $G_{ij}(s)$, respectively. The symbol $\star$ denotes the convolution; that is, for $x : [0, \infty) \to \mathbb{R}$ and $y : [0, \infty) \to \mathbb{R}$,

$$(x \star y)(t) = \int_0^t x(t - \lambda)\, y(\lambda)\, d\lambda, \quad t \geq 0.$$

Suppose that the input vector $\mathbf{f}$ is known only to the extent that for $i = 1, 2$, the input $f_i : [0, \infty) \to \mathbb{R}$ belongs to the possible set $\mathcal{P}_i$ where $\mathcal{P}_i \subseteq \mathbb{L}_\infty$ and, as usual, $\mathbb{L}_\infty$ denotes the set of all bounded functions

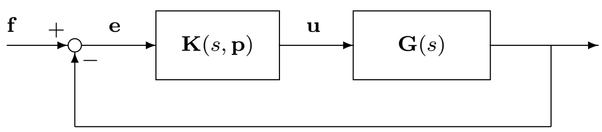

Fig.1: *The feedback system given in* (1).

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* The authors are with Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University, Bangkok, Thailand. Emails: tadchanon01@hotmail.com[1] and suchin.a@chula.ac.th[2].

defined on the half line $[0, \infty)$. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are the sets of input signals whose magnitude and whose slope satisfy certain bounding conditions; see Section 2 for details on various models of the possible sets $\mathcal{P}_i$. In this connection, define the Cartesian product of the possible sets as follows.

$$\boldsymbol{\mathcal{P}} \triangleq \mathcal{P}_1 \times \mathcal{P}_2.$$

For the system (1), define the performance measures

$$\hat{e}_i \triangleq \sup_{\mathbf{f} \in \boldsymbol{\mathcal{P}}} \|e_i\|_\infty \text{ and } \hat{u}_i \triangleq \sup_{\mathbf{f} \in \boldsymbol{\mathcal{P}}} \|u_i\|_\infty. \tag{2}$$

Note that $\hat{e}_i$ and $\hat{u}_i$ are called the peak values of e_i and u_i, respectively, in association with the Cartesian product space $\boldsymbol{\mathcal{P}}$. Clearly, the peaks $\hat{e}_i$ and $\hat{u}_i$ are functions of the design parameter $\mathbf{p}$.

The problem considered in the paper is to find a design parameter $\mathbf{p} \in \mathbb{R}^N$ such that the following criteria are satisfied.

$$\hat{e}_1 \le \mathcal{E}_1 \text{ and } \hat{e}_2 \le \mathcal{E}_2, \tag{3}$$

$$\hat{u}_1 \le \mathcal{U}_1 \text{ and } \hat{u}_2 \le \mathcal{U}_2 \tag{4}$$

where the bounds $\mathcal{E}_i$ and $\mathcal{U}_i$ are specified. One should note that the inequalities of the form (3) and (4) have long been investigated by many researchers [5–14] and have been used as design criteria for ensuring that $|e_i(t)|$ and $|u_i(t)|$ are kept within respective bounds $\mathcal{E}_i$ and $\mathcal{U}_i$ for all t and for all $\mathbf{f} \in \boldsymbol{\mathcal{P}}$.

Following previous work [6–9, 12–14], it is readily appreciated that satisfactory computational methods are available for obtaining solutions of the inequalities (3) and (4) for various models of the possible set $\mathcal{P}_i$ when every element of the plant transfer matrix $\mathbf{G}(s)$ is a rational function. Furthermore, for non-rational systems that are described by retarded delay differential equations and retarded fractional delay differential equations, the inequalities (3) and (4) can be solved by means of the computational tools presented in [15–17]; also see [18] for a design example. However, the numerical solution of the inequalities (3) and (4) for general non-rational systems is still an open research problem.

To alleviate the difficulty in solving the design problem for non-rational systems, Zakian [10–14] derived a criterion of approximation for single-input single-output (SISO) feedback systems. The criterion provides practical sufficient conditions for ensuring that the design solution for the nominal system (that is, the system obtained by replacing the plant transfer function $G(s)$ in the original system with a rational approximant $G^*(s)$) still provides satisfactory results in the sense that the original design criteria (3) and (4) are satisfied. The criterion was applied to the design of SISO control systems with a time delay [19, 20]. Furthermore, on the basis of Zakian's criterion, useful inequalities for designing control systems with plant uncertainties were derived in [11, 13, 14] and were used in designing robust control systems in [14, 21, 22]. Such inequalities have recently been extended and used in developing methods for designing feedback systems with a nonlinear element [23, 24]. It should be noted that Zakian's criterion is not applicable to the case of MIMO systems (1).

The main purpose of the paper is to derive a criterion of approximation for the system (1) where the plant transfer matrix $\mathbf{G}(s)$ is replaced with a rational approximant matrix $\mathbf{G}^*(s)$ during the design process. The criterion provides sufficient conditions for ensuring that when the controller $\mathbf{K}(s, \mathbf{p})$ that is obtained from the design carried out with $\mathbf{G}^*(s)$ is used in the original system (1), the design criteria (3) and (4) are satisfied. As will be seen later, the results obtained in the paper are generalizations of Zakian's results [10–12] for two-input two-output systems.

The structure of the paper is as follows. Section 2 provides a review of Zakian's criterion of approximation for SISO feedback systems. Section 3 derives the criterion of approximation for 2×2 systems; in fact, the main results of the paper are stated in Theorems 3 and 4. Section 4 derives sufficient conditions for the finiteness of the associated matrices (see Theorem 5); the conditions are used for enabling a numerical algorithm to search for a design solution in the space of design parameters. Section 5 illustrates the effectiveness of the developed method by carrying out an example of controller design for a binary distillation column using the criterion in conjunction with the method of inequalities. Finally, the conclusions and discussion are given in Section 6.

2. CRITERION OF APPROXIMATION FOR SISO SYSTEMS

This section recapitulates Zakian's criterion of approximation [10–14], which was derived for SISO feedback systems.

Consider the SISO feedback system described by

$$\begin{aligned} u &= e \star k, \\ e &= f - g \star u \end{aligned} \tag{5}$$

where $G(s)$ denotes the plant transfer function and $K(s, \mathbf{p})$ denotes the controller transfer function with the design parameter $\mathbf{p} \in \mathbb{R}^N$ (see Fig. 2). The responses e and u are the error and the control of the system, respectively, and $k : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ denote the inverse Laplace transforms of $K(s, \mathbf{p})$ and $G(s)$, respectively.

Suppose that f is a *possible input* (that is, input that happens or is likely to happen in practice) and is known only to the extent that it belongs to a set $\mathcal{P}$. Accordingly, the set $\mathcal{P}$ contains all possible inputs and is called the *possible set*. In this work, the set $\mathcal{P}$ is a subset of $\mathbb{L}_\infty$, which denotes the set of all bounded functions defined on $[0, \infty)$.

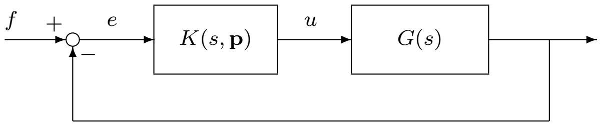

Fig.2: *The SISO feedback system given in* (5).

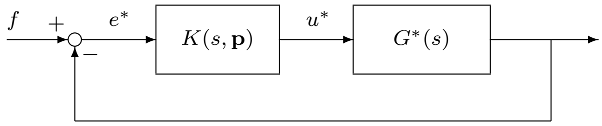

Fig.3: *The nominal system for* (5).

Note, in passing, that different models of the possible set $\mathcal{P}$ have been investigated by many researchers. For example, the set $\mathcal{P}$ given by

$$\mathcal{P} = \{f : \|f\|_\infty \le M_\infty \text{ and } \|\dot{f}\|_\infty \le D_\infty\} \tag{6}$$

was investigated by [5, 9, 14, 25], while the set $\mathcal{P}$ given by

$$\mathcal{P} = \{f : \|f\|_2 \le M_2 \text{ and } \|\dot{f}\|_2 \le D_2\} \tag{7}$$

was investigated in [6] (see also Chapter 2 of [14]) and was used in the controller design for buildings subject to seismic disturbances in [26]. Recently, the set $\mathcal{P}$ given by

$$\mathcal{P} = \{f : \|f\|_i \le M_i \text{ and } \|\dot{f}\|_i \le D_i, \ \ i = 2, \infty\} \tag{8}$$

has been considered in [7, 8]. For details on the characterization of the above sets $\mathcal{P}$ and their implications, see, for example, [6, 8, 14] and the references therein.

In connection with the possible set $\mathcal{P}$, define the performance measures $\hat{e}$ and $\hat{u}$ as follows.

$$\hat{e} \triangleq \sup_{f\in\mathcal{P}} \|e\|_\infty \quad \text{and} \quad \hat{u} \triangleq \sup_{f\in\mathcal{P}} \|u\|_\infty. \tag{9}$$

Note that $\hat{e}$ and $\hat{u}$ are the error peak and the controller output peak for the possible set $\mathcal{P}$. Methods for computing such peak values in association with the possible sets described by (6), (7) and (8) are readily available. For the details of such methods, readers are referred to [6, 8, 14] and also the references cited therein.

Suppose that the design problem for the system (5) is to determine a design parameter $\mathbf{p} \in \mathbb{R}^N$ such that the following design criteria are satisfied.

$$\hat{e} \le E_{\max} \quad \text{and} \quad \hat{u} \le U_{\max} \tag{10}$$

where the bounds $E_{\max} > 0$ and $U_{\max} > 0$ are specified.

Let $G^*(s)$ be an approximant of the original plant transfer function $G(s)$. In connection with the system (5), the nominal system (shown in Fig. 3) is described by

$$\begin{aligned} u^* &= e^* \star k, \\ e^* &= f - g^* \star u^* \end{aligned} \tag{11}$$

where e^* and u^* are the error and the control of the nominal system (11), respectively, and $g^* : [0, \infty) \to \mathbb{R}$ be the inverse Laplace transform of $G^*(s)$. Let $\hat{e}^*$ and $\hat{u}^*$ denote the peak values of e^* and u^*, respectively, for the possible set $\mathcal{P}$. That is to say,

$$\hat{e}^* \triangleq \sup_{f\in\mathcal{P}} \|e^*\|_\infty \quad \text{and} \quad \hat{u}^* \triangleq \sup_{f\in\mathcal{P}} \|u^*\|_\infty. \tag{12}$$

Let μ denote the approximation index and be defined by

$$\mu \triangleq \|w\|_1$$

where $w : [0, \infty) \to \mathbb{R}$ is the inverse Laplace transform of $W(s)$ given by

$$W(s, \mathbf{p}) = \frac{K(s, \mathbf{p})}{1 + K(s, \mathbf{p})G^*(s)} \left[G(s) - G^*(s)\right].$$

Now it is ready to state the main theorem on the criterion of approximation for SISO systems.

Theorem 1: [10] For the nominal system (11), suppose that $\hat{e}^* < \infty$ and $\hat{u}^* < \infty$. Let $\mu < 1$. Then the original design criteria (10) for the system (5) are satisfied if the following inequalities hold.

$$\frac{\hat{e}^*}{1-\mu} \le E_{\max} \quad \text{and} \quad \frac{\hat{u}^*}{1-\mu} \le U_{\max}. \tag{13}$$

Furthermore, it follows that

$$\begin{aligned} \frac{\hat{e}^*}{1+\mu} \le \hat{e} \le \frac{\hat{e}^*}{1-\mu}, \\ \frac{\hat{u}^*}{1+\mu} \le \hat{u} \le \frac{\hat{u}^*}{1-\mu}. \end{aligned} \tag{14}$$

Proof: See [10]. ∎

Since the approximant $G^*(s)$ and the controller $K(s, \mathbf{p})$ are rational functions, the peaks $\hat{e}^*$ and $\hat{u}^*$ of the nominal system (11) are easily obtainable for various cases of possible sets in (6)–(8); for more details, see [8, 9, 14] and the references therein. When tools for stability analysis and computing time-responses for non-rational systems are not available, inequalities (13) are more computationally tractable than the original design criteria (10).

Following the method of inequalities [10, 12–14, 27], it is readily appreciated that in solving the inequalities (13) by numerical methods, it is necessary that a search algorithm should start from a point $\mathbf{p} \in \mathbb{R}^N$ such that $\mu(\mathbf{p}) < \infty$. In this connection, the following theorem provides a practical sufficient condition that enables the algorithm to start from an arbitrary point in $\mathbb{R}^N$.

Define Λ as the set of all of the finite poles of the transfer function

$$\frac{F(s)}{U^*(s)} = \frac{K(s, \mathbf{p})}{1 + K(s, \mathbf{p})G^*(s)}$$

where $F(s)$ and $U^*(s)$ are the Laplace transforms of f and u^*, respectively.

Theorem 2: [10,12,13] Assume that $G^*(s)$ and $K(s,\mathbf{p})$ are rational transfer functions. Then $\mu < \infty$ if the two conditions hold.

(a) $\|z\|_1 < \infty$, where $z \triangleq g - g^*$.

(b) $\mathrm{Re}\,\lambda(\mathbf{p}) < 0$ for all $\lambda(\mathbf{p}) \in \Lambda$.

Proof: See [10,12,13]. ■

It is clear from Theorem 2 that with an appropriate choice of approximant $G^*(s)$, condition (a) is always satisfied. Consequently, condition (b) provides a useful inequality for computing a point $\mathbf{p}$ satisfying $\mu(\mathbf{p}) < \infty$ by numerical methods. This is because the number $\mathrm{Re}\,\lambda(\mathbf{p})$ is finite for every $\mathbf{p} \in \mathbb{R}^N$. See [12] for more discussion.

3. CRITERION OF APPROXIMATION FOR 2×2 SYSTEMS

This section derives the criterion of approximation for the system (1). To this end, let $\mathbf{G}^*(s) \triangleq [G^*_{ij}(s)]_{2\times 2}$ be a rational approximant matrix of $\mathbf{G}(s)$. Then replacing $\mathbf{G}(s)$ with $\mathbf{G}^*(s)$ yields the resulting system which is called a nominal system (see Fig. 4) and described by

$$\left.\begin{aligned} u_i^* &= k_{i1} \star e_1^* + k_{i2} \star e_2^* \\ e_i^* &= f_i - g_{i1} \star u_1^* - g_{i2} \star u_2^* \end{aligned}\right\}, \quad i = 1,2 \qquad (15)$$

where $\mathbf{e}^* \triangleq [e_1^*, e_2^*]^T$ and $\mathbf{u}^* \triangleq [u_1^*, u_2^*]^T$ are the error vector and the control vector of the nominal system (15), respectively, and $g^*_{ij} : [0,\infty) \to \mathbb{R}$ denotes the inverse Laplace transform of $G^*_{ij}(s)$.

Let $\hat{e}_i^*$ and $\hat{u}_i^*$ denote the peak values of e_i^* and u_i^*, respectively, in association with the Cartesian product space $\boldsymbol{\mathcal{P}}$. That is,

$$\hat{e}_i^* \triangleq \sup_{\mathbf{f}\in\boldsymbol{\mathcal{P}}} \|e_i^*\|_\infty \quad \text{and} \quad \hat{u}_i^* \triangleq \sup_{\mathbf{f}\in\boldsymbol{\mathcal{P}}} \|u_i^*\|_\infty. \qquad (16)$$

Now, define

$$\mathbf{X}(s) = [X_{ij}(s)]_{2\times 2} \triangleq [I + \mathbf{G}^*(s)\mathbf{K}(s,\mathbf{p})]^{-1}, \qquad (17)$$

$$\mathbf{Z}(s) = [Z_{ij}(s)]_{2\times 2} \triangleq \mathbf{G}(s) - \mathbf{G}^*(s), \qquad (18)$$

$$\mathbf{W}(s) = [W_{ij}(s)]_{2\times 2} \triangleq \mathbf{X}(s)\mathbf{Z}(s)\mathbf{K}(s,\mathbf{p}), \qquad (19)$$

$$\mathbf{V}(s) = [V_{ij}(s)]_{2\times 2} \triangleq \mathbf{K}(s,\mathbf{p})\mathbf{X}(s)\mathbf{Z}(s), \qquad (20)$$

$$\mathbf{M} = [\mu_{ij}]_{2\times 2}, \quad \mu_{ij} \triangleq \|w_{ij}\|_1, \qquad (21)$$

$$\mathbf{N} = [\nu_{ij}]_{2\times 2}, \quad \nu_{ij} \triangleq \|v_{ij}\|_1 \qquad (22)$$

where $w_{ij} : [0,\infty) \to \mathbb{R}$ and $v_{ij} : [0,\infty) \to \mathbb{R}$ are the inverse Laplace transforms of $W_{ij}(s)$ and $V_{ij}(s)$, respectively. The following results provide useful mathematical expressions for upper bounds and lower bounds of $\hat{e}_i$ and $\hat{u}_i$.

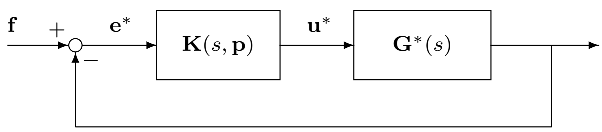

Fig.4: *The nominal system for* (1).

Lemma 1: Suppose that for the nominal system (15), $\hat{e}_1^* < \infty$ and $\hat{e}_2^* < \infty$. Let $\mu_{11} < 1$, $\mu_{22} < 1$ and $\det(I - \mathbf{M}) > 0$. Then it follows that

$$\frac{(1+\mu_{22})\hat{e}_1^* - \mu_{12}\hat{e}_2^*}{\det(I+\mathbf{M})} \le \hat{e}_1 \le \frac{(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I-\mathbf{M})},$$
$$\frac{(1+\mu_{11})\hat{e}_2^* - \mu_{21}\hat{e}_1^*}{\det(I+\mathbf{M})} \le \hat{e}_2 \le \frac{(1-\mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I-\mathbf{M})}. \qquad (23)$$

Proof: From (1), (15) and (19), it is easy to verify that

$$E_i(s) = E_i^*(s) - \sum_{k=1}^{2} W_{ik}(s)E_k(s). \qquad (24)$$

Consequently,

$$e_i(t) = e_i^*(t) - \sum_{k=1}^{2} \int_0^t w_{ik}(\tau)e_k(t-\tau)d\tau \qquad (25)$$

and, by the definition of μ_{ij} in (21), we have

$$\begin{aligned} \sup_{\tau\in[0,t]} |e_1(\tau)| \;\le\; & \sup_{\tau\in[0,t]} |e_1^*(\tau)| + \sup_{\tau\in[0,t]} |e_1(\tau)|\mu_{11} \\ & + \sup_{\tau\in[0,t]} |e_2(\tau)|\mu_{12}, \end{aligned} \qquad (26)$$

$$\begin{aligned} \sup_{\tau\in[0,t]} |e_2(\tau)| \;\le\; & \sup_{\tau\in[0,t]} |e_2^*(\tau)| + \sup_{\tau\in[0,t]} |e_2(\tau)|\mu_{22} \\ & + \sup_{\tau\in[0,t]} |e_1(\tau)|\mu_{21}. \end{aligned} \qquad (27)$$

When $\mu_{11} < 1$, $\mu_{22} < 1$ and $\det(I - \mathbf{M}) > 0$, it follows from (26) and (27) that letting $t \to \infty$ yields

$$\begin{aligned} \|e_1\|_\infty &\le \frac{(1-\mu_{22})\|e_1^*\|_\infty + \mu_{12}\|e_2^*\|_\infty}{\det(I-\mathbf{M})}, \\ \|e_2\|_\infty &\le \frac{(1-\mu_{11})\|e_2^*\|_\infty + \mu_{21}\|e_1^*\|_\infty}{\det(I-\mathbf{M})}. \end{aligned} \qquad (28)$$

Using (21) and (25), one can see that for each i,

$$\|e_i^* - e_i\|_\infty \le \sum_{j=1}^{2} \mu_{ij}\|e_j\|_\infty. \qquad (29)$$

Notice that $e_i^*(t) = [e_i^*(t) - e_i(t)] + e_i(t)$ for all t. Then it is clear that

$$\|e_i^*\|_\infty \le \|e_i^* - e_i\|_\infty + \|e_i\|_\infty.$$

Consequently, it follows from (29) that

$$\begin{aligned} \|e_1^*\|_\infty &\le \mu_{11}\|e_1\|_\infty + \mu_{12}\|e_2\|_\infty + \|e_1\|_\infty, \\ \|e_2^*\|_\infty &\le \mu_{21}\|e_1\|_\infty + \mu_{22}\|e_2\|_\infty + \|e_2\|_\infty. \end{aligned} \qquad (30)$$

Then it readily follows from (30) that

$$\begin{aligned} \|e_1\|_\infty &\geq \frac{(1+\mu_{22})\|e_1^*\|_\infty - \mu_{12}\|e_2^*\|_\infty}{\det(I+\mathbf{M})}, \\ \|e_2\|_\infty &\geq \frac{(1+\mu_{11})\|e_2^*\|_\infty - \mu_{21}\|e_1^*\|_\infty}{\det(I+\mathbf{M})}. \end{aligned} \tag{31}$$

Thus, the lemma is obtained as a consequence of (28) and (31). ■

Lemma 2: Suppose that for the nominal system (15), $\hat{u}_1^* < \infty$ and $\hat{u}_2^* < \infty$. Let $\nu_{11} < 1$, $\nu_{22} < 1$ and $\det(I - \mathbf{N}) > 0$. Then it follows that

$$\begin{aligned} \frac{(1+\nu_{22})\hat{u}_1^* - \nu_{12}\hat{u}_2^*}{\det(I+\mathbf{N})} \leq \hat{u}_1 \leq \frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})}, \\ \frac{(1+\nu_{11})\hat{u}_2^* - \nu_{21}\hat{u}_1^*}{\det(I+\mathbf{N})} \leq \hat{u}_2 \leq \frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})}. \end{aligned} \tag{32}$$

Proof: From (1), (15) and (20), it is easy to verify that

$$U_i(s) = U_i^*(s) - \sum_{k=1}^{2} V_{ik}(s) U_k(s). \tag{33}$$

By using the same technique as in Lemma 1, the result readily follows from (33). ■

Now, it is ready to state the main results in this section.

Theorem 3: Suppose that for the nominal system (15), $\hat{e}_1^* < \infty$ and $\hat{e}_2^* < \infty$. Let $\mu_{11} < 1$, $\mu_{22} < 1$ and $\det(I - \mathbf{M}) > 0$. Then the original design criteria (3) for the system (1) are satisfied if the following inequalities hold.

$$\begin{aligned} \frac{(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I-\mathbf{M})} &\leq \mathcal{E}_1, \\ \frac{(1-\mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I-\mathbf{M})} &\leq \mathcal{E}_2. \end{aligned} \tag{34}$$

Proof: The theorem readily follows from Lemma 1. ■

Theorem 4: Suppose that for the nominal system (15), $\hat{u}_1^* < \infty$ and $\hat{u}_2^* < \infty$. Let $\nu_{11} < 1$, $\nu_{22} < 1$ and $\det(I - \mathbf{N}) > 0$. Then the original design criteria (4) for the actual system (1) are satisfied if the following inequalities hold.

$$\begin{aligned} \frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})} &\leq \mathcal{U}_1, \\ \frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})} &\leq \mathcal{U}_2. \end{aligned} \tag{35}$$

Proof: The theorem readily follows from Lemma 2. ■

In connection with Theorems 3 and 4, it is important to note that the left-hand expressions of the inequalities (34) and (35) are in fact upper bounds of $\hat{e}_i$ and $\hat{u}_i$. They are expressed explicitly in terms of μ_{ij}, ν_{ij} and the peaks $\hat{e}_i^*$ and $\hat{u}_i^*$ of the nominal system (15). In addition, when the system (1) becomes the scalar system (5), the conditions in Lemmas 1 and 2 and those in Theorems 3 and 4 turn out to be identical to the results stated in Theorem 1.

Since $\mathbf{G}^*(s)$ and $\mathbf{K}(s, \mathbf{p})$ are rational matrices, satisfactory tools for stability analysis and computing time-responses are readily available for the case of the nominal system (15). Hence, when such tools for the non-rational system (1) are not readily available, the inequalities (34) and (35) are more computationally tractable than the original design criteria (3) and (4). For this reason, the inequalities (34) and (35) are called the *surrogate* design criteria.

4. FINITENESS CONDITIONS

Following the method of inequalities [9, 10, 12, 13, 27], it is readily appreciated that in solving the inequalities (34) and (35) by numerical methods, a search algorithm needs to start from a point $\mathbf{p} \in \mathbb{R}^N$ such that $\hat{e}_i(\mathbf{p}) < \infty$ and $\hat{u}_i(\mathbf{p}) < \infty$ for $i = 1, 2$. In this regard, the following proposition provides useful sufficient conditions for ensuring that $\hat{e}_i < \infty$ and $\hat{u}_i < \infty$ for $i = 1, 2$.

Proposition 1: Consider the original system (1) and the nominal system (15). Suppose that $\{A_{G^*}, B_{G^*}, C_{G^*}, 0\}$ and $\{A_K, B_K, C_K, D_K\}$ are state-space realizations of $\mathbf{G}^*(s)$ and $\mathbf{K}(s, \mathbf{p})$, respectively. Let Λ_{cl} denote the set of all the eigenvalues of the closed-loop system matrix

$$A_{cl} = \left[\begin{array}{c|c} A_K & -B_K C_{G^*} \\ \hline B_G C_K & A_{G^*} - B_{G^*} D_K C_{G^*} \end{array}\right].$$

Then $\hat{e}_i < \infty$ and $\hat{u}_i < \infty$ for $i = 1, 2$ if the following conditions hold.

(a) $\mathrm{Re}\,\lambda(\mathbf{p}) < 0$ for all $\lambda(\mathbf{p}) \in \Lambda_{cl}$.

(b) $\mu_{ii} < 1$ and $\nu_{ii} < 1$ for $i = 1, 2$.

(c) $\det(\mathbf{M}) > 0$ and $\det(\mathbf{N}) > 0$.

Proof: Condition (a) is a well-known result of bounded-input bounded-output (BIBO) stability for rational systems. It implies that $\hat{e}_i^* < \infty$ and $\hat{u}_i^* < \infty$ for any bounded input. Therefore, it follows from (23) and (32) that if conditions (a), (b) and (c) hold, then $\hat{e}_i < \infty$ and $\hat{u}_i < \infty$ for each i. ■

An important consequence of Proposition 1 is that the satisfaction of conditions (a), (b) and (c) guarantee the finiteness of the peaks $\hat{e}_i$ and $\hat{u}_i$ for the non-rational system (1). In this case, no stability conditions for the non-rational system are involved. Further, one should note that condition (a) implies that the nominal system is BIBO stable, thereby implying that the peaks $\hat{e}_i^*$ and $\hat{u}_i^*$ are finite. Also, the finiteness of $\mathbf{M}$ and $\mathbf{N}$ is necessary for the satisfaction of conditions (b) and (c).

In the following, sufficient conditions for ensuring that $\mathbf{M}$ and $\mathbf{N}$ are finite are provided. To this end, the following notations are useful. For given integers i, j, k and l, define

$$\alpha_{ij}^{kl}(\mathbf{p}) \triangleq \max_{\lambda\in\Lambda_{ij}^{kl}} \operatorname{Re}\lambda(\mathbf{p})$$

where Λ_{ij}^{kl} denotes the set of all the poles of the function $K_{ij}(s,\mathbf{p})X_{kl}(s)$. Let z_{kl} be the inverse Laplace transform of $Z_{kl}(s)$.

Lemma 3: Assume that $\mathbf{G}^*(s)$ and $\mathbf{K}(s,\mathbf{p})$ are rational matrices. Then, for each i and j, $\mu_{ij} < \infty$ if the two conditions hold.

(a) $\|z_{kl}\|_1 < \infty\ \forall\, k,l = 1,2.$

(b) $\alpha_{lj}^{ik}(\mathbf{p}) < 0,\ \forall\, k,l = 1,2.$

Proof: It is easy to see from (17), (18) and (19) that

$$W_{ij}(s) = \sum_{k=1}^{2}\sum_{l=1}^{2} K_{lj}(s,\mathbf{p})X_{ik}(s)Z_{kl}(s).$$

From the definition of μ_{ij} in (21), one can easily see that

$$\mu_{ij} = \left\|\sum_{k=1}^{2}\sum_{l=1}^{2}\mathcal{L}^{-1}\{K_{lj}(s,\mathbf{p})X_{ik}(s)Z_{kl}(s)\}\right\|_1 . \quad (36)$$

Evidently, the product $K_{lj}X_{ik}$ is a proper rational function of s for any $i,j,k,l = 1,2$. Thus,

$$K_{lj}(s,\mathbf{p})X_{ik}(s) = a_{lj}^{ik} + R_{lj}^{ik}(s) \quad (37)$$

where $a_{lj}^{ik} \in \mathbb{R}$ and R_{lj}^{ik} is a strictly proper function of s. Then it follows from (36) and (37) that

$$\mu_{ij} \le \sum_{k=1}^{2}\sum_{l=1}^{2} |a_{lj}^{ik}|\cdot||z_{kl}||_1 + ||r_{lj}^{ik}\star z_{kl}||_1 \quad (38)$$

where r_{lj}^{ik} denotes the inverse Laplace transform of $R_{lj}^{ik}(s)$. By a well-known result (see, for example, [28]), it follows that if $||r_{lj}^{ik}||_1 < \infty$ and $||z_{kl}||_1 < \infty$, then

$$||r_{lj}^{ik}\star z_{kl}||_1 < \infty. \quad (39)$$

Condition (b) is necessary and sufficient for $||r_{lj}^{ik}||_1 < \infty$ for all $k,l = 1,2$. ■

Lemma 4: Assume that $\mathbf{G}^*(s)$ and $\mathbf{K}(s,\mathbf{p})$ are rational matrices. Then, for each i and j, $\nu_{ij} < \infty$ if the two conditions hold.

(a) $\|z_{lj}\|_1 < \infty\ \forall\, l = 1,2.$

(b) $\alpha_{ik}^{kl}(\mathbf{p}) < 0,\ \forall\, k,l = 1,2.$

Proof: It is easy to see from (17), (18) and (20) that

$$V_{ij}(s) = \sum_{k=1}^{2}\sum_{l=1}^{2} K_{ik}(s,\mathbf{p})X_{kl}(s)Z_{lj}(s).$$

From the definition of ν_{ij} in (22), the proof can be completed by the technique used in Lemma 3. ■

Now it is ready to state the main theorem for the finiteness of the matrices $\mathbf{M}$ and $\mathbf{N}$.

Theorem 5: Assume that $\mathbf{G}^*(s)$ and $\mathbf{K}(s,\mathbf{p})$ are rational matrices. Then all elements of $\mathbf{M}$ and $\mathbf{N}$ are finite if the two conditions hold.

(a) $\|z_{kl}\|_1 < \infty\ \forall\, k,l = 1,2.$

(b) $\alpha_{ij}^{kl}(\mathbf{p}) < 0,\ \forall\, i,j,k,l = 1,2.$

Proof: The theorem readily follows from Lemmas 3 and 4. ■

From Theorems 3 and 4, one can see that a solution of the inequalities (3) and (4) can be obtained by solving the inequalities (34) and (35). With an appropriate choice of approximant matrix $\mathbf{G}^*(s)$, condition (a) of Theorem 5 is always satisfied. Consequently, condition (b) of Theorem 5 provides a practical and useful sufficient condition for the finiteness of the matrices $\mathbf{M}$ and $\mathbf{N}$, thereby enabling the algorithm to start from an arbitrary point in $\mathbb{R}^N$.

From the above, one can easily deduce that the solution of inequalities (34) and (35) involves three phases of computation as follows.

- Phase I: With an arbitrary starting point, find $\mathbf{p}_0$ satisfying

$$\alpha_{ij}^{kl}(\mathbf{p}) \le -\varepsilon\ \ \forall\, i,j,k,l = 1,2 \quad (40)$$

where $0 < \varepsilon \ll 1$ is given.

- Phase II: By starting from $\mathbf{p}_0$, find $\mathbf{p}_1$ satisfying

$$\begin{gathered}\alpha_{ij}^{kl}(\mathbf{p}) \le -\varepsilon\ \ \forall\, i,j,k,l = 1,2,\\ \mu_{11} < 1 \text{ and } \mu_{22} < 1,\\ \nu_{11} < 1 \text{ and } \nu_{22} < 1,\\ \det(I-\mathbf{M}) > 0 \text{ and } \det(I-\mathbf{N}) > 0.\end{gathered} \quad (41)$$

- Phase III: By starting from $\mathbf{p}_1$, find $\mathbf{p}$ satisfying the design criteria (34), (35), and the inequalities (41).

5. NUMERICAL EXAMPLE

In this section, the linearized model of a binary distillation column obtained from linearization about an operating point [29] is used. The plant transfer matrix $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-2s}}{14.4s+1}\end{bmatrix}. \quad (42)$$

Let $\mathbf{K}(s,\mathbf{p})$ take the form

$$\mathbf{K}(s,\mathbf{p}) = \begin{bmatrix} \dfrac{(p_2 s+p_1)}{s} & p_5 \\ p_6 & \dfrac{(p_4 s+p_3)}{s}\end{bmatrix}$$

where $\mathbf{p} \triangleq [p_1,p_2,p_3,p_4,p_5,p_6]^T$ is a design parameter. Assume that the input f_1 belongs to the set $\mathcal{P}$

where

$$\mathcal{P} \triangleq \{f_1 : \|f_1\|_\infty \leq 0.2 \text{ and } \|\dot{f}_1\|_\infty \leq 0.2\}. \quad (43)$$

For simplicity, let $f_2 = 0$. It may be noted that if f_2 is not zero, one can use the principle of superposition to compute the peaks due to both inputs f_1 and f_2.

Suppose that the plant transfer matrix $\mathbf{G}(s)$ is replaced by a strictly proper rational approximant matrix $\mathbf{G}^*(s)$. Here, we replace $e^{-\tau s}$ with its [1/2] Padé approximant[1], which is

$$e^{-\tau s} \approx \frac{1 - \tau s/3}{(\tau s)^2/6 + 2\tau s/3 + 1}.$$

Then the resultant approximant matrix $\mathbf{G}^*(s)$ is given by

$$\begin{aligned} G_{11}^*(s) &= \frac{-25.6(s-3)}{(16.7s+1)(s^2+4s+6)}, \\ G_{12}^*(s) &= \frac{37.8(s-1)}{(21.0s+1)(3s^2+4s+2)}, \\ G_{21}^*(s) &= \frac{-13.2(7s-3)}{(10.9s+1)(49s^2+28s+6)}, \\ G_{22}^*(s) &= \frac{19.4(2s-3)}{(14.4s+1)(2s^2+4s+3)}. \end{aligned}$$

The impulse responses of the original system and the nominal system are shown in Fig. 5. Obviously, the chosen approximants $G_{ij}^*(s)$ are appropriate in the sense that the impulse responses g_{ij}^* are reasonably close to the impulse responses g_{ij}.

The main control objective is to ensure that, during the operation (that is, whenever $f_1 \in \mathcal{P}$),

- the top product deviation e_1 stays within ± 0.35 mol%,
- the bottom product deviation e_2 stays within ± 0.20 mol%,
- the deviation of the reflux rate u_1 stays within ± 0.10 lb/min,
- the deviation of the reboiler rate u_2 stays within ± 0.10 lb/min.

Accordingly, the principal design criteria can be expressed as

$$\begin{aligned} \hat{e}_1 &\leq 0.35, \quad \hat{e}_2 \leq 0.20, \\ \hat{u}_1 &\leq 0.10, \quad \hat{u}_2 \leq 0.10. \end{aligned} \quad (44)$$

Following Theorems 3, 4 and 5, it readily follows that the controller is obtained by determining a value of $\mathbf{p}$ satisfying the following inequalities.

$$\alpha_{ij}^{kl}(\mathbf{p}) \leq -10^{-6} \;\; \forall\, i,j,k,l = 1,2. \quad (45)$$

[1]The $[M/N]$ Padé approximant to a function $h(s)$ is defined as the rational function $P(s)/Q(s)$ that satisfies certain properties, where P and Q are polynomials of degree M and N, respectively. See, for example, [30] for the details.

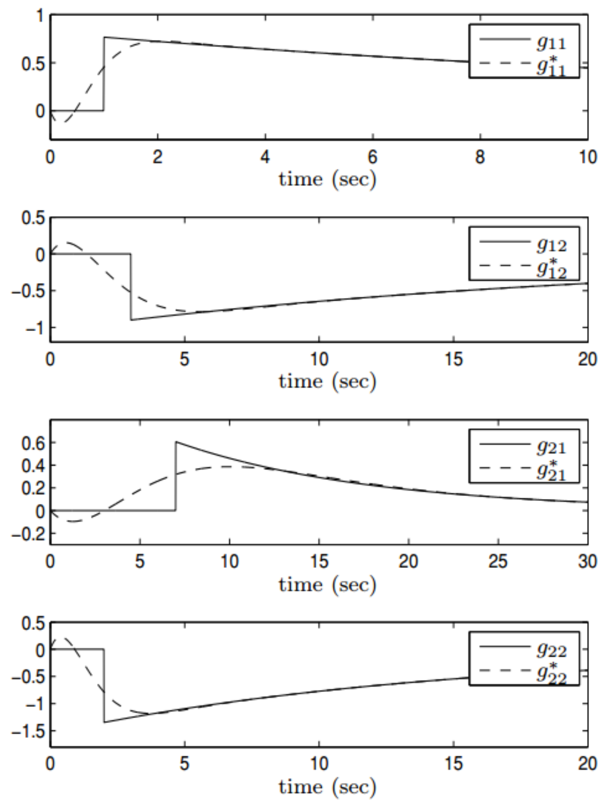

Fig.5: *Comparison of the impulse responses of* $\mathbf{G}(s)$ *and* $\mathbf{G}^*(s)$.

$$\begin{aligned} \mu_{11}(\mathbf{p}) &\leq 0.5, \\ \mu_{22}(\mathbf{p}) &\leq 0.5, \\ \nu_{11}(\mathbf{p}) &\leq 0.5, \\ \nu_{22}(\mathbf{p}) &\leq 0.5, \\ -\det(I-\mathbf{M}) &\leq -0.5, \\ -\det(I-\mathbf{N}) &\leq -0.5. \end{aligned} \quad (46)$$

$$\begin{aligned} \frac{(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I-\mathbf{M})} &\leq 0.35 \text{ mol\%}, \\ \frac{(1-\mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I-\mathbf{M})} &\leq 0.2 \text{ mol\%}. \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})} &\leq 0.1 \text{ lb/min}, \\ \frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})} &\leq 0.1 \text{ lb/min}. \end{aligned} \quad (48)$$

From the above, note that the inequalities (45) ensure the finiteness of $\mathbf{M}$ and $\mathbf{N}$, whereas the inequalities (46) ensure that $\hat{e}_1$, $\hat{e}_2$, $\hat{u}_1$ and $\hat{u}_2$ are finite. Furthermore, the inequalities (47) and (48) are sufficient conditions for ensuring that the design criteria (44) are satisfied.

In this work, the inequalities (45)–(48) are solved by using a numerical search algorithm called the moving-boundaries-process (MBP). The detail of the

MBP algorithm can be found in [14, 27]. Alternatively, other algorithms for solving a set of inequalities may be used (see, for example, Chapters 6–8 in [14] and the references therein). In addition, the nominal peaks $\hat{e}_i^*$ and $\hat{u}_i^*$ associated with the possible set (43) are computed by the method developed in [8].

After a number of iterations, the MBP algorithm locates a design solution $\mathbf{p}$ resulting in the following controller

$$\begin{aligned} K_{11}(s,\mathbf{p}) &= \frac{0.0021s+0.0152}{s}, \\ K_{12}(s,\mathbf{p}) &= 0.0309, \\ K_{21}(s,\mathbf{p}) &= -0.0389, \\ K_{22}(s,\mathbf{p}) &= \frac{-0.0624s-0.0196}{s}. \end{aligned}$$

And the corresponding performance measures are

$$\mu_{11}(\mathbf{p}) = 0.0586, \qquad \mu_{22}(\mathbf{p}) = 0.1534,$$
$$\nu_{11}(\mathbf{p}) = 0.0899, \qquad \nu_{22}(\mathbf{p}) = 0.1468,$$
$$\det(I-\mathbf{M}) = 0.7851, \quad \det(I-\mathbf{N}) = 0.7576,$$

$$\frac{(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I-\mathbf{M})} = 0.3345 \text{ mol\%},$$
$$\frac{(1-\mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I-\mathbf{M})} = 0.1722 \text{ mol\%},$$
$$\frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})} = 0.0519 \text{ lb/min},$$
$$\frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})} = 0.0522 \text{ lb/min}.$$

To verify the design, a simulation is carried out for the case in which the control system is subjected to a test input $f = [\bar{f}_1, 0]^T$, where $\bar{f}_1$ is generated randomly so that its magnitude and slope satisfy (43). The waveform of $\bar{f}_1$ and the corresponding responses e_1, e_2, u_1 and u_2 are displayed in Figure 6. Clearly, the design objectives are satisfied.

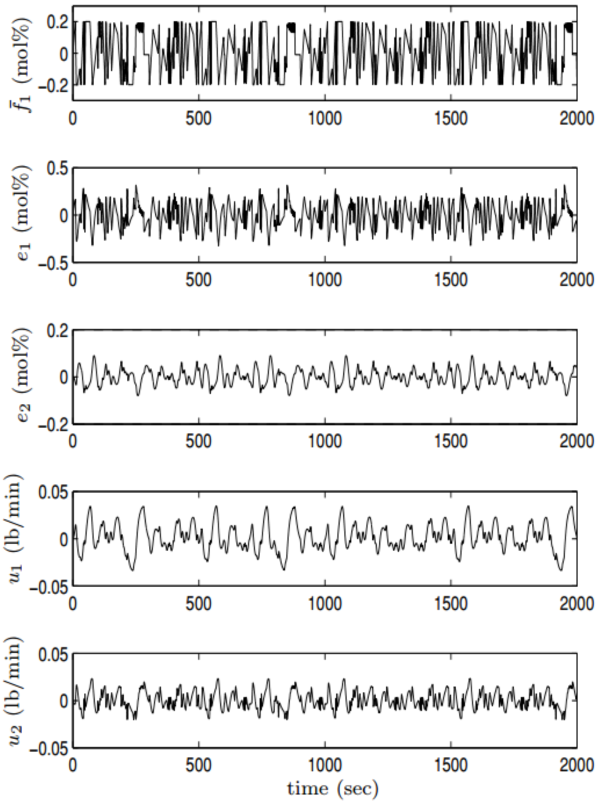

Fig.6: *Waveforms of e_1, e_2, u_1 and u_2 in response to the test input $\bar{f}_1$.*

6. CONCLUSIONS AND DISCUSSION

This paper derives a criterion of approximation for 2×2 feedback systems that are subjected to possible inputs satisfying certain bounding conditions. The design objective is to ensure that the error peaks $(\hat{e}_1, \hat{e}_2)$ and the control peaks $(\hat{u}_1, \hat{u}_2)$ always stay within the specified bounds $(\mathcal{E}_1, \mathcal{E}_2)$ and $(\mathcal{U}_1, \mathcal{U}_2)$, respectively. For a chosen rational approximant $\mathbf{G}^*(s)$ to the plant transfer matrix $\mathbf{G}(s)$, the criterion provides useful sufficient conditions that are expressed as readily computable inequalities, thereby providing the surrogate design criteria (34) and (35) to be used instead of the original criteria (3) and (4). The surrogate criteria are particularly useful when computational tools for the non-rational system (1) are not readily available. When the plant and the controller transfer matrices become scalar transfer functions, the results obtained in the paper turn out to be identical to those reported in Zakian's articles [10–14]. To illustrate the usefulness of the results, a controller design for a binary distillation column is carried out successfully by using the criterion in conjunction with the method of inequalities.

Although the system treated in the numerical example is a time-delay system, the method developed in the paper is readily applicable to other types of non-rational systems as long as the impulse response matrix $[g_{ij}]$ of the plant is obtained, where the Laplace transform of g_{ij} is $G_{ij}(s)$. For various non-rational systems that can be found in practice, see [31]. It is also interesting to note that the generalization of the criterion of approximation to the case of $n \times n$ feedback systems is currently being undertaken and will be published elsewhere.

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References

[1] L. G. Gibilaro and F. P. Lees, "The reduction of complex transfer function models to simple models using the method of moments," *Chemical Eng. Sci.*, vol. 24, no. 1, pp. 85–93, 1969.

[2] V. Zakian, "Simplification of linear time-invariant systems by moment approximants," *Int. J. Control*, vol. 18, no. 3, pp. 455–460, 1973.

[3] J. Lam, "Model reduction of delay systems using Padé approximants," *Int. J. Control*, vol. 57, no. 2, pp. 377–391, 1993.

[4] G. Gu, P. P. Khargonekar and E. B. Lee, "Approximation of infinite-dimensional systems," *IEEE Trans. Automat. Control*, vol 34, no. AC-6, pp. 610–618, 1989.

[5] B. J. Birch and R. Jackson, "The behaviour of linear systems with inputs satisfying certain bounding conditions," *J. Electron. Control*, vol. 6, no. 4, pp. 366–375, 1959.

[6] P. G. Lane, "The principle of matching: a necessary and sufficient condition for inputs restricted in magnitude and rate of change," *Int. J. Control*, vol. 62, no. 4, pp. 893–915, 1995.

[7] W. Silpsrikul and S. Arunsawatwong, "Computation of peak output for inputs restricted in $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms using finite difference schemes and convex optimization," *Int. J. Automation Computing*, vol. 6, no. 1, pp. 7–13, 2009.

[8] W. Silpsrikul and S. Arunsawatwong, "Computation of peak output for inputs satisfying many bounding conditions on magnitude and slope," *Int. J. Control*, vol. 83, no. 1, pp. 49–65, 2010.

[9] V. Zakian, "New formulation for the method of inequalities," *Proc. Institution Elect. Engineers*, vol. 126, no. 6, pp. 579–584, 1979.

[10] V. Zakian, "A criterion of approximation for the method of inequalities," *Int. J. Control*, vol. 37, no. 5, pp. 1103–1111, 1983.

[11] V. Zakian, A framework for design: theory of majorants, *Control Systems Centre Report No. 604*, University of Manchester Institute of Science and Technology, Manchester, UK, 1984.

[12] V. Zakian, "A performance criterion," *Int. J. Control*, vol. 43, no. 3, pp. 921–931, 1986.

[13] V. Zakian, "Perspectives of the principle of matching and the method of inequalities," *Int. J. Control*, vol. 65, no. 1, pp. 147–175, 1996.

[14] V. Zakian, *Control Systems Design: A New Framework*, Springer-Verlag, London, 2005.

[15] S. Arunsawatwong, "Stability of retarded delay differential systems," *Int. J. Control*, vol. 65, no. 2, pp. 347–364, 1996.

[16] S. Arunsawatwong, "Stability of Zakian I_{MN} recursions for linear delay differential equations," *BIT Numerical Math.*, vol. 38, no. 2, pp. 219–233, 1998.

[17] S. Arunsawatwong and V. Q. Nguyen, "Design of retarded fractional delay differential systems using the method of inequalities," *Int. J. Automation Computing*, vol. 6, no. 1, pp. 22–28, 2009.

[18] W. Silpsrikul and S. Arunsawatwong, "Design of fractional PI controllers for a binary distillation column with disturbances restricted in magnitude and slope," In *Proc. 18th IFAC World Congr.*, Milan, Italy, pp. 7702–7707, 2011.

[19] A. T. Bada, "Design of delayed control systems with Zakian's method," *Int. J. Control*, vol. 40, pp. 773–781, 1984.

[20] A. T. Bada, "Design of delayed control systems using Zakian's framework," *Proc. Institution Elect. Engineers*, vol. 132, no. 6, pp. 251–256, 1985.

[21] A. T. Bada, "Robust brake control for a heavy-duty truck," *Proc. Institution Elect. Engineers*, vol. 134, no. 1, pp. 1–8, 1987.

[22] O. Taiwo, "The design of robust control systems for plants with recycle," *Int. J. Control*, vol. 43, pp. 671–678, 1986.

[23] V. S. Mai, S. Arunsawatwong, and E. H. Abed, "Design of uncertain nonlinear feedback systems with inputs and outputs satisfying bounding conditions," In *Proc. 18th IFAC World Congr.*, Milan, Italy, pp. 10970–10975, 2011.

[24] H. H. Nguyen, *Numerical Design of Feedback Systems with Backlash for Inputs Restricted in Magnitude and Slope*, Master's thesis, Chulalongkorn University, Bangkok, Thailand, 2014.

[25] P. G. Lane, *Design of Control Systems with Input and Output Satisfying Certain Bounding Conditions*, PhD thesis, University of Manchester Institute of Science and Technology, Manchester, UK, 1992.

[26] S. Arunsawatwong, "Critical control of building under seismic disturbance," in *Control Systems Design: A New Framework*, V. Zakian, editor, Springer-Verlag, London, pp. 339–353, 2005.

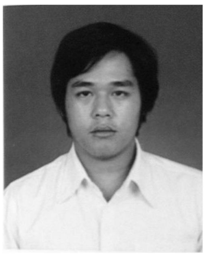
[27] V. Zakian and U. Al-Naib, "Design of dynamical and control systems by the method of inequalities," *Proc. Institution Elect. Engineers*, vol. 120, no. 11, pp. 1421–1427, 1973.

[28] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, USA, 1975.

[29] R. K. Wood and M. W. Berry, "Terminal composition control of binary distillation column," *Chemical Engineering Sci.*, vol. 28, no. 9, pp. 1707–1717, 1973.

[30] G. A. Baker Jr. and P. Graves-Morris, *Padé Approximants*, Cambridge University Press, New York, USA, 1996.

[31] R. Curtain and K. Morris, "Transfer functions of distributed parameter systems: A tutorial," *Automatica*, vol. 45, no. 5, pp. 1101–1116, 2009.



Tadchanon Chuman received the B.Eng. and M.Eng. degrees in electrical engineering from Chulalongkorn University, Thailand in 2012 and 2014, respectively. He is currently a Ph.D. student at the Department of Electrical Engineering, Chulalongkorn University. His main research interests include computer-aided control systems design using the method of inequalities and the principle of matching.

Suchin Arunsawatwong received his B.Eng. and M.Eng. degrees in electrical engineering from Chulalongkorn University, Thailand, in 1985 and 1988, respectively, and Ph.D. degree in control engineering from the Control Systems Centre, University of Manchester Institute of Science and Technology, U.K., in 1995. He is currently an assistant professor at the Department of Electrical Engineering, Chulalongkorn University. His main research interests include delay differential systems, fractional-order control systems, numerical solution of differential equations, and control systems design by the method of inequalities and the principle of matching.