Optimal Minimax Controller for Plants with Four Oscillatory Modes Using Gröbner Basis

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ABSTRACT

Optimal minimax rate feedback controller design problems was proposed and partially solved by R.S. Bucy *et al* in 1990. The application of the problem have found in the oscillation suppressor design of large space structure with multiple oscillatory/resonance modes. By employing Gröbner basis technique, the complete symbolic solution for the case when the cardinality of the plant oscillatory mode is three or fewer was later found by N.K. Bose and the author. In this paper, the case when the cardinality is four is considered based on the use of Gröbner bases. In general, the higher order (four or more) problem is analytically intractable and suboptimal solutions based on numerical techniques are then the only recourse.

In addition, it is also shown that, for a specified generic plant, by incorporating in rate feedback controller the additional parameter available in the basic design procedure, significant improvement in feedback system performance over what was believed to be possible can be realized . This proposed additional design parameter expands the searching space for optimal solution (i.e. provides higher degree of freedom). Various numerical examples are shown to illustrate the effectiveness of the proposed method.

Keywords: Gröbner basis, Hurwitz polynomial, Minimal line, Minimax controller, Optimal rate feedback

1. INTRODUCTION

Studies on future space missions have indicated that the need for large space antennas and structures ranging from 30 meter to 20 kilometer in size: specific missions have been pinpointed and future requirements have been identified for large space antennas for communications, earth sensing and radio astronomy [13]. In order to obtain a strict performance specification, the antennas must be controlled to specified precision in attitude and shape. The plant modelling of the supporting and maneuvering system often involves oscillatory characteristic. The problem of minimax control using rate feedback is proposed in the control of oscillations of these large space flexible structures. The method can be also used to design the controller for marginally stable plants with four or fewer oscillatory modes, e.g. higher-degree of freedom inverse pendulum problem. The path breaking contribution in this area was reported more than a decade back [8]. The oscillatory plant transfer function can generally be derived in the form of rational polynomial with complex-conjugate poles on the $j\omega-$ axis. That is the denominator is in the form $(s^2+\beta_1)(s^2+\beta_2)\cdots(s^2+\beta_m)$, where m is the number of oscillatory modes. The actual number of oscillatory modes is often known imprecisely and the control engineer usually computes using the static and dynamic laws of motion with various assumptions. In many cases, the modelling of the plant is done by linearization or model reduction and system identification process which can cause error in number of oscillatory modes estimation. This loss of accuracy generally can lead to the unpredictable performance especially in the region, out of linearization assumption. As a result, the system robustness obtained from the appropriate minimax controller design becomes even more crucial since one of the goals in the minimax controller design is to maximize the stability margin of the closed-loop system to avoid uncertainty from hidden or un-modelled oscillatory modes.

By employing Gröbner basis technique, the complete symbolic solution for the case when the cardinality of the plant oscillatory mode is three or fewer was later found by N.K. Bose and the author. In this paper, the case when the cardinality is four is considered based on the use of Gröbner bases. In general, the higher order (four or more) problem is analytically intractable and suboptimal solutions based on numerical techniques are then the only recourse. In [9], the improved minimax controller design was proposed based on the extension of the searching space by adding an extra allowable design parameter. The proposed technique (also shown in Section 4, here) showed that the stability margin of closed-loop system is significantly increased.

When the plant to be controlled is marginally stable i.e. it contains pole(s) on the imaginary axis, a stabilizing controller is needed. The problem can be further complicated if the plant contains many os-

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cillatory modes as the rise-time and settling time of the unit step response of closed loop system will depend greatly on the slowest mode (i.e. the nearest close loop pole(s) to the imaginary axis). Therefore, it is desirable to have all poles as far away as possible from the imaginary axis. This optimal controller design is a type of minimax problem since its objective is to maximize the distance, from the imaginary axis, of poles associated with the slowest mode response. The resulting optimal minimax controller would significantly improve the unit step response of the closed loop system, e.g. shorter rise-time and settling time.

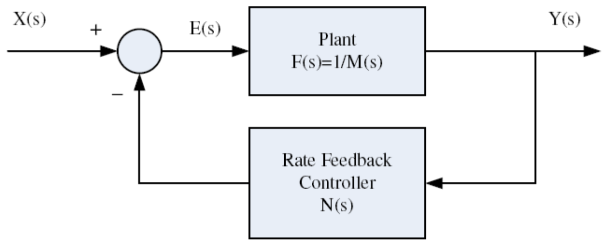

Fig.1: *Block diagram of Rate Feedback Control System*

As shown in Figure 1, given a plant $F(s) = 1/M(s)$, where

$$M(s) = \prod_{i=1}^{m}(s^2+\beta_i), \quad 0 < \beta_i < \beta_{i+1}$$

is an even polynomial with simple roots on the imaginary axis, in the complex s-plane, the problem is to find from the uncountably infinite set of odd polynomials $\{N_i(s)\}$, whose generic element has the form

$$N_i(s) = k_i s \prod_{l=1}^{m-1}(s^2+\gamma_{l(i)}),$$

$$k_i > 0, \beta_l < \gamma_{l(i)} < \beta_{l+1}$$

the one, denoted for brevity, by

$$N(s) = ks\prod_{l=1}^{m-1}(s^2+\gamma_l), \quad k > 0, \beta_l < \gamma_l < \beta_{l+1} \qquad (1)$$

so that the characteristic polynomial, $M(s)+N(s)$, of the resulting optimal rate feedback system has its rightmost roots located farthest to the left of the imaginary axis in comparison to similar roots for any other polynomial in the set $\{M(s)+N_i(s)\}$. In other words, the characteristic polynomial, $M(s)+N(s)$, is not only strict Hurwitz but, among an uncountably infinite set of such possibilities, is the one whose closest root (roots) to the imaginary axis is (are) farthest away from it [8]. This minimax system is then said to have the fastest slowest mode or maximum decay rate. Solutions to this challenging problem for the $m = 3$ case, given in [6], necessitated the use of Gröbner bases [5, Ch.6][1]. The purpose of this paper is

1. to find the algebraic algorithm for solving the minimax controller design using rate feedback in the $m = 4$ case
2. to show that for a prescribed number m of fixed modes it is possible to significantly improve the solution by modifying the strategy originally deployed in [8].

The solution for the $m = 1$ and $m = 2$ cases were obtained in [8] in a straightforward manner.

Definition 1[8] A line parallel to the imaginary axis in the complex s-plane is called the minimal line of a minimax system (optimal rate feedback system) if and only if (a) at least one pole of the minimax system lies on the line and (b) all the remaining poles of the minimax system lie to the left of the line.

Fact 1[8] For the $m \leq 4$ case, where n is the number of modes of the known plant, a minimax system is an optimal rate feedback system which has a pole of multiplicity greater than one on the minimal line.

This paper is organized as follows: in Section 2. , the solution for the case when $m = 3$ is first reviewed and the solving procedure for the case when $m = 4$ is carefully described. Numerical examples are illustrated in Section 3. In Section 4,the improved version of minimax solution is shown to be possible by increasing the order of the reate feedback controller by two without violating any underline constraints. Lastly, the conlusion and future research direction is given in Section 5.

2. MAIN RESULTS

2.1 The case when $m = 3$

Fact 1 implies that the characteristic polynomial

$$\begin{aligned} M(s)+N(s) &= (s^2+\beta_1)(s^2+\beta_2)(s^2+\beta_3) \\ &\quad +ks(s^2+\gamma_1)(s^2+\gamma_2), \qquad (2) \end{aligned}$$

(with $0 < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \beta_3 < \infty$) of the minimax system must be of the form

$$(s^2+As+B)^2(s^2+Cs+D), \qquad (3)$$

where A, B, C, D are positive real numbers. The elementary symmetric functions of the known plant modes $\beta_i > 0, i = 1, 2, 3$, are

$$\begin{aligned} c_1 &\triangleq \beta_1+\beta_2+\beta_3, \\ c_2 &\triangleq \beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1, \qquad (4) \\ c_3 &\triangleq \beta_1\beta_2\beta_3. \end{aligned}$$

The following result has been proved in [6] based on the usage of Gröbner basis [1, 11].

Theorem: The minimax solution for the $m = 3$ case is calculated recursively as follows so that each element of the solution set is real and positive.

Case 1: When $AD - BC = 0$, the first equation below is solved for a real, positive B and the remaining parameters are computed successively. The procedure is repeated for any remaining $B > 0$.

$$B^6 - c_2B^4 - 2c_3B^3 + 3c_1c_3B^2 - 2c_2c_3B + c_3^2 = 0, \quad (5)$$

$$A = \sqrt{\frac{B(-B^3 + c_2B - 2c_3)}{3c_3}}, \quad (6)$$

$$D = \frac{c_3}{B^2}, \quad (7)$$

$$C = \frac{AD}{B}, \quad (8)$$

$$E = \sqrt{C^2 - A^2}. \quad (9)$$

If at any stage of computation, either B or A or E is not real and positive, then the recursive mode of computation given in Case 2 below (valid when $E = 0$ or $A = C$) is used to construct the minimax solution. Note that in the case when there are more than one real, positive sets of solutions, the one with the largest value of parameter A is selected.

Case 2: When $E = 0$ or $A = C$, the parameters $B > 0, D > 0$ and $C = A > 0$ can be successively computed from

$$B^6 - 2c_1B^5 + 3c_2B^4 - 2c_3B^3 - c_1c_3B^2 + c_3^2 = 0,$$

$$D = \frac{c_3}{B^2},$$

$$C = A = \sqrt{\frac{c_1 - 2B - D}{3}}.$$

Once the solution set $\{A, B, C, D\}$ is calculated using the procedure detailed above, the corresponding solution set $\{k, \gamma_1, \gamma_2\}$ can be computed from the set of equations below. These equations are obtained by performing straightforward algebraic manipulations on the set of constraints generated by equating the coefficients of powers of s^5, s^3, s^1 in Eq.(2) and Eq.(3),

$$k = 2A + C$$

$$R \triangleq \frac{2AD + A^2C + 2BC + 2AB}{2k},$$

$$S \triangleq \frac{2ABD + B^2C}{k},$$

$$\gamma_1 = R - \sqrt{R^2 - S},$$

$$\gamma_2 = \frac{S}{\gamma_1}.$$

2.2 The case when $m = 4$

Fact 1 implies that the characteristic polynomial $M(s) + N(s)$ given by

$$(s^2 + \beta_1)(s^2 + \beta_2)(s^2 + \beta_3)(s^2 + \beta_4) + ks(s^2 + \gamma_1)(s^2 + \gamma_2)(s^2 + \gamma_3), \quad (10)$$

$(0 < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \beta_3 < \gamma_3 < \beta_4 < \infty)$ of the minimax system must be of the form

$$(s^2 + As + B)^2(s^2 + Cs + D)(s^2 + Es + F), \quad (11)$$

where A, B, C, D, E and F are positive real numbers with $A \leq C$ and $A \leq E$. Let define the elementary symmetric functions of the known plant modes $\beta_i > 0, i = 1, 2, 3, 4$, as

$$c_1 \triangleq \beta_1 + \beta_2 + \beta_3 + \beta_4,$$
$$c_2 \triangleq \beta_1\beta_2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_3 + \beta_2\beta_4 + \beta_3\beta_4, \quad (12)$$
$$c_3 \triangleq \beta_1\beta_2\beta_3 + \beta_1\beta_2\beta_4 + \beta_1\beta_3\beta_4 + \beta_2\beta_3\beta_4,$$
$$c_4 \triangleq \beta_1\beta_2\beta_3\beta_4.$$

The constraints obtained from equating coefficients of powers of s^6, s^4, s^2 and s^0 in Eq.(10) and Eq.(11) are

$$f_1 \triangleq D + 2AE + A^2 + 2AC + CE + F + 2B - c_1 = 0, \quad (13)$$

$$f_2 \triangleq A^2(D + F + CE) + A(2BE + 2BC + 2CF + 2DE) + 2BCE + 2BD + 2BF + B^2 + DF - c_2 = 0, \quad (14)$$

$$f_3 \triangleq 2BDF + 2ABDE + B^2CE + A^2DF + 2ABCF + B^2D + B^2F - c_3 = 0, \quad (15)$$

$$f_4 \triangleq B^2DF - c_4 = 0. \quad (16)$$

Furthermore, the conditions, $0 < A \leq C$ and $0 < A \leq E$ (by Fact 1) imply the existence of real numbers P and Q such that

$$f_5 \triangleq A^2 - C^2 + P^2 = 0 \quad (17)$$

$$f_6 \triangleq A^2 - E^2 + Q^2 = 0. \quad (18)$$

Actually, the minimax solution, characterized by A, B, C, D, E, F, P and Q is obtained from the maximization (minimization) of the cost function $A(\cdot)$ $(-A(\cdot))$ over the eight-dimensional parameter space subject to the constraints given by Eqs.(13)-(18). Note that the dot in the parenthesis indicates implicit dependence on the parameters $k_i, \gamma_{1i}, \gamma_{2i}, \gamma_{3i}$. In this case, the Lagrangian constrained optimization problem is set to minimize the cost function

$$G \triangleq -A(\cdot) + \sum_{i=1}^{6} \lambda_i f_i(\cdot),$$

where the λ_i's are the Lagrange multipliers. Since dependence on $k_i, \gamma_{1i}, \gamma_{2i}$ and γ_{3i} is not explicit, the goal of minimization of G and the finding of the Lagrange multipliers will be implemented by setting to zero each of the partial derivatives of G with respect to $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $E(\cdot)$, $F(\cdot)$,

$P(\cdot), Q(\cdot)$, and $\lambda_i, i = 1, 2, ..., 6$. This results in 14 polynomial equations in 14 unknowns. To reduce the complexity of the problem, consider a subset of these equations that are linear in the λ_i's and obtain conditions for existence of the Lagrange multipliers. Taking the partial derivatives of G with respect to $A(\cdot), B(\cdot), C(\cdot), D(\cdot), E(\cdot), F(\cdot), P(\cdot)$, and $Q(\cdot)$, the resulting set of linear equations (in the λ_i's) can be written in matrix-vector form as

$$\mathbf{X}\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t, \tag{19}$$

where Λ denotes a vector $[\lambda_1 \ \lambda_2 \ \cdots \ \lambda_6]^t$; $(\cdot)^t$ denotes the transpose operation; and $\mathbf{X}$ denotes the 8×6 coefficient matrix whose elements are polynomials in A, B, C, D, E, F, P and Q . Let $\tilde{\mathbf{X}}$ be a 7×6 matrix formed by the last 7 rows of the coefficient matrix $\mathbf{X}$. The homogeneous equation,

$$\tilde{\mathbf{X}}\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t, \tag{20}$$

is extracted from Eq.(19). In order to guarantee the existence of a nontrivial solution set for the λ_i's, the matrix $\tilde{\mathbf{X}}$ cannot be of full rank. This results in seven additional constraints involving the seven major determinants of $\tilde{\mathbf{X}}$. These major determinants were computed using a symbolic algebra program. Five of the seven expressions contain PQ as a factor, and the remaining two have either P or Q as a factor. Therefore from the following four special cases that yield real positive solution sets for $\{A, B, C, D, E, F\}$, the one containing the maximum value of A is identified with the location of the minimal line.

Case 1: When $P = 0$ and $Q = 0$, all the poles of the minimax system are on the minimal line. By using Eqs.(17)-(18), one gets

$$C = A \quad \text{and} \quad E = A.$$

Substituting the preceding constraints in Eqs.(13) - (16), it is possible to generate a set of four polynomial equations in A, B, D, and F. This set can be solved by using the technique of Gröbner basis similar to the case when $m = 3$[6]. The Gröbner basis of the ideal generated by the four polynomials referred to using the degree reverse lexicographical ordering[11] with $B \succ A \succ F \succ D$, yields the following relationships.

$$\begin{aligned} 31B^{12} &- 76c_1B^{11} + (129c_2 + 46c_1^2)B^{10} \\ &+ (2c_1^3 - 176c_3 - 168c_1c_2)B^9 \\ &+ (145c_4 + 241c_1c_3 + 144c_2^2 - 4c_1^2c_2)B^8 \\ &+ (6c_1^2c_3 - 200c_1c_4 - 408c_2c_3)B^7 \\ &+ (366c_2c_4 - 21c_1^2c_4 + 289c_3^2)B^6 \\ &+ (32c_1c_2c_4 - 544c_3c_4)B^5 \\ &+ (289c_4^2 - 51c_1c_3c_4)B^4 + 36c_1c_4^2B^3 \\ &+ 17c_2c_4^2B^2 - 17c_4^3 = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} 17B^2A^4 &+ 12B^3A^2 - 3c_1B^2A^2 + 3B^4 \\ &- 2c_1B^3 + c_2B^2 - c_4 = 0. \end{aligned} \tag{22}$$

Eq.(21) is solved for $B > 0$ and then Eq.(22) is solved for $A > 0$. After A and B are calculated, the remaining parameters, D and F, are successively computed from the following equations,

$$D = -3A^2 - B + \frac{1}{2}c_1 \pm \sqrt{M}, \tag{23}$$

$$F = \frac{c_4}{B^2D}, \tag{24}$$

where

$$\begin{aligned} M \triangleq\ & 18BA^4 + 12B^2A^2 - 6c_1BA^2 + 2B^3 \\ & -2c_1B^2 + \frac{1}{2}c_1^2B - \frac{2}{B}c_4. \end{aligned}$$

Expressions in Eqs.(23)-(24) are obtained by solving Eq.(13) and Eq.(16) for D and F.

Case 2: When $Q = 0, P \neq 0$, and P is real, all but one pair of complex conjugate poles are on the minimal line.

From Eq.(17), $Q = 0$ implies $E = A$. After substituting $Q = 0$ and $E = A$ into Eqs.(13)-(16) and the seven constraints for $\tilde{\mathbf{X}}$ in Eq.(20) to be not of full rank, the necessary and sufficient conditions for the minimax solutions to exist in this case are

$$c_1 = D + 3A^2 + 3AC + F + 2B, \tag{25}$$

$$\begin{aligned} c_2 = & CA^3 + (3D + F + 2B)A^2 + \\ & (4BC + 2CF)A + B^2 + 2BD \\ & +DF + 2BF, \end{aligned} \tag{26}$$

$$\begin{aligned} c_3 = & 2BDF + 2A^2BD + B^2CA + A^2DF \\ & +2ABCF + B^2D + B^2F, \end{aligned} \tag{27}$$

$$c_4 = B^2DF, \tag{28}$$

$$\begin{aligned} 0 = & (F - B)[(C + A)(D - F)B^2 + (2A^3D \\ & +F^2C + 2A^2CD - FDC + 2FAD \\ & -FA^2C - 3FAC^2 - 2AD^2)B + AD \times \\ & (A^2F - 4A^2D + FD + 3FCA - F^2)]. \end{aligned} \tag{29}$$

The expression on the right-hand side of Eq.(29) is associated with the only major determinant of $\tilde{\mathbf{X}}$ that does not have Q for a factor.

Eq.(29) can be satisfied when either $F = B$ or the other factor is zero. Consider each of these cases separately.

Case 2.1 $F = B$.

For the sake of brevity, the detailed procedure is omitted. In this special case, substituting $F = B$ in Eqs.(25)-(28) results in a system of four polynomial equations which again is solved by computing the Gröbner basis of the ideal generated by each of the four polynomial constraints. The following set of equations which can be use to solve for B, A, C, and D, successively, are generated using the Gröbner basis algorithm.

$$\begin{aligned} 7B^{12} &- 8c_1B^{11} + 9c_2B^{10} - 8c_3B^9 \\ &+ (-7c_4 - c_3c_1)B^8 + 16c_1c_4B^7 \\ &+ (c_3^2 + c_1^2c_4 - 18c_2c_4)B^6 + 16c_3c_4B^5 \\ &+ (-7c_4^2 - c_1c_3c_4)B^4 - 8c_1c_4^2B^3 \\ &+ 9c_2c_4^2B^2 - 8c_3c_4^2B + 7c_4^3 = 0, \end{aligned} \quad (30)$$

$$A = \sqrt{\frac{2B^5 - c_1B^4 + c_3B^2 - 2c_4B}{3(c_4 - B^4)}}, \quad (31)$$

$$C = \frac{c_1 - c_4/B^3 - 3A^2 - 3B}{3A}, \quad (32)$$

$$D = c_4/B^3. \quad (33)$$

Case 2.2 $F \neq B$.

The minimax solution for this case can be computed by solving for A, B, C, D and F using Eqs.(25)-(28) and

$$\begin{aligned} 0 = &(C + A)(D - F)B^2 + (2A^3D - 2AD^2 \\ &+F^2C + 2A^2CD - FDC + 2FAD \\ &-FA^2C - 3FAC^2)B + AD(A^2F \\ &-4A^2D + FD + 3FCA - F^2). \end{aligned} \quad (34)$$

Due to the complexity of computing the Gröbner basis in this case, the solutions are, then, generated using numerical techniques on any specified problem.

Case 3: $P = 0, Q \neq 0$, and Q is real.

This case is the complement of Case 2 and is tackled similarly by interchanging $C \leftrightarrow E$ and $D \leftrightarrow F$ in Eqs.(25)-(34).

Case 4: When $P \neq 0$ and $Q \neq 0$, the pair of complex conjugate poles on the minimal line is of multiplicity two and the remaining poles are strictly to its left.

It is conjectured from numerous simulations that Case 4 is vacuous. However, a convincing analytic approach for resolving this conjecture is lacking.

When the parameters A, B, C, D, E, and F are already computed, the minimax solution in terms of k, γ_1, γ_2, and γ_3 can be calculated by solving the set of conditions, obtained by equating coefficients of powers of s^7, s^5, s^3 and s^1 in Eq.(10) and Eq.(11).

The parameters k can be computed directly using the equation,

$$k = 2A + C + E, \quad (35)$$

and γ_1, γ_2, and γ_3 are the three roots of the third degree polynomial equation,

$$s^3 + v_2s^2 + v_1s + v_0 = 0, \quad (36)$$

where v_0, v_1, and v_2 are defined as,

$$v_0 = -\frac{2ABDF + B^2(CF + DE)}{k}, \quad (37)$$

$$\begin{aligned} v_1 = &\frac{1}{k}[(C + E)B^2 + 2(AF + AD + ACE \\ &+CF + DE)B + A(ACF + ADE \\ &+2DF)], \end{aligned} \quad (38)$$

$$\begin{aligned} v_2 = &-\frac{1}{k}[(C+E)A^2+2(B+D+F+CE)A \\ &+2BC + CF + DE + 2BE]. \end{aligned} \quad (39)$$

3. EXAMPLES

The examples below illustrate the procedure. The implementation of the procedure involves symbolic as well as numerical computations and the result could be sensitive to the data. The examples illustrate this sensitivity. In the first example, the choice of the solution set from the set of candidate solution sets is very dependent on the accuracy of computations. In the second example a slight perturbation of the data gives a robust solution.

3.1 Example 1.

Let the specified plant modes $\beta_1, \beta_2, \beta_3$, and β_4 be $\frac{1}{16}, \frac{1}{9}, \frac{1}{4}$ and 1, respectively. The objective is to construct a minimax controller characterized by the parameters $k, \gamma_1, \gamma_2, \gamma_3$, such that the overall feedback system has the fastest slowest mode.

From Eq.(12),

$$c_1 = \frac{205}{144}, \quad c_2 = \frac{273}{576},$$

$$c_3 = \frac{15}{288}, \quad c_4 = \frac{1}{576}.$$

Next from Case 1, assuming that $P = Q = 0$, the parameters B, A, D, F, are calculated recursively using Eqs. (21)-(24). Substituting c_1, c_2, c_3 and c_4 into Eq.(21) and using an accurate root finder, the real positive solutions of the parameter B are 0.088804, 0.257390 and 1.332363. For each case, the parameters A, D, and F can be solved successively by using Eqs.(22) to (24). In this case, there are two candidate solution sets $\{A, B, D, F\}$

$$\{0.08958, 0.0888, 0.2267, 0.9712\},$$

and

$$\{0.08958, 0.0888, 0.9712, 0.2267\}.$$

Note that a candidate solution set is implied to have each of its elements real and positive.

The second case ($Q = 0$ and $P \neq 0$) is considered next. Further assuming that $F = B$ (Case 2.1), the parameter B can be computed directly by solving Eq.(30). Using a root finder, the real and positive solutions of the parameters B are $0.054566, 0.108041, 0.25$, and 1.14603. For every real and positive solution of the parameter B, the remaining parameters A, C, and D are calculated successively by direct substitutions of the previously calculated parameters in Eqs.(31)-(33). The only candidate solution set $\{A, B, C, D\}$, is

$$\{0.4330, 0.2500, 1.082 \times 10^{-8}, 0.1111\}.$$

Note that the last candidate solution set is not a solution set because the associated parameter P is not real because $A > C > 0$. Case 2.2 is, then, applied. By using a numerical optimization program package to solve Eqs.(25)-(28) and (34), the parameters A, B, C, D, and F are 0.0897528, 0.0905779, 0.313231, 0.0898412, and 0.2355363 respectively. Since $Q = 0$, therefore $E = A = 0.0897528$. This candidate solution set contains the largest value of A (by a very slender margin) in comparison with each of the candidate solution sets in Case 1 and Case 2.1 above.

The corresponding minimax solution set $\{k, \gamma_1, \gamma_2, \gamma_3\}$ can be computed by using the Eqs.(35) and (36).

$$k = 0.5824906, \quad \gamma_1 = 0.08673205,$$
$$\gamma_2 = 0.01812798, \quad \gamma_3 = 0.51400550.$$

The closed loop poles of the optimal rate feedback system are located at

$$-0.044877 \pm 0.29760i \text{ (each of double multiplicity)},$$
$$-0.044877 \pm 0.48324i \text{ and } -0.15662 \pm 0.934817i.$$

It is noted that first three complex conjugate pairs are on the minimal line.

3.2 Example 2.

Here the plant modes β_1, β_2, and β_4 are the same as in the previous example while β_3, here, is $\frac{1}{8}$ instead of $\frac{1}{4}$.

The counterparts of the results in the previous example are summarized next. The candidate solution sets $\{A, B, D, F\}$ from Case 1 are

$$\{0.02026, 0.11702, 0.06348, 0.99863\},$$

and

$$\{0.02026, 0.11702, 0.99863, 0.06348\}.$$

The candidate solution set $\{A, B, C, D, F\}$, from Case 2.1 and 2.2 are, respectively,

$$\{0.03460, 0.11614, 3.78102, 0.55418, 0.11614\},$$

and

$$\{0.02844, 0.12580, 3.41268, 0.67130, 0.08212\}.$$

Therefore, the solution set $\{A, B, C, D, E, F\}$, is easily identified to be,

$$\{ 0.03460, 0.11614, 3.78102, 0.55418, 0.03460, 0.11614 \}.$$

The corresponding minimax solution set $\{k, \gamma_1, \gamma_2, \gamma_3\}$ can be computed by using the Eqs.(35) and (36).

$$k = 3.8848121, \quad \gamma_1 = 0.1014424,$$
$$\gamma_2 = 0.1174253, \quad \gamma_3 = 0.1447472.$$

4. IMPROVING MINIMAX SOLUTION BY EXTENDING ALLOWABLE SEARCHING SPACE

In order to satisfy the simple alternating (interlacing) constraint,

$$0 < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \beta_3 < \gamma_3 < \beta_4 \cdots \quad (40)$$

involving the known plant modes $\{\beta_i\}$ and the unknown controller modes $\{\gamma_i\}$, we require here the finding of the polynomial $ks\prod_{i=1}^{m}(s^2+\gamma_i)$ instead of $ks\prod_{i=1}^{m-1}(s^2+\gamma_i)$ in Eq.(**??**). The additional parameter γ_m, then, is capable of providing a better solution as demonstrated next. Note that a further increase in the number of controller parameters is not possible without violating the interlacing constraint in Eq.(40).

When the plant is second-order, characterized by the polynomial $(s^2+\beta_1), \beta_1 > 0$ ($m = 1$ case) the procedure in [8] produces a closed-loop characteristic polynomial,

$$(s^2+\beta_1) + 2s\sqrt{\beta_1} = (s+\sqrt{\beta_1})^2,$$

of the minimax feedback system. It is shown below that by modifying the strategy for solving the problem posed in [8] in the manner suggested here, the solution will be improved significantly from that in [8]. The modified characteristic polynomial for the $m = 1$ case is

$$(s^2+\beta_1)+ks(s^2+\gamma_1) = k(s^3+\frac{1}{k}s^2+\gamma_1 s+\frac{\beta_1}{k}), \quad (41)$$

where the monic polynomial shown within brackets on the right-hand side is of interest. The following special cases associated with two types of pole configurations for the feedback system are considered.

Case 1: The feedback system has a pair of complex conjugate poles and one real-valued pole.

With $A > 0, B > 0$, and $C > 0$, the generic monic characteristic polynomial in Eq.(41) is of the form

$$(s+A)(s^2+Bs+C) = s^3+(A+B)s^2+(AB+C)s+AC, \quad (42)$$

where $B^2 - 4C \leq 0$, which require consideration of two subcategories relevant to this problem. The coefficients of s^2, s^1 and s^0 in the right-hand sides of Eq.(41) and Eq.(42) generate the following identities.

$$(A+B) = \frac{1}{k}, \quad (AB+C) = \gamma_1, \quad AC = \frac{\beta_1}{k}. \quad (43)$$

The solution set $\{k, \gamma_1\}$ can then be determined from the solution set $\{A, B, C\}$ using the first two relations in Eq.(43). The finding of the solution set $\{A, B, C\}$ require the imposition of the constraint carried in the third relation of Eq.(43), rewritten below as

$$AC = \beta_1(A+B) \quad (44)$$

in a Largrangian minimization problem described below for each of the subcategories for this case.

Case 1a: $A \geq \frac{B}{2}$. The cost function to be minimized is $-\frac{B}{2}$ (note that the closed-loop poles closest to the imaginary axis are at a distance $\frac{B}{2}$ from it) subject to the constraints in Eq.(44) and Eq.(45) below.

The conditions, $A \geq \frac{B}{2} > 0$ and $B^2 - 4C \leq 0$ imply the existence of real numbers P and Q such that

$$A - \frac{B}{2} - P^2 = 0, \quad B^2 - 4C + Q^2 = 0. \quad (45)$$

The unconstrained objective function G for finding the solution set $\{A, B, C\}$ then becomes

$$\begin{aligned} G &= -\frac{B}{2} + \lambda_1(AC - \beta_1(A+B)) + \lambda_2(A - \frac{B}{2} - P^2) \\ &\quad +\lambda_3(B^2 - 4C + Q^2), \end{aligned}$$

where $\lambda_i, i = 1, 2, 3$ are the Largrange multipliers. The goal of minimization of G is implemented by setting to zero each of the partial derivatives of G with respect to $A, B, C, P, Q, \lambda_1, \lambda_2$ and λ_3. Routine algebra leads to the minimax solution set $\{A, B, C, P, Q\}$ in terms of the known plant mode β_1 as given below.

$$\{A = \sqrt{3\beta_1}, \quad B = \sqrt{12\beta_1}, \quad C = 3\beta_1, \quad P = Q = 0\}.$$

Case 1b: $A \leq \frac{B}{2}$.

Similar to Case 1a, the unconstrained objective function G in this case is

$$\begin{aligned} G &= -A + \lambda_1(AC - \beta_1(A+B)) + \lambda_2(A - \frac{B}{2} + P^2) \\ &\quad +\lambda_3(B^2 - 4C + Q^2). \end{aligned}$$

The solution set $\{A, B, C, P, Q\}$, obtained as in Case 1a, is

$$\{A = \sqrt{3\beta_1}, \quad B = \sqrt{12\beta_1}, \quad C = 3\beta_1, \quad P = Q = 0\}.$$

Note that the minimax solution set in this case is identical to that in Case 1a and all poles turns out to be real with triple multiplicity (i.e. the imaginary parts of the complex poles converge to zero).

Case 2: The feedback system has three real-valued poles.

The generic monic characteristic polynomial is then of the form

$$\begin{aligned} (s+A)(s+D)(s+E) &= s^3 + (A+D+E)s^2 + (AD \\ &\quad +DE+AE)s + ADE, \quad (46) \end{aligned}$$

where $E \geq D \geq A > 0$. By equating the coefficients of s^2, s^1 and s^0 in the right-hand sides of Eq.(41) and Eq.(6), the counterparts of Eq.(43) and Eq.(44) are Eq.(47) and Eq.(48) below.

$$\begin{aligned} k &= \frac{1}{A+D+E}, \quad \gamma_1 = (AD+DE+AE) \quad (47) \\ ADE &= \beta_1(A+D+E). \quad (48) \end{aligned}$$

The unconstrained objective function G to solve for $\{A, D, E, P, Q\}$ is, in this case,

$$\begin{aligned} G &= -A + \lambda_1(ADE - \beta_1(A+D+E)) \\ &\quad +\lambda_2(A - D + P^2) + \lambda_3(A - E + Q^2), \end{aligned}$$

which can be shown to be minimized when $A = D = E$. The minimax solution set $\{A, D, E, P, Q\}$ is

$$\{A = D = E = \sqrt{3\beta_1}, \quad P = Q = 0\}.$$

Note that in all cases, the minimal line (a line parallel to the imaginary axis in the complex s-plane with the conditions that every pole lies to the left of it and at least one pole of the system lies on the line [8]) is situated at a distance of $\sqrt{3\beta_1}$ from the imaginary axis in comparison with $\sqrt{\beta_1}$ obtained via the solution strategy in [8]. Thus, a significant improvement in performance is generated in the rate feedback minimax control of a specified plant.

5. CONCLUSION

In this paper, the minimax controller design problem using rate feedback is investigated when the number m of distinct coupled oscillator frequencies for the plant characterizing a large space structure does not exceed four. Gröbner bases in polynomial ideal theory are used for the purpose. Complete analytic characterization and solution construction is possible when $m \leq 3$. The extent to which such a characterization and recursive mode of solution construction is possible, is shown for the $m = 4$ case. When the plant is of higher order, the construction of an optimal solution is analytically and computationally intractable necessitating the need for numerical methods for generating satisfactory suboptimal solutions.

Here, it is also demonstrated how a modification of the solution strategy by the incorporation of one extra allowable parameter in the controller can lead to a substantial improvement in the feedback system performance over that based on the conventional approach[8]. Note that, the result obtained by the modification suggested here leads necessarily to the optimal minimax solution because further increase in the number of controller parameters will violate the interlacing condition of the plant and controller modes in Eq.(40). The proposed approach can be extended to the higher order case ($m > 1$).

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APPENDIX

Derivation of minimax controller design procedure for $m = 4$ case

It is required to solve the Lagrangian constrained optimization problem which requires the minimization of

$$G \triangleq -A(\cdot) + \sum_{i=1}^{6} \lambda_i f_i(\cdot),$$

where the λ_i's are the Lagrange multipliers. Consider a subset of these equations that are linear in the λ_i's and obtain conditions for existence of the Lagrange multipliers. Taking the partial derivatives of G with respect to $A(\cdot), B(\cdot), C(\cdot), D(\cdot), E(\cdot), F(\cdot), P(\cdot)$, and $Q(\cdot)$, the resulting set of linear equations (in the λ_i's) can be written in matrix-vector form as

$$\mathbf{X}\Lambda = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^t, \qquad (49)$$

where Λ denotes a vector $[\lambda_1\ \lambda_2\ \cdots\ \lambda_6]^t$ and $\mathbf{X}$ denotes the 8×6 coefficient matrix given below.

$$\mathbf{X}=\begin{bmatrix} A+C+E & x_{12} & x_{13} & 0 & A & A \\ 1 & x_{22} & x_{23} & BDF & 0 & 0 \\ 2A+E & x_{32} & x_{33} & 0 & -2C & 0 \\ 1 & x_{42} & x_{43} & B^2F & 0 & 0 \\ 2A+C & x_{52} & x_{53} & 0 & 0 & -2E \\ 1 & x_{62} & x_{63} & B^2D & 0 & 0 \\ 0 & 0 & 0 & 0 & P & 0 \\ 0 & 0 & 0 & 0 & 0 & Q \end{bmatrix},$$

where

$$\begin{aligned}
x_{12} &= (CD + D + F)A + BC + BE + CF + DE \\
x_{22} &= AC + AE + CE + B + D + F \\
x_{32} &= A^2E + 2AB + 2AF + 2BE \\
x_{42} &= A^2 + 2AE + 2B + F \\
x_{52} &= A^2C + 2AB + 2AD + 2BC \\
x_{62} &= A^2 + 2AC + 2B + D \\
x_{13} &= BDE + ADF + BCF \\
x_{23} &= ACF + ADE + BCE + BD + BF + DF \\
x_{33} &= B^2E + 2ABF \\
x_{43} &= A^2F + 2ABE + B^2 + 2BF \\
x_{53} &= 2ABD + B^2C \\
x_{63} &= A^2D + 2ABC + B^2 + 2BD.
\end{aligned}$$

Let $\tilde{\mathbf{X}}$ be a 7×6 matrix formed by the last 7 rows of the coefficient matrix $\mathbf{X}$. The homogeneous equation,

$$\tilde{\mathbf{X}}\Lambda = [0\ 0\ 0\ 0\ 0\ 0\ 0]^t\,, \tag{50}$$

is extracted from Eq.(49). In order to guarantee the existence of a nontrivial solution set for the λ_i's, the matrix $\tilde{\mathbf{X}}$ cannot be of full rank. This results in seven additional constraints involving the seven major determinants $w_i,\ i = 1, 2, \ldots, 7$ of $\tilde{\mathbf{X}}$ listed below.

$$\begin{aligned}
w_1 &= 4A^4B^3PQ(C^2F - CDE - CEF + D^2 \\
&\quad +DE^2 - 2DF + F^2) \\
w_2 &= -2A^2BDPQ(-A^3CDF + A^3DEF \\
&\quad -A^2BC^2F + A^2BCDE + A^2BCEF \\
&\quad -A^2BDE^2 + 2A^2BDF - 2A^2BF^2 \\
&\quad -2A^2D^2F + 2A^2DF^2 + AB^2C^2E \\
&\quad -AB^2CD - AB^2CE^2 + 2AB^2CF \\
&\quad -2AB^2DE + AB^2EF - 2ABCF^2 \\
&\quad +2ABD^2E + ACDF^2 - AD^2EF \\
&\quad -B^3C^2 + B^3E^2 + 2B^2C^2F - 2B^2DE^2 \\
&\quad -BC^2F^2 + BD^2E^2)
\end{aligned}$$

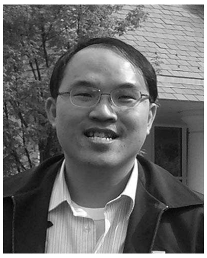
$$\begin{aligned}
w_3 &= -2A^2BPQ(A^3D^2F - A^3DF^2 + A^2BCDF \\
&\quad +A^2BCF^2 - A^2BD^2E - A^2BDEF \\
&\quad -2A^2CDF^2 + 2A^2D^2EF - AB^2C^2F \\
&\quad -AB^2CEF + 2AB^2DE^2 - AB^2DF \\
&\quad +AB^2F^2 + 2ABC^2F^2 - 2ABD^2E^2 \\
&\quad +2ABD^2F - 2ABDF^2 - AC^2DF^2 \\
&\quad +ACD^2EF - AD^3F + AD^2F^2 + B^3CD \\
&\quad -B^3CF + B^3DE - B^3EF - B^2C^3F \\
&\quad +B^2C^2DE - B^2C^2EF - B^2CD^2 + \\
&\quad B^2CDE^2 + B^2CF^2 - 2B^2D^2E \\
&\quad +2B^2DEF + BC^3F^2 - BCD^2E^2 \\
&\quad +BCD^2F - BCDF^2 + BD^3E - BD^2EF)
\end{aligned}$$

$$\begin{aligned}
w_4 &= -2A^2BPQ(A^3D^2F - A^3DF^2 + A^2BCDF \\
&\quad +A^2BCF^2 - A^2BD^2E - A^2BDEF \\
&\quad -2A^2CDF^2 + 2A^2D^2EF - 2AB^2C^2F \\
&\quad +AB^2CDE - AB^2D^2 + AB^2DE^2 \\
&\quad +AB^2DF + 2ABC^2F^2 - 2ABD^2E^2 \\
&\quad +2ABD^2F - 2ABDF^2 - ACDEF^2 \\
&\quad +AD^2E^2F - AD^2F^2 + ADF^3 \\
&\quad +B^3CD - B^3CF + B^3DE - B^3EF \\
&\quad -B^2C^2EF + B^2CDE^2 - 2B^2CDF \\
&\quad -B^2CE^2F + 2B^2CF^2 - B^2D^2E \\
&\quad +B^2DE^3 + B^2EF^2 + BC^2EF^2 \\
&\quad +BCDF^2 - BCF^3 - BD^2E^3 \\
&\quad +BD^2EF - BDEF^2)
\end{aligned}$$

$$\begin{aligned}
w_5 &= -2A^2BFPQ(-A^3CDF + A^3DEF \\
&\quad +A^2BC^2F - A^2BCDE - A^2BCEF \\
&\quad +2A^2BD^2 + A^2BDE^2 - 2A^2BDF \\
&\quad -2A^2D^2F + 2A^2DF^2 + AB^2C^2E \\
&\quad -AB^2CD - AB^2CE^2 + 2AB^2CF \\
&\quad -2AB^2DE + AB^2EF - 2ABCF^2 \\
&\quad +2ABD^2E + ACDF^2 - AD^2EF \\
&\quad -B^3C^2 + B^3E^2 + 2B^2C^2F \\
&\quad -2B^2DE^2 - BC^2F^2 + BD^2E^2)
\end{aligned}$$

$$\begin{aligned}
w_6 &= -4A^2BCQ(A^3D^2F - A^3DF^2 \\
&\quad +A^2BCDF + A^2BCF^2 - A^2BD^2E \\
&\quad -A^2BDEF - 2A^2CDF^2 + 2A^2D^2EF \\
&\quad -AB^2C^2F - AB^2CEF + 2AB^2DE^2 \\
&\quad -AB^2DF + AB^2F^2 + 2ABC^2F^2 \\
&\quad -2ABD^2E^2 + 2ABD^2F - 2ABDF^2 \\
&\quad -AC^2DF^2 + ACD^2EF - AD^3F \\
&\quad +AD^2F^2 + B^3CD - B^3CF + B^3DE \\
&\quad -B^3EF - B^2C^3F + B^2C^2DE \\
&\quad -B^2C^2EF - B^2CD^2 + B^2CDE^2 \\
&\quad +B^2CF^2 - 2B^2D^2E + 2B^2DEF \\
&\quad +BC^3F^2 - BCD^2E^2 + BCD^2F \\
&\quad -BCDF^2 + BD^3E - BD^2EF)
\end{aligned}$$

$$
\begin{aligned}
w_7 \;=\;& 4A^2BEP(A^3D^2F - A^3DF^2 \\
&+A^2BCDF + A^2BCF^2 - A^2BD^2E \\
&-A^2BDEF - 2A^2CDF^2 + 2A^2D^2EF \\
&-2AB^2C^2F + AB^2CDE - AB^2D^2 \\
&+AB^2DE^2 + AB^2DF + 2ABC^2F^2 \\
&-2ABD^2E^2 + 2ABD^2F - 2ABDF^2 \\
&-ACDEF^2 + AD^2E^2F - AD^2F^2 + ADF^3 \\
&+B^3CD - B^3CF + B^3DE - B^3EF \\
&-B^2C^2EF + B^2CDE^2 - 2B^2CDF \\
&-B^2CE^2F + 2B^2CF^2 - B^2D^2E \\
&+B^2DE^3 + B^2EF^2 + BC^2EF^2 \\
&+BCDF^2 - BCF^3 - BD^2E^3 \\
&+BD^2EF - BDEF^2).
\end{aligned}
$$



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