# Noise analysis of single stage fractional-order low-pass filter using stochastic and fractional Calculus

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## ABSTRACT

In this paper, we present the noise analysis of a simple single stage low-pass filter (SSLPF) with the fractional-order capacitor, using stochastic differential equations (SDE). The input noise is considered to be white and various solution statistics of output namely mean, variance, auto-correlation and power spectral density (PSD) are obtained using tools from both stochastic and fractional calculus. We investigate the change in these statistics with the change in the capacitor order. The closed form solutions of the step response of the fractional filter are also provided and it has been found that filters with capacitor order greater than unity have a faster step response but suffer from higher output noise and although, the filters with capacitor order less than unity enjoy advantage of less output noise, but they have a sluggish step response. And hence, an appropriate fractional capacitor can be chosen for the desired circuit behavior. A brief study of more generic class of single stage fractional-order high-pass and all-pass filtering functions has been included. The idea can be extended to more complex and practical fractional-order circuits.


**Keywords**: Single-stage fractional low-pass filter, Noise, Stochastic differential equation (SDE), Fractional calculus, Mean and Variance

## 1. INTRODUCTION

Traditional circuit analysis/design techniques are based on integer-order energy storage elements. In the classical RC circuit design, the capacitor follows the current-voltage relation:

$$i(t) = \frac{dv(t)}{dt} \tag{1}$$

A fractional-order capacitor is an element whose current and voltage relationship is given by,

$$i(t) = \frac{d^{\alpha}v(t)}{dt^{\alpha}} \tag{2}$$

where $\alpha$ is the fractional order of the capacitor, and when $\alpha = 1$ the all-familiar integer-order relationship follows. Use of the fractional-order elements would generalize the classical circuit analysis. Recently, there has been some promising efforts towards synthesis of fractional capacitors [1-6]. Undoubtedly, developments in the synthesis of such fractional order elements would impact the future of circuit design. And therefore, an accurate noise analysis of such fractional-order circuits is important.

The method of SDEs has been used extensively for noise analysis of traditional circuits with integer-order storage elements [7-9]. Here we extend the time-domain method of SDE for noise analysis of a simple single stage low-pass filter (SSLPF), wherein the capacitor is fractional order. We analyze the changes in the solution statistics of the output noise of the system for different values of the order of the capacitor. For the stochastic model being used in this paper, the external input noise is assumed to be white. Although the assumption of white Gaussian noise is an idealization, it may be justified because of the existence of many random input effects. According to the central limit theorem, when the uncertainty is due to additive effects of many random factors, the probability distribution of such random variables is Gaussian. It may be difficult to isolate and model each factor that produces uncertainty in the circuit analysis. Therefore, the noise sources are assumed to be white with a flat power spectral density. Also, the intrinsic noise could be modelled on similar lines as explained in the subsequent sections. Since the involved differential equations governing the output noise variance and auto-correlation are fractional order, hence finding the analytical solutions for these output noise statistics requires some techniques from fractional calculus. We will use the 'Riemann-Louivelle' definition of the fractional integrals throughout the paper [10].

The rest of the paper is organized as follows. Section 3 derives the differential equations governing noise variance processes and solves them in time domain. Section 4 provides with the numerical simulation results, discusses high-pass and all-pass fractional filters and we conclude in section 5. Section 6 and 7 include the appendix. We provide the necessary background of stochastic calculus in Appendix A and provide with the some necessary details of fractional

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calculus in Appendix B.

## 2. NOISE ANALYSIS

In this section we present the time domain and frequency domain analysis of the single-stage low-pass filter as shown in Fig. 1, using fractional calculus. The noise can enter into the circuit via two paths: either from external sources (extrinsic noise) or from within the circuit (intrinsic noise, example in the resistor). Both noise sources are assumed to be white and could be modelled as as a single equivalent noise source. Extensive work on the intrinsic noise analysis of integral order RLC circuits have been done previously by Rawat et al [8-9]. The fractional-order differential equation governing the circuit is given by

$$\frac{d^\alpha(v_0(t))}{dt^\alpha} + \frac{1}{RC}v_0(t) = \frac{v_i(t)}{RC} \tag{3}$$

For the noise analysis we consider the input to the fractional circuit $v_i = \sigma n(t)$, where $n(t)$ represents Gaussian white noise process and $\sigma^2$ is the magnitude of power spectral density (PSD) of noise process (modelled as summation of both intrinsic and extrinsic noise); hence (3) can be rewritten as

$$\frac{d^\alpha(v_0(t))}{dt^\alpha} + \frac{1}{RC}v_0(t) = \frac{\sigma n(t)}{RC} \tag{4}$$

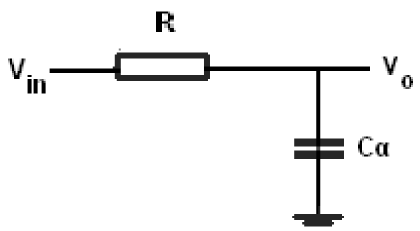


***Fig.1:*** *Single-stage fractional RC LPF*

### 2.1 Mean

Taking expected values on both sides of (4) and considering the fact that $E[n(t)] = 0$, we get

$$\frac{d^\alpha E[v_0(t)]}{dt^\alpha} + \frac{1}{RC}E[v_0(t)] = 0 \tag{5}$$

For $\alpha = 1$, the above equation results into a simple first order differential equation whose solution is well known to be:

$$E[v_0(t)] = c_1 e^{-t/RC} \tag{6}$$

where $c_1$ is a constant whose value depends on the initial conditions.

But for $\alpha$ to be a non negative fraction the solution is not that easy and would require some techniques from fractional calculus for finding the solution. Considering $E[v_0(t)] = Y(t)$ and taking the Laplace transform on both sides of (5) we get

$$Y(s) = \frac{c_2}{1 + s^\alpha RC} \tag{7}$$

where $c_2$ is a constant whose value is $D^{-(1-\alpha)}v_0(0)$ (Appendix B) and which is initial condition dependent. For taking the inverse Laplace transform of (7) to get the solution for mean in time-domain would require the use of fractional calculus; which has been dealt in Appendix B.

Considering $\alpha = \frac{p}{q}$ and $v = \frac{1}{q}$ where p and q are non negative integers and using some standard known results from fractional calculus (Appendix B), the solution of (7) is

$$E[v_0(t)] = Y(t) = \frac{c_2}{pRC}\sum_{k=1}^{p}\sum_{j=1}^{q}(\beta_k)^j E_t(jv-1,(\beta_k)^q) \tag{8}$$

where in this case $\beta_k$ are the p roots of $\frac{-1}{RC}$. As a special case of (8) if $RC = 1$ then $\beta_k$ would be the roots of -1.

If $RC = 1$ and $p = 1$ then $\alpha = \frac{1}{q}$ so that (8) becomes

$$E[v_0(t)] = c_2\sum_{j=1}^{q}(-1)^{j-1}E_t(j\alpha-1,(-1)^q) \tag{9}$$

Clearly, if $\alpha = 1$ the the solution got from (9) confirms with our previous answer (6) (This is for $E_t(0,a) = e^{at}$) (Appendix B)

### 2.2 Second Moment and Auto-Correlation

Next we turn our attention to finding the auto-correlation function and variance. For $\alpha = 1$ the governing equation of the circuit is

$$d(v_0(t)) = -\frac{1}{RC}v_0(t)dt + \frac{\sigma dW(t)}{RC} \tag{10}$$

where $W(t)$ is the Wiener motion process, a continuous, but not differentiable process and is given by $dW(t) = n(t)dt$.

The solution for second moment $E[v_0^2(t)]$ can be easily got by solving the following first order differential equation (Appendix A)

$$\frac{dE[v_0^2(t)]}{dt} + \frac{2}{RC}E[v_0^2(t)] = \frac{\sigma^2}{R^2C^2} \tag{11}$$

Considering $RC = 1$ as before the second moment is given by

$$E[v_0^2(t)] = \frac{\sigma^2}{2} + c_3 e^{-2t/RC} \tag{12}$$

where $c_3$ is a constant whose value depends on initial conditions. If initial conditions are zero, the variance is given by

$$E[v_0^2(t)] = \frac{\sigma^2}{2}(1 - e^{-2t/RC}) \tag{13}$$

For $\alpha$ to be a fractional we use a two-step procedure for finding the auto-correlation function of $v_0(t)$ through which variance can be determined. Consider (3) at time $t = t_1$ with initial conditions $R_{v_0,v_0}(0,t_2) = E[v_0(t_1)v_0(t_2)]|_{t_1=0} = 0$. Multiplying both sides of (3) with $v_0(t_2)$ and then taking the expectation, we obtain

$$\frac{d^\alpha R_{v_0,v_0}(t_1,t_2)}{dt_2^\alpha} + \frac{R_{v_0,v_0}(t_1,t_2)}{RC} = \frac{R_{v_i,v_0}(t_1,t_2)}{RC} \tag{14}$$

Again considering (3) at time $t = t_2$ with initial conditions $R_{v_i,v_0}(t_1,0) = E[v_i(t_1)v_0(t_2)]|_{t_2=0} = 0$. Multiplying both sides of (3) with $v_i(t_1)$ and then taking the expectation, we obtain

$$\frac{d^\alpha R_{v_i,v_0}(t_1,t_2)}{dt_2^\alpha} + \frac{R_{v_i,v_0}(t_1,t_2)}{RC} = \frac{R_{v_i,v_i}(t_1,t_2)}{RC} \tag{15}$$

For simplicity we consider $RC = 1$ . With this assumption the closed-form solution for auto-correlation function has been arrived at. It is known that as input is white noise therefore $R_{v_i,v_i}(t_1,t_2) = \sigma^2\delta(t_1-t_2)$. Now considering $t_1 = t+\tau$ and $t_2 = t$ and using (15) and (14), we obtain the auto-correlation for large value of time $t$, to be

$$R_{vo,vo}(\tau) = \frac{\sigma^2}{1-(-1)^\alpha}\pounds^{-1}\Big(\frac{1}{s^\alpha + 1} - \frac{1}{(-1)^{-\alpha} + s^\alpha}\Big) \tag{16}$$

For simplicity we consider $\alpha = \frac{1}{q}$ where $q$ is a non negative integer; then auto-correlation $R_{vo,vo}(\tau)$ for all $\tau$ becomes:-

$$R_{vo,vo}(\tau) = \frac{\sigma^2}{1-(-1)^\alpha}\Big(\sum_{j=1}^{q}(-1)^{j-1}E_\tau(j\alpha-1,(-1)^q)u(\tau) + \sum_{j=1}^{q}(-1)^{-(\alpha+1)(j-1)}E_\tau(j\alpha-1,(-1)^{-(\alpha+1)q})u(-\tau)\Big) \tag{17}$$

If $\alpha = 1$, (17) changes into a familiar expression of auto-correlation $R_{vo,vo}(\tau) = \frac{\sigma^2}{2}e^{-|\tau|}$. The second moment of the output for the fractional circuit is found using the equation

$$E[v_0^2(t)] = \sigma^2\int_0^t |h(t)|^2 dt \tag{18}$$

where $h(t)$ is given by

$$h(t) = \pounds^{-1}\Big(\frac{1}{s^\alpha + 1}\Big) \tag{19}$$

For $\alpha = \frac{1}{q}$ where $q$ is a non negative integer, as before, the expression for the variance is

$$E[v_0^2(t)] = \sigma^2\int_0^t |(\sum_{j=1}^{q}(-1)^{j-1}E_\tau(j\alpha-1,(-1)^q)|^2 dt \tag{20}$$

For $\alpha = 1$, the variance got from (20) confirms with the solution (13).

## 2.3 Power Spectral Density (PSD)

The PSD of the output for the fractional-order SSLPF is found to be

$$S_{yy}(\omega) = \frac{\sigma^2}{\omega^{2\alpha}R^2C^2 + 2RC\omega^\alpha cos(\pi\alpha/2) + 1} \tag{21}$$

For $\alpha = 1$ and $RC = 1$, (18) changes to a familiar expression for the PSD:

$$S_{yy}(\omega) = \frac{\sigma^2}{\omega^2 + 1} \tag{22}$$

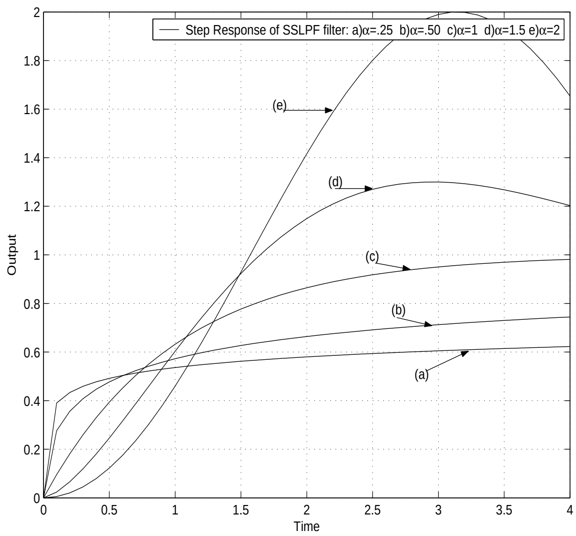


***Fig.2:*** *Time response of fractional-order RC LPF for (a)* $\alpha = 0.25$ *(b)* $\alpha = 0.50$ *(c)* $\alpha = 1.0$ *(d)* $\alpha = 1.5$ *(e)* $\alpha = 2.0$

## 3. SIMULATION RESULTS AND DISCUSSION ON OTHER SINGLE-STAGE FRACTIONAL FILTERS

To supplement the theoretical results numerical simulations have been carried out (from the derived closed-form solutions) in MATLAB. For simulations,

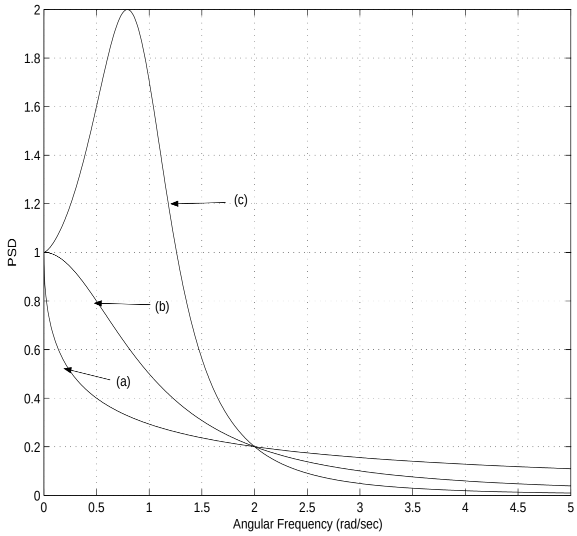


***Fig.3:*** *Variation of PSD of the output of SSLPF with frequency for different orders (a)* $\alpha = 0.50$*, (b)* $\alpha = 1.0$*, (c)* $\alpha = 1.5$

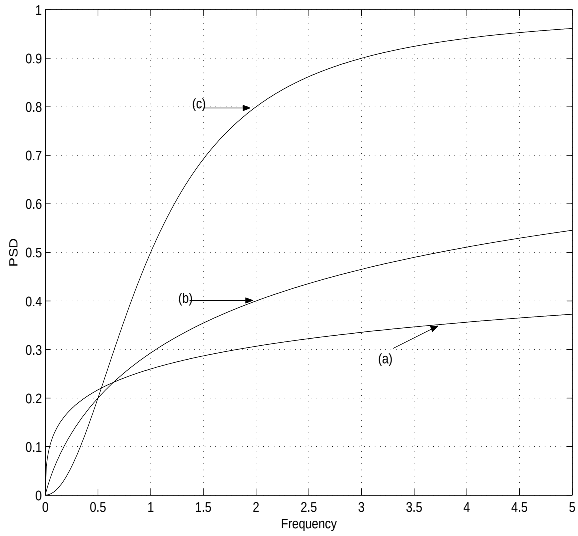


***Fig.5:*** *Variation of PSD of the output of SSHPF with frequency for different orders (a)* $\alpha = 0.25$*, (b)* $\alpha = 0.5$*, (c)* $\alpha = 1.0$

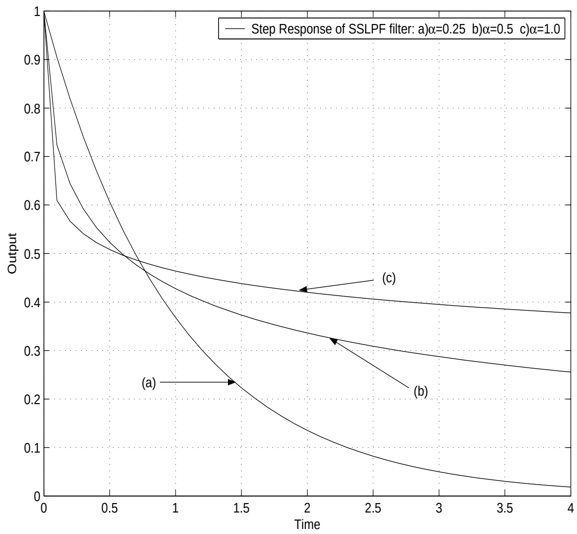


***Fig.4:*** *Time response of fractional-order SSHPF for (a)* $\alpha = 1.0$*, (b)* $\alpha = 0.50$*, (c)* $\alpha = 0.25$

the value of RC has been taken to be unity and the value of $\sigma$ is also taken to be unity. The step response of the fractional-order SSLPF for various values of capacitor order $\alpha$ is shown in Fig. 2. The variation of PSD with the order of the capacitor ($\alpha$) is shown in Fig. 3.

Following (53) and (54) (Appendix B), we find that SSLPF with the capacitor order $\alpha > 1$ has a faster step response than the conventional first-order low-pass filter and also, exhibits an overshooting behavior as is evident in Fig. 2; but from Fig. 3 it is clear that the PSD of the output noise in SSLPF with $\alpha > 1$ is more than the conventional circuit with $\alpha = 1$; and hence the fractional circuits with $\alpha > 1$ will have a higher average power of output noise. Again following (53), (54) (Appendix B) we find that SSLPF with the capacitor order less than unity has a sluggish response (Fig. 2), but as is evident from Fig 3 that for such circuits ($\alpha < 1$) the PSD of output noise is less than conventional circuit. Therefore, the order of the capacitor should be chosen appropriately for the desired circuit performance (compromise between the speed of response and noise performance of the circuit). For convenience we provide the closed form solutions of the step response of the SSLPF for different values of $\alpha$. They have been found using (53) and (54) (Appendix B).

(a) $\alpha = 0.25$

$$v_0(t) = \pounds^{-1}\Big(\frac{1}{s(s^{.25}+1)}\Big) = E_t(.25, 1) - E_t(.5, 1) + E_t(.75, 1) - E_t(1, 1) \tag{23}$$

(b) $\alpha = 0.5$

$$v_0(t) = \pounds^{-1}\Big(\frac{1}{s(s^{.5}+1)}\Big) = E_t(.5, 1) - E_t(1, 1) \tag{24}$$

(c) $\alpha = 1$

$$v_0(t) = 1 - e^{-t} \tag{25}$$

(d) $\alpha = 1.5$

$$v_0(t) = \pounds^{-1}\Big(\frac{1}{s(s^{1.5}+1)}\Big) = \frac{1}{3}(-E_t(.5, 1) + E_t(1, 1) + w_1E_t(.5, w_1^2) + w_1^2E_t(1, w_1^2) + w_2E_t(.5, w_2) + w_2^2E_t(1, w_2^2)) \tag{26}$$

where $w_1 = 0.5 + 0.866i$ and $w_2 = 0.5 - 0.8660i$ which along with -1, are the three roots of -1.

The method presented here, is general and could be extended to more complex and practical fractional circuits. Since, in the recent past many active implementations of the fractional capacitor [3-4] have come up, the method could be used by appropriately considering the intrinsic noise in all the active or passive elements and modelling it by a single equivalent noise source as in this paper.

### 3.1 Single stage high-pass and all-pass filtering functions

The transfer function of single stage high-pass filter (SSHPF) and single stage all-pass filter (SSAPF) filtering functions are given as

$$HP: \quad H(s) = \frac{s^\alpha RC}{1 + (s)^\alpha RC} \tag{27}$$

and

$$AP: \quad H(s) = \frac{s^\alpha RC - 1}{1 + s^\alpha RC} \tag{28}$$

Moreover, it is also known that the HP and AP filter outputs in time-domain are related to the LP filter output by $v_{HP} = 1 - v_{LP}$ and $v_{AP} = 1 - 2v_{LP}$. For the SSAPF, we focus on the phase-response rather than the step response (which is constant). The phase response of the SSAPF system is given as

$$\angle H_{AP}(j\omega) = \pi - 2tan^{-1}\Big(\frac{\omega^\alpha \sin(\pi\alpha/2)}{\omega^\alpha \cos(\pi\alpha/2) + \frac{1}{RC}}\Big) \tag{29}$$

The PSD of the output for the fractional-order SSHPF is found to be

$$S_{yy}(\omega) = \frac{\sigma^2\omega^{2\alpha}R^2C^2}{\omega^{2\alpha}R^2C^2 + 2RC\omega^\alpha cos(\pi\alpha/2) + 1} \tag{30}$$

The step response of the fractional-order SSHPF for various values of capacitor order $\alpha$ is shown in Fig. 4 and the variation of PSD with the order of the capacitor ($\alpha$) is shown in Fig. 5. It is evident from Fig. 4 and 5 that, the step response becomes more sluggish as the order of the capacitor decreases and the PSD of the output noise is lower for smaller values of order of capacitors.

## 4. CONCLUSIONS

Noise in a single stage fractional-order RC low-pass filter, is analyzed using stochastic differential equations. Closed-form solution for various solution statistics like mean, auto-correlation, power spectral density is obtained and the effect of order $\alpha$ on average power of output noise is investigated. It has been shown that although by allowing the capacitor to acquire an order greater than unity one can achieve the advantages of fast response but such a circuit suffers from a higher output noise power and if reduced noise characteristics is required the order of capacitor may be taken to be less than unity. Therefore the results obtained can be used for design process wherein an appropriate fractional capacitor can be chosen for any desired circuit behavior. The stability aspects of the aforementioned circuits have not been discussed and the authors plan to address these issues in the near future.

## 5. APPENDIX A-DERIVATION OF $E[X_T^2]$ IN THE SDE

In this Appendix, we will derive (12). The necessary background is introduced to make this section as self-contained as possible.

Thermal noise is often modelled as white noise (denoted as n(t)), whose PSD is flat across all frequencies upto infinity. However, white noise is not a physical process because it has infinite power. Therefore to treat noise rigorously, we need to definite its integral, called Wiener process, which can be approximated by physical processes

$$W(t) = \int_0^t n(s)ds \tag{31}$$

A Wiener process has a continuous sample path and independent Gaussian increments. However, sample paths of a Wiener process have unbounded variation (or infinite length), so it is difficult to find the solution of a Wiener process. Ito's stochastic calculus is invented precisely to solve this problem [11-15].

For the general linear case of equation

$$d\mathbf{X}(t)=[\mathbf{f}(t)+\mathbf{F}(t)\mathbf{X}(t)]dt+\sum_{i=1}^{m}[\mathbf{g}_i(t)+\mathbf{G}_i(t)\mathbf{X}(t)]dW_i(t) \tag{32}$$

where $\mathbf{X}(t)$ represents a random n vector process, $W(t) = [W_1(t), ...., W_m(t)]^T$ is an m-vector standard Wiener process, $\mathbf{F}(t)$ and the $\mathbf{G_i(t)}$ are $n \times n$ matrix functions, and $\mathbf{f(t)}$ and the $\mathbf{g_i(t)}$ are n vector functions, respectively.

Let $\phi(t)$ be the fundamental matrix of the corresponding to (32) homogenous equation

$$d\phi(t) = F(t)\phi(t)dt + \sum_{i=1}^{m} G_i(t)\phi(t)dW_i(t) \tag{33}$$

that is $\phi(t)$ is the $n \times n$ matrix solution of (33) which satisfies $\phi(0) = I$. The solution of (32) can be written as

$$\mathbf{X}(t) = \Phi(t)\{\mathbf{X}(0)+\int_0^t \phi^{-1}(s)[\mathbf{f}(s)-\sum_{i=1}^{m}\mathbf{G}_i(s)\mathbf{g}_i(s)]ds+$$

$$\int_0^t \Phi^{-1}(s)\sum_{i=1}^m \mathbf{g}_i(s)dW_i(s)\} \tag{34}$$

In the narrow-sense linear case ($\mathbf{G}_i(t) = 0$ for all $i$ ; which is exactly the case dealt in this paper); (34) has the simplified form

$$\mathbf{X}(t) = \Phi(t)\{\mathbf{X}(0) + \int_0^t \Phi^{-1}(s)\mathbf{f}(s)ds +$$

$$\int_0^t \Phi^{-1}(s)\sum_{i=1}^m \mathbf{g}_i(s)dW_i(s)\} \tag{35}$$

and the fundamental matrix $\phi(t)$ corresponds to the solution of the deterministic initial value problem

$$\frac{d\Phi(t)}{dt} = \mathbf{F}(t)$$

and $\phi(0) = I$ ($n \times n$ identity matrix), that is, $\phi(t)$ is the fundamental matrix for the deterministic part. Taking the first moment $\mathbf{m}(t) = E[\mathbf{X}(t)]$ and the second moment $\mathbf{M}(t) = E[\mathbf{X}(t)\mathbf{X}^T(t)]$. Towards solving the second moment matrix M(t); recall that Ito's formula leads to the "Product Rule" for stochastic differentials, which, in this case takes the form

$$d(\mathbf{X}(t)\mathbf{X}^T(t)) = \mathbf{X}(t)d\mathbf{X}^T(t) + d\mathbf{X}(t)\mathbf{X}^T(t) +$$

$$\sum_{i=1}^m \Big(\mathbf{G}_i(t)\mathbf{X}(t) + \mathbf{g}_i(t)\Big)\Big(\mathbf{X}^T(t)\mathbf{G}_i^T(t) + \mathbf{g}_i^T(t)\Big)dt \tag{36}$$

Substituting for $d\mathbf{X}(t)$ and $d\mathbf{X}^T(t)$ in (36), writing in integral form, and taking expected value leads to the integral equation equivalent to the initial value problem

$$\frac{d\mathbf{M}(t)}{dt} = \mathbf{F}(t)\mathbf{M}(t) + \mathbf{M}(t)\mathbf{F}^T(t) +$$

$$\sum_{i=1}^m \mathbf{G}_i(t)\mathbf{M}(t)\mathbf{G}_i^T(t) + \mathbf{f}(t)\mathbf{m}^T(t) + \mathbf{m}(t)\mathbf{f}^T(t) +$$

$$\sum_{i=1}^m \Big(\mathbf{G}_i(t)\mathbf{m}(t)\mathbf{g}_i^T(t) + \mathbf{g}_i(t)\mathbf{m}^T(t)\mathbf{G}_i^T(t) + \mathbf{g}_i(t)\mathbf{g}_i^T(t)\Big) \tag{37}$$

and $\mathbf{M}(0) = E[\mathbf{X}(0)\mathbf{X}^T(0)]$.

## 6. APPENDIX B-FRACTIONAL CALCULUS

In this section we give all the necessary background about fractional calculus and the details about transcendental functions used in the paper. We also provide details about finding the inverse laplace transform of the functions of form: $\frac{1}{s^w - a}$ where $'w'$ is not an integer.

For $Re(v) > 0$ and f be continuous on $J' = (0, \infty)$ and integrable on any finite sub-interval of $J = [0, \infty]$ then for $t > 0$ we call

$$_0D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-\zeta)^{v-1} f(\zeta)d\zeta \tag{38}$$

the Riemman-Louivelle fractional integral of f of order v [10]. Since we mainly shall be considering integrals of the form (38), the notation will be simplified by dropping the subscripts 0 and t on $_0D_t^{-v}f(t)$. Considering $f(t) = e^{at}$ we get

$$D^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-\zeta)^{v-1} e^{a\zeta}d\zeta \tag{39}$$

If we make the change of variables $x = t - \zeta$ (39) becomes

$$D^{-v} e^{at} = \frac{e^{at}}{\Gamma(v)} \int_0^t x^{v-1} e^{-ax}dx \tag{40}$$

Clearly (40) is not an elementary function. But it is closely related to the transcendental function known as the incomplete gamma function [10]. For $Re(v) > 0$ the incomplete gamma function $\gamma^*(v, t)$ is defined as

$$\gamma^*(v, t) = \frac{1}{\Gamma(v)t^v} \int_0^t (\zeta)^{v-1} e^{-\zeta}d\zeta \tag{41}$$

Thus we can write (41) as

$$D^{-v} e^{at} = t^v e^{at} \gamma^*(v, t) \tag{42}$$

Since the right hand side of (42) is the fractional integral of an exponential, it is not surprising that this function arises frequently in the study of fractional calculus. We shall call $D^{-v}e^{at} = E_t(v, a)$ [10].

For numerical simulation $\gamma^*(v, t)$ may be defined as

$$\gamma^*(v, t) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+v+1)} \tag{43}$$

It is sometimes convenient to consider the incomplete gamma function in the form $\gamma^*(v, at)$ where a is an arbitrary constant. Substituting $at$ as $t$ in (43) we get

$$\gamma^*(v, at) = e^{-at} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+v+1)} \tag{44}$$

Using (44), (42) becomes

$$E_t(v, a) = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+v+1)} \tag{45}$$

We also see immediately from (45) that $E_t(0, a) = e^{at}$ and $E_t(v, 0) = \frac{t^v}{\Gamma(v+1)}$

Since we are interested in finding the inverse laplace transform of $\frac{1}{s^w - a}$ where 'w' is not an integer, let us introduce Laplace Transform applied to fractional integrals. One of the most useful properties of the Laplace Transform is embodied in convolution theorem. Thus if F(s) and G(s) are the Laplace Transforms of f(t) and g(t), respectively, then

$$\pounds(\int_0^t f(t-\zeta)g(\zeta)d\zeta) = F(s)G(s) \tag{46}$$

Looking carefully the left hand side of (46) is the Laplace Transform of Riemman-Louivelle fractional integral of f of order v.

$$\pounds(D^{-v}f(t)) = \frac{1}{\Gamma(v)}\pounds(t^{v-1})\pounds(f(t)) = s^{-v}F(s) \tag{47}$$

Taking $f(t) = e^{at}$ as before, (47) becomes

$$\pounds(E_t(v,a)) = \frac{1}{s^v(s-a)} \tag{48}$$

With all the above information about fractional calculus we are now in the position to find out the inverse laplace transform of $\frac{1}{s^w - a}$ where 'w' is not an integer.

Let p and q be relatively prime positive numbers and let

$$r = \frac{p}{q}, v = \frac{1}{q}$$

Let $a \neq 0$ be a real or complex number, and let $\beta_k, k = 1, 2, 3...., p$ be the p, pth roots of of a. The partial fraction of $(x^p - a)^{-1}$ is

$$\frac{1}{x^p - a} = \frac{1}{ap}\sum_{k=1}^{p}\frac{\beta_k}{x-\beta_k} \tag{49}$$

and $(x^q - \beta^q)$ can be factored as

$$x^q - \beta^q = (x-\beta)\sum_{j=1}^{q}\beta^{j-1}x^{q-j} \tag{50}$$

Now substitute $(x-\beta)$ from (50) (with $\beta$ replaced by $\beta_k$) in (49) and let $x = s^v$ to obtain:-

$$\frac{1}{s^r - a} = \frac{1}{ap}\sum_{k=1}^{p}\sum_{j=1}^{q}\frac{\alpha_k^j}{s^{jv-1}(s-\alpha_k^j)} \tag{51}$$

A very useful version of (51) occurs when p=1. That is, when $r = v = \frac{1}{q}$, then (51) becomes:-

$$\frac{1}{s^r - a} = \sum_{j=1}^{q}\frac{a^{j-1}}{s^{jv-1}(s-a^q)} \tag{52}$$

Now using (48) the inverse laplace transform of (52) can be computed by the relation given by:

$$\pounds^{-1}\Big(\frac{1}{s^v - a}\Big) = \sum_{j=1}^{q}a^{j-1}E_t(jv-1, a^q) \tag{53}$$

Also for $Re(u+v) > 0$

$$\pounds^{-1}\Big(\frac{1}{s^u(s^v - a)}\Big) = \sum_{j=1}^{q}a^{j-1}E_t(jv-1+u, a^q) \tag{54}$$

For a general r such that $r = \frac{p}{q}$ , following (51) and (55) the general formula for the output voltage would be :-

$$v_0(t) = \frac{1}{ap}\sum_{k=1}^{p}\sum_{j=1}^{q}(\beta_k)^j E_t(jv, (\beta_k)^q) \tag{55}$$

Having introduced the Laplace transform of Fractional integral above, we state the expressions for Laplace Transform of the fractional integral, and which are

For $0 < v \leq 1$

$$\pounds(D^v f(t)) = s^v F(s) - D^{-(1-v)}f(0) \tag{56}$$

and for $1 < v \leq 2$

$$\pounds(D^v f(t)) = s^v F(s) - sD^{-(2-v)}f(0) - D^{-(1-v)}f(0), \tag{57}$$

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