Robust $\mathcal{H}_\infty$ State-Feedback Control Design for Nonlinear Time-Varying Delay Systems Based on An LMI Approach

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ABSTRACT

This paper examines the problem of designing a robust $\mathcal{H}_\infty$ state-feedback controller for a class of nonlinear systems with time-varying delay described by a Takagi-Sugeno (TS) fuzzy model. Based on a linear matrix inequality (LMI) approach, we develop a robust $\mathcal{H}_\infty$ state-feedback controller which guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output to be less than some prescribed value for this class of nonlinear systems. A numerical example is provided to illustrate the design developed in this paper.

Keywords: $\mathcal{H}_\infty$ control, Time-varying delay, Fuzzy systems, Linear Matrix Inequality (LMI).

1. INTRODUCTION

Over the past few decades, the nonlinear $\mathcal{H}_\infty$-control theory has been extensively studied by many researchers; e.g., [1]-[5]. The nonlinear $\mathcal{H}_\infty$-control problem can be stated as follows: given a dynamic system with the exogenous input noise and the measured output, find a controller such that the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than or equal to a prescribed value. Clearly, it is known that delays may appear in many dynamic nonlinear systems and are often a source of instability and encountered in control analysis and design problem. In recent years, the $\mathcal{H}_\infty$ problem for delayed system has also received considerably attention by a number of researchers; e.g., [1]-[3]. Until now, there are two commonly used approaches for providing solutions to the nonlinear $\mathcal{H}_\infty$-control problems. The first approach is based on the dissipativity theory and theory of differential games; see [4]-[7]. The second approach is based on the nonlinear version of classical Bounded Real Lemma; see [5]–[8]. Both approaches show that the solution of the nonlinear $\mathcal{H}_\infty$-control problem is in fact related to the solvability of Hamilton-Jacobi inequalities (HJIs). So far to our best knowledge, there exists no computation to solve those inequalities.

Recently, a great amount of effort has been made on the design of fuzzy $\mathcal{H}_\infty$ control for a class of nonlinear systems which can be represented by a Takagi-Sugeno (TS) fuzzy model; e.g., [9]-[10]. Fuzzy system theory enables us to utilize qualitative, linguistic information about a highly complex nonlinear system to construct a mathematical model for it. Recent studies [11]-[18] show that a fuzzy linear model can be used to approximate global behaviors of a highly complex nonlinear system. In this fuzzy linear model, local dynamics in different state space regions are represented by local linear systems. The overall model of the system is obtained by "blending" of these linear models through nonlinear fuzzy membership functions. Unlike conventional modelling which uses a single model to describe the global behavior of a system, fuzzy modelling is essentially a multi-model approach in which simple sub-models (linear models) are combined to describe the global behavior of the system. Recently, the design of fuzzy $\mathcal{H}_\infty$ control for a class of nonlinear systems without delays has been significantly considered and many results have been reported; e.g., [9]-[10]. However, there have been also some attempts in [12]-[15] in which robust fuzzy control analysis and synthesis for nonlinear time-delay systems have been examined. Nevertheless, so far, to the best of our knowledge, the global robust $\mathcal{H}_\infty$ fuzzy state-feedback control problem for a class of uncertain nonlinear systems with time-varying delays via LMIs approach has not yet been considered in the literature.

What we intend to do in this paper is to design a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller for a class of nonlinear systems with time-varying delay. First, we approximate this class of nonlinear systems with time-varying delay by a Takagi-Sugeno fuzzy model. Then based on an LMI approach, we develop a technique for designing a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller such that the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value. This paper is organized as follows. In Section 2, system descriptions and definitions are presented. In Section 3, based on an LMI approach, we develop a technique for designing a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller such that the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value for the system described in Section 2.

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The validity of this approach is demonstrated by an example from a literature in Section 4. Finally in Section 5, conclusions are given.

2. SYSTEM DESCRIPTIONS AND DEFINITIONS

The class of nonlinear systems under consideration is described by the following fuzzy system model:
Plant Rule i:
IF $\nu_1(t)$ is M_{i1} and $\cdots$ and $\nu_\vartheta(t)$ is $M_{i\vartheta}$ THEN

$$\begin{aligned}\dot{x}(t) &= [A_i+\Delta A_i]x(t)+A_{d_i}x(t-\tau(t)) \\ &\quad +[B_{1_i}+\Delta B_{1_i}]w(t)+[B_{2_i}+\Delta B_{2_i}]u(t), \\ z(t) &= [C_{1_i}+\Delta C_{1_i}]x(t)+[D_{12_i}+\Delta D_{12_i}]u(t) \\ x(t) &= \psi(t), \quad t\in[-\tau,0], \ \ \tau(t)\le\tau \end{aligned} \tag{1}$$

where $M_{ij}(j=1,2,\cdots,\vartheta)$ are fuzzy sets, $x(t)\in\Re^n$ is the state vector, $u(t)\in\Re^m$ is the input, $w(t)\in\Re^p$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, $z(t)\in\Re^s$ is the controlled output, the matrices $A_i, A_{d_i} B_{1_i}, B_{2_i}, C_{1_i}$ and D_{12_i} are of appropriate dimensions, r is the number of IF-THEN rules, $\tau(t)\le\tau$ is the bounded time-varying delay in the state and $\psi(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau,0]$. The matrices ΔA_i, ΔB_{1_i}, ΔB_{2_i}, ΔC_{1_i} and ΔD_{12_i} represent the uncertainties in the system and satisfy the following assumption.

Assumption 1:

$$\Delta A_i = F(x(t),t)H_{1_i}, \ \ \Delta B_{1_i} = F(x(t),t)H_{2_i},$$

$$\Delta B_{2_i} = F(x(t),t)H_{3_i}, \ \ \Delta C_{1_i} = F(x(t),t)H_{4_i}$$

$$\text{and } \ \Delta D_{12_i} = F(x(t),t)H_{5_i}$$

where $H_{j_i}, j=1,2,\cdots,5$ are known matrix functions which characterize the structure of the uncertainties. Furthermore, the following inequality holds:

$$\|F(x(t),t)\|\le\rho \tag{2}$$

for any known positive constant ρ.

Let $\varpi_i(\nu(t)) = \prod_{k=1}^{\vartheta} M_{ik}(\nu_k(t))$ and $\mu_i(x(t)) = \frac{\varpi_i(\nu(t))}{\sum_{i=1}^r \varpi_i(\nu(t))}$ where $M_{ik}(\nu_k(t))$ is the grade of membership of $\nu_k(t)$ in M_{ik}. It is assumed in this paper that

$$\varpi_i(\nu(t))\ge 0, \quad i=1,2,...,r; \qquad \sum_{i=1}^r \varpi_i(\nu(t))>0$$

for all t. Therefore,

$$\mu_i(\nu(t))\ge 0, \quad i=1,2,...,r; \qquad \sum_{i=1}^r \mu_i(\nu(t))=1$$

for all t. For the convenience of notations, we let $\varpi_i=\varpi_i(\nu(t))$ and $\mu_i=\mu_i(\nu(t))$.

The resulting fuzzy system model is inferred as the weighted average of the local models of the form:

$$\begin{aligned}\dot{x}(t) &= \textstyle\sum_{i=1}^r \mu_i\Big[[A_i+\Delta A_i]x(t)+A_{d_i}x(t-\tau(t)) \\ &\quad +[B_{1_i}+\Delta B_{1_i}]w(t)+[B_{2_i}+\Delta B_{2_i}]u(t)\Big], \\ z(t) &= \textstyle\sum_{i=1}^r \mu_i\Big[[C_{1_i}+\Delta C_{1_i}]x(t) \\ &\quad +[D_{12_i}+\Delta D_{12_i}]u(t)\Big]. \end{aligned} \tag{3}$$

We recall the following definitions.

Definition 1: Suppose γ is a given positive number. A system (3) is said to have $\mathcal{L}_2[0,T_f]$ gain less than or equal to γ if

$$\int_0^{T_f} z^T(t)z(t)dt \le \gamma^2\left[\int_0^{T_f} w^T(t)w(t)dt\right]. \tag{4}$$

In this paper, we consider the following $\mathcal{H}_\infty$ fuzzy state-feedback which is inferred as the weighted average of the local models of the form:

$$u(t)=\sum_{j=1}^r \mu_j K_j x(t). \tag{5}$$

where K_j is the controller gain.

Before ending this section, we describe the problem under our study as follows.

Problem Formulation: Given a prescribed $\mathcal{H}_\infty$ performance $\gamma>0$, design a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller of the form (5)such that the inequality (4) holds.

Note that for the symmetric block matrices, we use $(*)$ as an ellipsis for terms that are induced by symmetry.

3. ROBUST $\mathcal{H}_\infty$ FUZZY STATE-FEEDBACK CONTROL DESIGN

The following theorem provides a way of designing a fuzzy state-feedback controller.

Theorem 1: Consider the system (3). Given a prescribed $\mathcal{H}_\infty$ performance $\gamma>0$ and a positive constant δ, if there exist symmetric matrices $P>0$, $W>0$, and matrices Y_j, $j=1,2,\cdots,r$, satisfying the following linear matrix inequalities:

$$\Omega_{ii} < 0, \quad i=1,2,\cdots,r \tag{6}$$

$$\Omega_{ij}+\Omega_{ji} < 0, \quad i<j\le r \tag{7}$$

where

$$\Omega_{ij}=\begin{pmatrix} \Phi_{ij} & (*)^T & (*)^T & (*)^T \\ \tilde{B}_{1_i}^T & -\gamma^2 I & (*)^T & (*)^T \\ P & 0 & -W & (*)^T \\ \tilde{C}_{1_i}P+\tilde{D}_{12_i}Y_j & 0 & 0 & -I \end{pmatrix} \tag{8}$$

and

$$\Phi_{ij}=A_iP+PA_i^T+B_{2_i}Y_j+Y_j^TB_{2_i}^T+A_{d_i}WA_{d_i}^T \tag{9}$$

with

$$\tilde{B}_{1_i} = \begin{bmatrix} \delta I & I & \delta I & B_{1_i} \end{bmatrix},$$

$$\tilde{C}_{1_i} = \begin{bmatrix} \frac{\gamma\rho}{\delta} H_{1_i}^T & 0 & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{bmatrix}^T,$$

$$\tilde{D}_{12_i} = \begin{bmatrix} 0 & \frac{\gamma\rho}{\delta} H_{3_i}^T & \sqrt{2}\lambda\rho H_{5_i}^T & \sqrt{2}\lambda D_{12_i}^T \end{bmatrix}^T,$$

$$\lambda = \left(1 + \rho^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \left[\|H_{2_i}^T H_{2_j}\| \right] \right)^{\frac{1}{2}}$$

then the inequality (4) holds. Furthermore, a suitable choice of the fuzzy controller is

$$u(t) = \sum_{j=1}^{r} \mu_j K_j x(t) \tag{10}$$

where

$$K_j = Y_j P^{-1}. \tag{11}$$

Proof: The state space form of the fuzzy system model (3) with the controller (10) is given by

$$\begin{aligned} \dot{x}(t) &= \textstyle\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big([A_i + B_{2_i}K_j]x(t) \\ &\quad + A_{d_i}x(t-\tau(t)) + \tilde{B}_{1_i}\tilde{w}(t) \Big) \end{aligned} \tag{12}$$

where

$$\tilde{B}_{1_i} = \begin{bmatrix} \delta I & I & \delta I & B_{1_i} \end{bmatrix},$$

and the disturbance $\tilde{w}(t)$ is

$$\tilde{w}(t) = \begin{bmatrix} \frac{1}{\delta}F(x(t),t)H_{1_i}x(t) \\ F(x(t),t)H_{2_i}w(t) \\ \frac{1}{\delta}F(x(t),t)H_{3_i}K_jx(t) \\ w(t) \end{bmatrix}. \tag{13}$$

Let choose a Lyapunov-Kranovskii functional $V(x(t))$ as

$$V(x(t)) = x^T(t)Qx(t) + \int_{t-\tau(t)}^{t} x^T(v)Sx(v)dv$$

where $Q = P^{-1} > 0$ and $S = W^{-1} > 0$. Taking the differentiate $V(x(t))$ along the closed-loop system (12) and using the fact that for any vector $x_1(t)$ and $x_2(t)$ and a matrix G

$$\begin{aligned} &x_1^T(t)Gx_2(t) + x_2^T(t)G^Tx_1(t) \\ &\qquad \le x_1^T(t)GR^{-1}G^Tx_1(t) + x_2^T(t)Rx_2(t) \end{aligned} \tag{14}$$

where R is a positive definite matrix, we have

$$\begin{aligned} \dot{V}(x(t)) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big[x^T(t)\Big((A_i + B_{2_i}K_j)^TQ \\ &\quad + Q(A_i + B_{2_i}K_j) + S\Big)x(t) \\ &\quad + x^T(t)QA_{d_i}x(t-\tau(t)) \\ &\quad + x^T(t-\tau(t))A_{d_i}^TQx(t) \\ &\quad - x^T(t-\tau(t))Sx(t-\tau(t)) \\ &\quad + \tilde{w}^T(t)\tilde{B}_{1_i}^TQx(t) + x^T(t)Q\tilde{B}_{1_i}\tilde{w}(t)\Big] \\ &\le \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big[x^T(t)\Big((A_i + B_{2_i}K_j)^TQ \\ &\quad + Q(A_i + B_{2_i}K_j) + S\Big)x(t) \\ &\quad + x^T(t)QA_{d_i}S^{-1}A_{d_i}^TQx(t) \\ &\quad + x^T(t-\tau(t))Sx(t-\tau(t)) \\ &\quad - x^T(t-\tau(t))Sx(t-\tau(t)) \\ &\quad + \tilde{w}^T(t)\tilde{B}_{1_i}^TQx(t) + x^T(t)Q\tilde{B}_{1_i}\tilde{w}(t)\Big] \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \Big[x^T(t)\Big((A_i + B_{2_i}K_j)^TQ \\ &\quad + Q(A_i + B_{2_i}K_j) \\ &\quad + QA_{d_i}S^{-1}A_{d_i}^TQ + S\Big)x(t) \\ &\quad + \tilde{w}^T(t)\tilde{B}_{1_i}^TQx(t) + x^T(t)Q\tilde{B}_{1_i}\tilde{w}(t)\Big]. \end{aligned} \tag{15}$$

Adding and subtracting $-\tilde{z}^T(t)\tilde{z}(t)$ $+\gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]$ to and from (15), we get

$$\begin{aligned} &\dot{V}(x(t)) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n \times \\ &\begin{bmatrix} x^T(t) & \tilde{w}^T(t) \end{bmatrix} \times \\ &\left(\begin{matrix} \left(\begin{matrix} (A_i + B_{2_i}K_j)^TQ \\ +Q(A_i + B_{2_i}K_j) \\ +QA_{d_i}S^{-1}A_{d_i}^TQ + S \\ +(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j)^T \\ \times(\tilde{C}_{1_m} + \tilde{D}_{12_m}K_n) \end{matrix} \right) & (*)^T \\ \tilde{B}_{1_i}^TQ & -\gamma^2 I \end{matrix} \right) \\ &\times \begin{bmatrix} x(t) \\ \tilde{w}(t) \end{bmatrix} - \tilde{z}^T(t)\tilde{z}(t) \\ &+\gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)] \end{aligned} \tag{16}$$

where

$$\tilde{z}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j[\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j]x(t) \tag{17}$$

with

$$\tilde{C}_{1_i} = \begin{bmatrix} \frac{\gamma\rho}{\delta} H_{1_i}^T & 0 & \sqrt{2}\lambda\rho H_{4_i}^T & \sqrt{2}\lambda C_{1_i}^T \end{bmatrix}^T$$

and $\tilde{D}_{12_i} = \begin{bmatrix} 0 & \frac{\gamma\rho}{\delta} H_{3_i}^T & \sqrt{2}\lambda\rho H_{5_i}^T & \sqrt{2}\lambda D_{12_i}^T \end{bmatrix}^T$.

Pre and post multiply (6)-(7) by $\begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$ yields

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_i)^T Q \\ +Q(A_i + B_{2_i}K_i) \\ +QA_{d_i}WA_{d_i}^T Q \end{pmatrix} & (*)^T & (*)^T & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma^2 I & (*)^T & (*)^T \\ I & 0 & -W & (*)^T \\ \tilde{C}_{1_i} + \tilde{D}_{12_i}K_i & 0 & 0 & -I \end{pmatrix} < 0, \quad (18)$$

$i = 1, 2, \cdots, r$, and

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_j)^T Q \\ +Q(A_i + B_{2_i}K_j) \\ +QA_{d_i}WA_{d_i}^T Q \end{pmatrix} & (*)^T & (*)^T & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma^2 I & (*)^T & (*)^T \\ I & 0 & -W & (*)^T \\ \tilde{C}_{1_i} + \tilde{D}_{12_i}K_j & 0 & 0 & -I \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} (A_j + B_{2_j}K_i)^T Q \\ +Q(A_j + B_{2_j}K_i) \\ +QA_{d_j}WA_{d_j}^T Q \end{pmatrix} & (*)^T & (*)^T & (*)^T \\ \tilde{B}_{1_j}^T Q & -\gamma^2 I & (*)^T & (*)^T \\ I & 0 & -W & (*)^T \\ \tilde{C}_{1_j} + \tilde{D}_{12_j}K_i & 0 & 0 & -I \end{pmatrix} < 0, \quad (19)$$

$i < j \leq r$, respectively. Applying the Schur complement on (18)-(19) and rearranging them, then we have

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_i)^T Q \\ +Q(A_i + B_{2_i}K_i) \\ +QA_{d_i}WA_{d_i}^T Q + W^{-1} \\ +(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_i)^T \\ \times(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_i) \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma^2 I \end{pmatrix} < 0, \quad (20)$$

$i = 1, 2, \cdots, r$, and

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_j)^T Q \\ +Q(A_i + B_{2_i}K_j) \\ +QA_{d_i}WA_{d_i}^T Q + W^{-1} \\ +(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j)^T \\ \times(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j) \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} (A_j + B_{2_j}K_i)^T Q \\ +Q(A_j + B_{2_j}K_i) \\ +QA_{d_j}WA_{d_j}^T Q + W^{-1} \\ +(\tilde{C}_{1_j} + \tilde{D}_{12_j}K_i)^T \\ \times(\tilde{C}_{1_j} + \tilde{D}_{12_j}K_i) \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma^2 I \end{pmatrix} < 0, \quad (21)$$

$i < j \leq r$, respectively. Using (20)-(21) and the fact that

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n M_{ij}^T N_{mn} \leq \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j [M_{ij}^T M_{ij} + N_{ij}N_{ij}^T], \quad (22)$$

it is obvious that we have

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_{2_i}K_j)^T Q \\ +Q(A_i + B_{2_i}K_j) \\ +QA_{d_i}S^{-1}A_{d_i}^T Q + S \\ +(\tilde{C}_{1_i} + \tilde{D}_{12_i}K_j)^T \\ \times(\tilde{C}_{1_m} + \tilde{D}_{12_m}K_n) \end{pmatrix} & (*)^T \\ \tilde{B}_{1_i}^T Q & -\gamma^2 I \end{pmatrix} < 0 \quad (23)$$

where $W = S^{-1}$ and $i, j = 1, 2, \cdots, r$. Since (23) is less than zero and the fact that $\mu_i \geq 0$ and $\sum_{i=1}^{r}\mu_i = 1$, then (16) becomes

$$\dot{V}(x(t)) \leq -\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]. \quad (24)$$

Integrate both sides of (24) yields

$$\int_0^{T_f} \dot{V}(x(t))dt \leq \int_0^{T_f} \Big[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]\Big] dt$$

$$V(x(t)) + V(x(0)) \leq \int_0^{T_f} \Big[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]\Big] dt.$$

Using the fact that $x(0) = 0$ and $V(x(t)) \geq 0$, we get

$$\int_0^{T_f} \tilde{z}^T(t)\tilde{z}(t)dt \leq \gamma^2 \left[\int_0^{T_f} \sum_{i=1}^r \sum_{j=1}^r \sum_{m=1}^r \sum_{n=1}^r \mu_i\mu_j\mu_m\mu_n [\tilde{w}^T(t)\tilde{w}(t)]dt \right]. \quad (25)$$

Putting $\tilde{z}(t)$ and $\tilde{w}(t)$ respectively given in (17) and (13) into (25) and using the fact that $\|F(x(t),t)\| \leq \rho$, $\lambda^2 = \left(1 + \rho^2 \sum_{i=1}^r \sum_{j=1}^r [\|H_{2_i}^T H_{2_j}\|]\right)$ and (22), we have

$$\int_0^{T_f} \sum_{i=1}^r \sum_{j=1}^r \mu_i\mu_j \Big(2\lambda^2 x^T(t)[C_{1_i} + D_{12_i}K_j]^T \times [C_{1_i} + D_{12_i}K_j]x(t) + (2\lambda^2\rho^2 \times x^T(t)[H_{4_i} + H_{5_i}K_j]^T[H_{4_i} + H_{5_i}K_j]x(t))\Big)\, dt \leq \gamma^2\lambda^2 \left[\int_0^{T_f} w^T(t)w(t)\, dt \right]. \quad (26)$$

Adding and subtracting

$$\lambda^2 z^T(t)z(t) = \lambda^2 \sum_{i=1}^r \sum_{j=1}^r \mu_i\mu_j \times \Big[x^T(t)\Big(C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{5_i}K_j\Big)^T \times \Big(C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{5_i}K_j\Big)x(t)\Big]$$

to and from (26), one obtains

$$\int_0^{T_f} \Big\{\lambda^2 z^T(t)z(t) + \sum_{i=1}^r \sum_{j=1}^r \mu_i\mu_j \Big(2\lambda^2 \times x^T(t)[C_{1_i} + D_{12_i}K_j]^T[C_{1_i} + D_{12_i}K_j]x(t) + 2\lambda^2\rho^2 x^T(t)[H_{4_i} + H_{5_i}K_j]^T[H_{4_i} + H_{5_i}K_j]x(t) - \lambda^2 x^T(t)[C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{5_i}K_j]^T \times [C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{5_i}K_j]x(t)\Big)\Big\}\, dt \leq \gamma^2\lambda^2 \left[\int_0^{T_f} w^T(t)w(t)\, dt \right] \quad (27)$$

Using the triangular inequality and the fact that $\|F(x(t),t)\| \leq \rho$, we have

$$\lambda^2 \sum_{i=1}^r \sum_{j=1}^r \mu_i\mu_j \Big(x^T(t)\Big[C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{5_i}K_j\Big]^T \times \Big[C_{1_i} + F(x(t),t)H_{4_i} + D_{12_i}K_j + F(x(t),t)H_{5_i}K_j\Big]x(t)\Big) \leq \sum_{i=1}^r \sum_{j=1}^r \mu_i\mu_j \Big(2\lambda^2 x^T(t)\Big[C_{1_i} + D_{12_i}K_j\Big]^T \times \Big[C_{1_i} + D_{12_i}K_j\Big]x(t) + \big(2\lambda^2\rho^2 \times x^T(t)\Big[H_{4_i} + H_{5_i}K_j\Big]^T\Big[H_{4_i} + H_{5_i}K_j\Big]x(t)\big)\Big) \quad (28)$$

Using (28) on (27), we obtain

$$\int_0^{T_f} z^T(t)z(t)\, dt \leq \gamma^2 \int_0^{T_f} w^T(t)w(t)\, dt. \quad (29)$$

Hence, the inequality (4) holds.

4. AN ILLUSTRATIVE EXAMPLE

Consider a modified nonlinear mass-spring-damper system as shown in Figure 1 which is governed by the following state equation [10]:

$$\begin{aligned} \dot{x}_1(t) &= -[0.1125 + \Delta R]x_1(t) - 0.0125x_1(t-\tau(t)) \\ &\quad -0.02x_2(t) - 0.67x_2^3(t) - 0.1x_2^3(t-\tau(t)) \\ &\quad -0.005x_2(t-\tau(t)) + u(t) + 0.1w_1(t) \\ \dot{x}_2(t) &= x_1(t) + 0.1w_2(t) \\ z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \quad (30)$$

where $x_1(t)$ and $x_2(t)$ are the state vectors which represent the velocity and distance, respectively, $u(t)$ is the control input, $w_1(t)$ and $w_2(t)$ are the disturbance input, $z(t)$ is the regulated output, ΔR is an uncertain term which is bounded in $[0 \;\; 0.1125]$, and the time-varying delay $\tau(t) = 4 + 0.5\cos(0.9t)$. It is assumed that

$$x_1(t) \in [-1.5 \;\; 1.5] \text{ and } x_2(t) \in [-1.5 \;\; 1.5].$$

Based on [10], the nonlinear term can be represented as

$$\begin{aligned} -0.67x_2^3(t) &= M_1 \cdot 0 \cdot x_2(t) \\ &\quad -(1-M_1) \cdot 1.5075x_2(t), \\ -0.1x_2^3(t-\tau(t)) &= M_1 \cdot 0 \cdot x_2(t-\tau(t)) \\ &\quad -(1-M_1) \cdot 0.225x_2(t-\tau(t)). \end{aligned}$$

Solving the above equations, M_1 is obtained as follows:

$$\begin{aligned} M_1(x_2(t)) &= 1 - \frac{x_2^2(t)}{2.25} \\ M_2(x_2(t)) &= 1 - M_1(x_2(t)) = \frac{x_2^2(t)}{2.25}. \end{aligned}$$

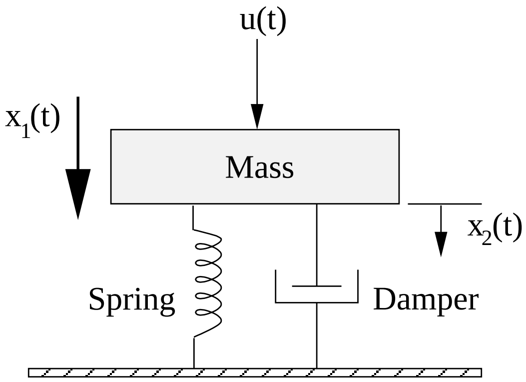

Fig.1: *Mass-spring-damper system.*

Note that $M_1(x_2(t))$ and $M_1(x_2(t))$ can be interpret as the membership functions of fuzzy set.

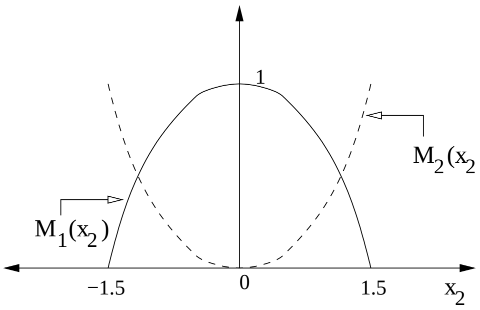

Fig.2: *Membership functions for two fuzzy set.*

Using these two fuzzy set, the uncertain nonlinear system with time-varying delay can be represented by the following TS fuzzy model:

Plant Rule 1: IF $x_2(t)$ is $M_1(x_2(t))$ THEN

$$\begin{aligned} \dot{x}(t) &= [A_1 + \Delta A_1]x(t) + A_{d_1}x(t-\tau(t)) \quad (31) \\ &\quad +B_1w(t) + B_2u(t), \quad x(0) = 0, \\ z(t) &= C_1x(t), \end{aligned}$$

Plant Rule 2: IF $x_2(t)$ is $M_2(x_2(t))$ THEN

$$\begin{aligned} \dot{x}(t) &= [A_2 + \Delta A_2]x(t) + A_{d_2}x(t-\tau(t)) \quad (32) \\ &\quad +B_1w(t) + B_2u(t), \quad x(0) = 0, \\ z(t) &= C_1x(t) \end{aligned}$$

where

$$A_1 = \begin{bmatrix} -0.113 & -0.02 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -0.113 & -1.528 \\ 1 & 0 \end{bmatrix},$$

$$A_{d_1} = \begin{bmatrix} -0.013 & -0.005 \\ 0 & 0 \end{bmatrix}, A_{d_2} = \begin{bmatrix} -0.013 & -0.23 \\ 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad B_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Delta A_1 = F(x(t),t)H_{1_1}, \quad \Delta A_2 = F(x(t),t)H_{1_2},$$

$$x(t) = [x_1^T(t)\ x_2^T(t)]^T \text{ and } w(t) = [w_1^T(t)\ w_2^T(t)]^T.$$

Now, by assuming that $\|F(x(t),t)\| \le \rho = 1$, we have

$$H_{1_1} = H_{1_2} = \begin{bmatrix} -0.1125 & 0 \\ 0 & 0 \end{bmatrix}.$$

Using the LMI optimization algorithm and Theorem **??** with $\gamma = 1$ and $\delta = 0.1$, we obtain

$$P = \begin{bmatrix} 0.471 & -0.174 \\ -0.174 & 0.137 \end{bmatrix}, \quad W = \begin{bmatrix} 3.009 & -0.124 \\ -0.124 & 2.889 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} -1.818 & -0.122 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -2.081 & 0.084 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -7.940 & -11.002 \end{bmatrix}, K_2 = \begin{bmatrix} -7.944 & -9.507 \end{bmatrix}.$$

The resulting fuzzy controller is

$$u(t) = \sum_{j=1}^{2} \mu_j K_j x(t) \quad (33)$$

where

$$\mu_1 = M_1(x_2(t)) \text{ and } \mu_2 = M_2(x_2(t)).$$

Remark 1: The fuzzy controller (33) guarantees that the inequality (4) holds. The histories of the state variables, $x_1(t)$ and $x_2(t)$ are given in Figure 3. The disturbance input signal, $w(t)$, which was used during simulation, is shown in Figure 4. The ratio of the regulated output energy to the disturbance input noise energy obtained by using the $\mathcal{H}_\infty$ fuzzy controller (33) is depicted in Figure 5. After 5 seconds, the ratio of the regulated output energy to the disturbance input noise energy tends to a constant value which is about 0.735. So $\gamma = \sqrt{0.735} = 0.857$, which is less than the prescribed value 1.

5. CONCLUSION

This paper has presented a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller design procedure for a class of uncertain nonlinear systems with time-varying delay which is described by TS fuzzy model. Based on an LMI approach, we developed a technique for designing a robust $\mathcal{H}_\infty$ fuzzy controller which guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output to be less than some prescribed value. A numerical example has been given to show the synthesis procedure developed in this paper.

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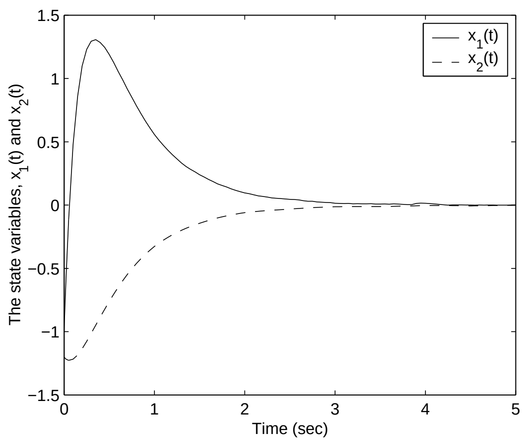

Fig.3: *The histories of the state variables, $x_1(t)$ and $x_2(t)$.*

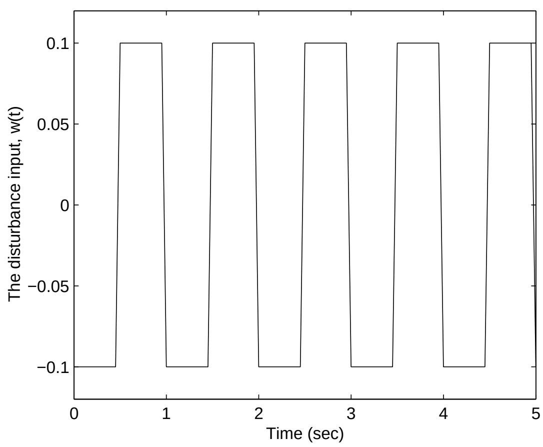

Fig.4: *The disturbance input noise, $w(t)$.*

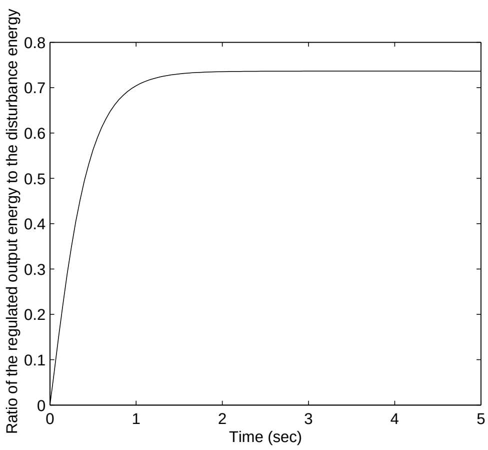

Fig.5: *The ratio of the regulated output energy to the disturbance noise energy, $\left(\frac{\int_0^{T_f} z^T(t)z(t)dt}{\int_0^{T_f} w^T(t)w(t)dt}\right)$.*

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