Achievement of low-sensitivity characteristics and robust stability condition for multi-variable systems having an uncertain number of right half plane poles

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ABSTRACT

In the present paper, we consider a design method that provides low-sensitivity control with robust stability for multiple-input/multiple-output continuous time-invariant systems having an uncertain number of right half plane poles. First, the class of uncertainty considered in the present paper is defined and the necessary and sufficient robust stability condition is presented for the system having this class of uncertainty under the assumption that the number of closed right half plane poles of the plant is equal to that of the nominal plant. The relationship between the plant and the nominal plant included in this class of uncertainty is clarified. Using this relationship, we will show the necessary and sufficient robust stability condition for the system having an uncertain number of right half plane poles.

Keywords: Robust Stability, The Relative Degree, Multiplicative Uncertainty, Uncertain Number of Right Half Plane Poles

1. INTRODUCTION

In the present paper, we examine a design method that provides low-sensitivity control systems with robust stability for multiple-input/multiple-output continuous time-invariant systems having an uncertain number of right half plane poles. Several studies have been conducted on the robust stabilization problem [1–7]. Doyle and Stein derived the basic solution for this problem [1, 2], and the necessary and sufficient conditions for the multiplicative uncertainty and additive uncertainty were shown. Chen and Desoer derived the complete proof of the solution presented by Doyle and Stein [3]. Kishore and Pearson clarified if a class of uncertainty is closed set then the gap between the necessary robust stability condition and the sufficient one exist[4]. In addition, if a class of uncertainty is open set then the necessary robust stability condition is equal to that of sufficient one [4]. In this way, the robust stability condition for the system with invariant number of right half plane poles completely clarified.

Kimura considered the robust stabilizability problem for single-input/single-output systems [8]. Vidyasagar and Kimura expanded the findings of Kimura for multiple-input/multiple-output systems [9].

According to various reports [1–3], in order to maintain stability for a large uncertainty, the complementary sensitivity function must be small. Ensuring that the complementary sensitivity function is small reduces the performance of the control systems by disturbance attenuation, etc., because the value of the sensitivity function increases. Since sum of the sensitivity function and the complementary sensitivity function is equal to 1, obtaining either low-sensitivity or high robust stability characteristics is impossible. However, controllers that achieve low-sensitivity characteristics do not always make the system unstable. Maeda et al. treated this problem as an infinite gain margin problem [10, 11]. Nogami et al. clarified the condition in which the hi-gain controller does not make the system unstable and proposed a design method [12] by reducing the robust passivity problem. Doyle et al. considered this low-sensitivity control problem from the another viewpoint: there exists a class of uncertainty that has low-sensitivity that makes the system robustly stable [13]. Therefore, from the uncertainty described by Doyle, Francis and Tannenbaum [13], we can construct low-sensitivity characteristics with robust stability. Thus, the uncertainty presented by Doyle, Francis and Tannenbaum is suitable for hi-performance robust control system design. However, this uncertainty cannot be applied to a system having an uncertain number of closed right half plane poles. There exist applications such that the number of right half plane poles changes. For example, the number of right half plane poles of a large flexible spacecraft changes when the configuration of the spacecraft is changed [9]. The problem of obtaining the robust stability condition for the system having an uncertain number of the

Manuscript received on July 6, 2009 ; revised on , .
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closed right half plane poles is difficult because the problem does not reduce to the small gain theorem.

In the present paper, we expand the uncertainty presented by Doyle, Francis and Tannenbaum [13, 14] to be applicable to the multiple-input/multiple-output system having an uncertain number of open right half plane poles. First, the class of uncertainty to be considered in the present paper is defined. If this class is assumed that the number of right half plane poles of the nominal plant is equal to that of the plant, this class of uncertainty is the same to that defined by Doyle, Francis and Tannenbaum [13]. The necessary and sufficient robust stability condition for the system of this class of uncertainty is presented under the assumption that the number of right half plane poles of the nominal plant is equal to that of the plant. Next, the condition under which the set of the plant is included in the above-mentioned class is clarified. Using this relationship between the nominal plant and the plant, the necessary and sufficient robust stability condition is obtained when the number of poles of the plant is not necessarily to be equal to that of the nominal plant. The robust stability condition described in the present paper is identical, whether or not the number of poles of the nominal plant is equal to that of the plant. Generally, in previous studies, the number of poles of the plant is assumed to be equal to that of the nominal plant [1–8]. Verma, Helton and Jonckheere considered the robust stabilizability problem [5], but they did not consider the class of uncertainty that is considered in the present paper. The robust stability condition discussed in the present paper is obtained using a kind of phase information, that is, the relative degree of the plant and the nominal plant. Therefore, the robust stability conditions considered in the present paper for the system having the same number of right half plane poles is identical to that having an uncertain number of right half plane poles. Conversely, the robust stability condition used by Verma, Helton and Jonckheere for the system having the same number of right half plane poles is different from that having an uncertain number of right half plane poles. Therefore, the findings presented by Verma, Helton and Jonckheere and those of the present paper are different.

Notation

R	the set of real numbers
R_{+e}	the set of real numbers with infinite
$\mathcal{C}$	the set of complex numbers
$R(s)$	the set of all real-rational transfer functions
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	represents the state space description of $C(sI-A)^{-1}B+D$
$\\|\cdot\\|_\infty$	H_∞ norm
$F_u(P,Q)$	upper LFT, that is $F_u(P,Q) = P_{22}+P_{21}Q(I-P_{11}Q)^{-1}P_{12}$, where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$
$F_l(P,Q)$	lower LFT, that is $F_l(P,Q) = P_{11} + P_{12}Q(I-P_{22}Q)^{-1}P_{21}$
$\Re\{\cdot\}$	real part of $\{\cdot\} \in \mathcal{C}$
$\bar{\sigma}\{\cdot\}$	the maximum singular value of $\{\cdot\}$

2. PROBLEM FORMULATION

Consider the control system as below.

$$\begin{cases} y(s) &= G(s)u(s) \\ u(s) &= C(s)\left(r(s)-y(s)\right) \end{cases} \tag{1}$$

Here, $G(s) \in R^{p\times m}(s)$ is the strictly proper multiple-input/multiple-output plant. $C(s) \in R^{m\times p}(s)$ is the controller, $r(s) \in R^p$ is the reference input, $y(s) \in R^m$ is the output and $u(s) \in R^m$ is the control input. The plant $G(s)$ is assumed to be stabilizable, detectable and $p \leq m$. The state space description of the plant $G(s)$ is denoted by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in R^{p\times m}(s).$$

The nominal plant of the plant $G(s)$ denotes

$$G_m(s) = \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & 0 \end{array}\right] \in R^{p\times m}(s).$$

Here, the number of zeroes of the nominal plant $G_m(s)$ in the closed right half plane is assumed to equal that of the plant $G(s)$. That is, the number of $s_0 \in \mathcal{C}$ in the closed right half plane satisfying

$$\text{rank} \begin{bmatrix} A-s_0I & B \\ C & 0 \end{bmatrix} < n+p \tag{2}$$

is equal to that of $\bar{s}_0 \in \mathcal{C}$ in the closed right half plane satisfying

$$\text{rank} \begin{bmatrix} A_m-\bar{s}_0I & B_m \\ C_m & 0 \end{bmatrix} < n+p \tag{3}$$

with counting multiplicity. (C_m, A_m, B_m) is assumed to be stabilizable and detectable. Let the plant $G(s)$ be denoted using the nominal plant $G_m(s)$ and the multiplicative uncertainty by

$$G(s) = \left(I+\Delta(s)\right)G_m(s). \tag{4}$$

Without loss of generality, $\Delta(s)$ is assumed to be stabilizable and detectable and $I+\Delta(s)$ is of normal full rank.

The sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$ of the control system (1) are denoted by

$$S(s) = \left(I+G_m(s)C(s)\right)^{-1} \tag{5}$$

and

$$\begin{aligned} T(s) &= I - S(s) \\ &= G_m(s)C(s)\left(I + G_m(s)C(s)\right)^{-1}, \quad (6) \end{aligned}$$

respectively.

In the present paper, we consider the robust stability problem for the following class of plants.

Definition 1: The set of plants is denoted by Ω. The elementary of the set Ω satisfies following expressions.

- The number of zeroes of the plant $G(s)$ in the closed right half plane is equal to that of the nominal plant $G_m(s)$.
- The number of right half plane poles of the plant $G(s)$ is not necessarily equal to that of the nominal plant $G_m(s)$.
-

$$\begin{aligned} &\bar{\sigma}\left\{\left(I+\Delta(j\omega)\right)^{-1}\Delta(j\omega)\right\} \\ &\quad < \ |W(j\omega)| \ \ (\omega \in R_{+e}), \end{aligned} \quad (7)$$

where $W(s) \in R(s)$ is a stable rational function.

If $\Delta(s)$ holds above expressions, we denote simply $\Delta(s) \in \Omega$.

As described in a later section, by adopting the set of uncertainty defined in Definition 1, the robust stability condition is directly related not to the complementary sensitivity function, but to the sensitivity function. In other words, a low-sensitivity controller can guarantee robust stability.

Before considering the robust stability condition for the class of the plant set Ω, the robust stability condition for the class of the plants defined in Definition 2. Here, the number of right half plane poles of the plant $G(s)$ is assumed to be equal to that of the nominal plant $G_m(s)$.

Definition 2: $\bar{\Omega}$ is the set of plants. The elementary of the set $\bar{\Omega}$ satisfies following expressions.

- The number of zeroes of the plant $G(s)$ in the closed right half plane is equal to that of the nominal plant $G_m(s)$.
- The plant $G(s)$ has the same number of right half plane poles as that of the nominal plant $G_m(s)$.
-

$$\begin{aligned} &\bar{\sigma}\left\{\left(I+\Delta(j\omega)\right)^{-1}\Delta(j\omega)\right\} \\ &\quad < \ |W(j\omega)| \ \ (\omega \in R_{+e}), \end{aligned} \quad (8)$$

where $W(s) \in R(s)$ is a stable rational function.

The robust stability condition for the set of plants $\bar{\Omega}$ is summarized in the following theorem.

Theorem 1: Assume that $C(s)$ stabilizes the nominal plant $G_m(s)$. $C(s)$ is a robust stabilizing controller for $\bar{\Omega}$ if and only if

$$\|S(s)W(s)\|_\infty \le 1 \quad (9)$$

holds.

Proof: Let $P(s)$ and $\bar{\Delta}(s)$ be

$$P(s) = \left[\begin{array}{c|c} W(s)I & W(s)G_m(s) \\ \hline I & G_m(s) \end{array}\right] \quad (10)$$

and

$$\bar{\Delta}(s) = \left(I+\Delta(s)\right)^{-1}\frac{\Delta(s)}{W(s)}, \quad (11)$$

respectively. Proof is immediately obtained by applying Theorem 3.3 in the book written by Mcfarlane and Glover [15] to obtain

$$F_u(P(s),\bar{\Delta}(s)) = \left(I+\Delta(s)\right)G_m(s) \quad (12)$$

$$\begin{aligned} F_l(P(s),-C(s)) &= \left(I+G_m(s)C(s)\right)^{-1}W(s) \\ &= S(s)W(s), \end{aligned} \quad (13)$$

thereby completing the proof of Theorem 1. ■

Theorem 1 shows that if the plant $G(s)$ can be placed in the form of Definition 2, low-sensitivity can be achieved for the robust stability condition.

3. RELATIONSHIP BETWEEN THE NOMINAL PLANT AND THE PLANT

In this section, the relationship between the nominal plant $G_m(s)$ and the plant $G(s)$ that satisfies Theorem 1 is described.

To maintain the internal stability condition, the system (1) must be well-posed. Therefore, the controller must be proper. Since (5) and the nominal plant $G_m(s)$ is assumed to be the strictly proper, when the controller $C(s)$ is proper, the sensitivity function has the property:

$$\lim_{\omega\to\infty}\left(S(j\omega)\right) = I. \quad (14)$$

By $\bar{\sigma}\{S(j\omega)\}$ is performed multiplication on both sides of (8), we have

$$\begin{aligned} &\bar{\sigma}\left\{S(j\omega)\left(I+\Delta(j\omega)\right)^{-1}\Delta(j\omega)\right\} \\ &\quad \le \ \bar{\sigma}\left\{S(j\omega)\right\}\bar{\sigma}\left\{\left(I+\Delta(j\omega)\right)^{-1}\Delta(j\omega)\right\} \\ &\quad < \ \bar{\sigma}\left\{S(j\omega)\right\}|W(j\omega)| \\ &\quad = \ \bar{\sigma}\left\{S(j\omega)W(j\omega)\right\}. \end{aligned} \quad (15)$$

In order to satisfy (9), from (14) and (15),

$$\begin{aligned} &\lim_{\omega\to\infty}\bar{\sigma}\left\{S(j\omega)\left(I+\Delta(j\omega)\right)^{-1}\Delta(j\omega)\right\} \\ &\quad < \ \lim_{\omega\to\infty}\bar{\sigma}\left\{S(j\omega)W(j\omega)\right\} \le 1 \end{aligned} \quad (16)$$

is required. Hence

$$\begin{aligned} &\lim_{\omega\to\infty}\bar{\sigma}\left\{\left(I+\Delta(j\omega)\right)^{-1}\Delta(j\omega)\right\} \\ &\quad < \ \lim_{\omega\to\infty}|W(j\omega)| \le 1 \end{aligned} \quad (17)$$

is required.

From (17), we obtain the following theorem.

Theorem 2: Necessary condition that there exist controllers to satisfy Theorem 1, that is necessary condition that $\Delta(s)$ satisfies (17), is that $I+\Delta(s)$ is biproper. That is, when $I+\Delta(s)$ is denoted by

$$I+\Delta(s)=\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right], \tag{18}$$

the necessary condition that there exist controllers to satisfy Theorem 1 is

$$\text{rank } D_d = p. \tag{19}$$

Proof: Proof is obtained by showing that if the nominal plant $G_m(s)$ is not biproper, that is, (19) is not satisfied, then (17) is not satisfied. For simplicity, let $\bar{\Delta}(s) = I+\Delta(s)$, then

$$\begin{aligned}(I+\Delta(s))^{-1}\Delta(s) &= \bar{\Delta}^{-1}(s)\left(\bar{\Delta}(s)-I\right) \\ &= I-\bar{\Delta}^{-1}(s). \end{aligned} \tag{20}$$

If $I+\Delta(s)$ is not biproper but proper, then $\bar{\Delta}^{-1}(s)$ is not proper. This implies that $I-\bar{\Delta}^{-1}(s)$ is also improper. We have

$$\lim_{\omega\to\infty}\bar{\sigma}\left\{(I+\Delta(j\omega))^{-1}\Delta(j\omega)\right\}=\infty. \tag{21}$$

Therefore (17) is not satisfied.

Conversely, if $I+\Delta(s)$ is improper, then $\bar{\Delta}^{-1}(s)$ is not biproper but proper. This implies that $I-\bar{\Delta}^{-1}(s)$ is proper,

$$\text{rank }\left\{\lim_{\omega\to\infty}\left(\bar{\Delta}^{-1}(j\omega)\right)\right\}<p$$

and at least one of the eigen value of

$$\lim_{\omega\to\infty}\left\{I-\bar{\Delta}^{-1}(j\omega)\right\}$$

is equal to 1. We have

$$\lim_{\omega\to\infty}\bar{\sigma}\left\{(I+\Delta(j\omega))^{-1}\Delta(j\omega)\right\}\geq 1. \tag{22}$$

This does not satisfy (17), thereby completing the proof of Theorem 2. ■

When $I+\Delta(s)$ is biproper, following theorem is satisfied.

Theorem 3: If $I+\Delta(s)$ is biproper, then following expressions hold:

- the number of right half plane zeroes of $I+\Delta(s)$ is sum of those of the plant $G(s)$ and right half plane poles of the nominal plant $G_m(s)$.
- the number of right half plane poles of $I+\Delta(s)$ is sum of those of the plant $G(s)$ and the number of right half plane zeroes of the nominal plant $G_m(s)$.

Proof of this theorem requires following lemmas.

Lemma 1: Let $\bar{G}(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A\in R^{n\times n}$. If

$$\text{rank }\ \bar{G}(s)=p, \tag{23}$$

then

$$\text{rank }\left[\begin{array}{cc} A-sI & B \\ C & D \end{array}\right]=n+p. \tag{24}$$

The matrix $\left[\begin{array}{cc} A-sI & B \\ C & D \end{array}\right]$ is called the system matrix of $\bar{G}(s)$.

Lemma 2: The zeroes of the system consists of the following four elements:

1. all transmission zeroes of the system
2. all uncontrollable and unobservable poles of the system
3. one or all uncontrollable and observable poles of the system
4. one or all controllable and unobservable poles of the system

Theorem 3 is proven using above lemmas.

Proof: Proof is to show

1. the number of right half plane zeroes of $I+\Delta(s)$ is equal to sum of those of the plant $G(s)$ and the number of right half plane poles of the nominal plant $G_m(s)$
2. the number of right half plane poles of $I+\Delta(s)$ is equal to that of the plant $G(s)$ and the number of right half plane zeroes of the nominal plant $G_m(s)$.

For this, it is sufficient only to show

1. right half plane zeroes of $I+\Delta(s)$ are consisted of those of the plant $G(s)$ and right half plane poles of the nominal plant $G_m(s)$. That is, for easy explanation, when $I+\Delta(s)$ is assumed to have no poles in the closed right half plane and some zeroes in the closed right half plane, the zero of $I+\Delta(s)$ in the closed right half plane is either a zero of the plant $G(s)$ or a pole of the nominal plant $G_m(s)$ will be proven.
2. right half plane poles of $I+\Delta(s)$ are consisted of that of the plant $G(s)$ and right half plane zeroes of the nominal plant $G_m(s)$. That is, for easy explanation, when $I+\Delta(s)$ is assumed to have no zeroes in the closed right half plane and some poles in the closed right half plane, the pole of $I+\Delta(s)$ in the closed right half plane is either a pole of the plant $G(s)$ or a zero of the nominal plant $G_m(s)$ will be proven.

At first it will be shown that right half plane zeroes of $I+\Delta(s)$ are that of the plant $G(s)$ or right half plane poles of the nominal plant $G_m(s)$. For easy explanation, $I+\Delta(s)$ is assumed to have no poles in the closed right half plane and only some zeroes in the closed right half plane. From Theorem 2, $I+\Delta(s)$ is denoted as

$$I+\Delta(s)=\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right], \tag{25}$$

where $A_d\in R^{n_d\times n_d}$, $B_d\in R^{n_d\times p}$, $C_d\in R^{p\times n_d}$ and $D_d\in R^{p\times p}$ is nonsingular. Therefore the state space description of $(I+\Delta(s))G_m(s)$ is written by

$$(I+\Delta(s))G_m(s) = \left[\begin{array}{cc|c} A_d & B_dC_m & 0 \\ 0 & A_m & B_m \\ \hline C_d & D_dC_m & 0 \end{array}\right], \quad (26)$$

Let s_0 a right half plane zero of $I+\Delta(s)$. We have

$$\text{rank}\left[\begin{array}{cc} A_d - s_0I & B_d \\ C_d & D_d \end{array}\right] < n_d + p. \quad (27)$$

From above equation, there exists $\left[\begin{array}{cc} \xi_1 & \xi_2 \end{array}\right] \neq 0$ satisfying

$$\left[\begin{array}{cc} \xi_1 & \xi_2 \end{array}\right]\left[\begin{array}{cc} A_d - s_0I & B_d \\ C_d & D_d \end{array}\right] = 0. \quad (28)$$

From (26) and (28), for the system matrix of $(I+\Delta(s))\,G_m(s)$, we have

$$\left[\begin{array}{ccc} \xi_1 & 0 & \xi_2 \end{array}\right]\left[\begin{array}{ccc} A_d - s_0I & B_dC_m & 0 \\ 0 & A_m - s_0I & B_m \\ C_d & D_dC_m & 0 \end{array}\right] = 0. \quad (29)$$

This equation implies s_0 is also a zero of $(I+\Delta(s))\,G_m(s)$. From Lemma 2, s_0 is either a zero of the plant $G(s)$, uncontrollable poles of $(I+\Delta(s))\,G_m(s)$ or unobservable poles of $(I+\Delta(s))\,G_m(s)$. When s_0 is not a right half plane zero of the plant $G(s)$, s_0 is either a pole of $I+\Delta(s)$ or a pole of the nominal plant $G_m(s)$. From the assumption that $I+\Delta(s)$ has no poles in the closed right half plane, s_0 is a pole of the nominal plant $G_m(s)$. Therefore s_0 is either a zero of the plant $G(s)$ or a pole of the nominal plant $G_m(s)$. From above discussion, right half plane zeroes of $I+\Delta(s)$ are equivalent to that of the plant $G(s)$ or right half plane poles of the nominal plant $G_m(s)$ was shown.

Next poles of $I+\Delta(s)$ in the closed right half plane are consisted of poles of the plant $G(s)$ or zeroes of the nominal plant $G_m(s)$ in the closed right half plane will be shown. For easy explanation, $I+\Delta(s)$ is assumed to have no zeroes in the closed right half plane and some poles in the closed right half plane. Since D_d is nonsingular, the state space description of the nominal plant $G_m(s)$ is rewritten by

$$\begin{aligned} G_m(s) &= (I+\Delta(s))^{-1}G(s) \\ &= \left[\begin{array}{c|c} A_d - B_dD_d^{-1}C_d & B_dD_d^{-1} \\ \hline -D_d^{-1}C_d & D_d^{-1} \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_d - B_dD_d^{-1}C_d & B_dD_d^{-1}C & 0 \\ 0 & A - s_0I & B \\ \hline -D_d^{-1}C_d & D_d^{-1}C & 0 \end{array}\right]. \end{aligned} \quad (30)$$

Let s_0 a right half plane pole of $I+\Delta(s)$. From

$$\begin{aligned} &\text{rank}\left[\begin{array}{cc} A_d - B_dD_d^{-1}C_d - s_0I & B_dD_d^{-1} \\ -D_d^{-1}C_d & D_d^{-1} \end{array}\right] \\ &= \text{rank}\left\{\left[\begin{array}{cc} I & B_d \\ 0 & I \end{array}\right]\left[\begin{array}{cc} A_d - s_0I & 0 \\ -D_d^{-1}C_d & D_d^{-1} \end{array}\right]\right\} \\ &= \text{rank}\left[\begin{array}{cc} A_d - s_0I & 0 \\ -D_d^{-1}C_d & D_d^{-1} \end{array}\right] \\ &< n_d + p, \end{aligned} \quad (31)$$

s_0 is also a zero of $(I+\Delta(s))^{-1}$. There exists $\left[\begin{array}{cc} \xi_1 & \xi_2 \end{array}\right] \neq 0$ satisfying

$$\left[\begin{array}{cc} \xi_1 & \xi_2 \end{array}\right]\left[\begin{array}{cc} A_d - B_dD_d^{-1}C_d - s_0I & B_dD_d^{-1} \\ -D_d^{-1}C_d & D_d^{-1} \end{array}\right] = 0. \quad (32)$$

From this equation and (30), for the system matrix of $(I+\Delta(s))^{-1}G(s)$, we have

$$\left[\begin{array}{c} \xi_1^T \\ 0 \\ \xi_2^T \end{array}\right]^T\left[\begin{array}{ccc} A_d - B_dD_d^{-1}C_d - s_0I & B_dD_d^{-1}C & 0 \\ 0 & A - s_0I & B \\ -D_d^{-1}C_d & D_d^{-1}C & 0 \end{array}\right] = 0. \quad (33)$$

This implies that s_0 is also a zero of $(I+\Delta(s))^{-1}G(s)$. Therefore form Lemma 2, s_0 is either a zero of the nominal plant $G_m(s)$, uncontrollable poles of $(I+\Delta(s))^{-1}G(s)$ or unobservable poles of $(I+\Delta(s))^{-1}G(s)$. When s_0 is not a zero of the nominal plant $G_m(s)$, s_0 is either a pole of $(I+\Delta(s))^{-1}$ or a pole of the plant $G(s)$. From the assumption that $I+\Delta(s)$ has no poles in the closed right half plane, s_0 is a pole of the plant $G(s)$. Therefore s_0 is either a pole of the plant $G(s)$ or a zero of the nominal plant $G_m(s)$. From above discussion, right half plane pole of $I+\Delta(s)$ is that of the plant $G(s)$ or right half plane zeroes of the nominal plant $G_m(s)$.

We have thus completed the proof of this theorem.

■

4. ROBUST STABILITY CONDITION FOR THE PLANTS HAVING UNCERTAIN NUMBER OF RIGHT HALF PLANE POLES

In this section, the robust stability problem in which the number of poles of the nominal plant $G_m(s)$ in the right half plane does not equal that of the plant $G(s)$ is considered using the results in the previous section. That is, the robust stability condition for the set of plants Ω is considered.

The robust stability condition for the set of plants Ω, is summarized below.

Theorem 4: Assume that $C(s)$ is a stabilizing controller for the nominal plant $G_m(s)$. $C(s)$ is a robust stabilizing controller for Ω if and only if

$$\|S(s)W(s)\|_\infty \leq 1. \quad (34)$$

The proof of Theorem 4 requires following lemmas.

Lemma 3: Suppose $M \in RH_\infty$ and $\|M\|_\infty > \gamma$. Then there exists a $\omega_0 > 0$ such that for any given $\omega \in [0, \omega_0]$ there exists a $\Delta(s) \in RH_\infty$ with $\|\Delta(s)\| < 1/\gamma$ such that $\det(I - M(s)\Delta(s))$ has a zero on the imaginary axis [16].

Lemma 4: Let $W(s)$ satisfy (17). The nominal plant $G_m(s)$ and the plant $G(s)$ are assumed to have p_m number of poles in the closed right half plane and p number of poles in the closed right half plane, respectively. Then, the Nyquist plot of $\det(I+\Delta(s))$

encircles the origin $(0,0)$ $p-p_m$ times in the counter-clockwise direction.

Proof: From the assumption that $W(s)$ satisfies (17) and Theorem 2, $I+\Delta(s)$ is biproper. From Theorem 3, the number of zeroes of $I+\Delta(s)$ in the closed right half plane is sum of those of the plant $G(s)$ and poles of the nominal plant $G_m(s)$ in the closed right half plane, and the number of poles of $I+\Delta(s)$ in the closed right half plane is sum of those of poles of the plant $G(s)$ in the closed right half plane and the number of zeroes of the nominal plant $G_m(s)$ in the closed right half plane. From the assumption that the number of zeroes of the plant $G(s)$ in the closed right half plane is equal to that of the nominal plant $G_m(s)$ in the closed right half plane, let p_z the number of zeroes of the plant $G(s)$ in the closed right half plane, $I+\Delta(s)$ has p_m+p_z of zeroes in the closed right half plane and $p+p_z$ of poles in the closed right half plane. According to argument principle, the Nyquist plot of $\det(I+\Delta(s))$ encircles the origin $(0,0)$ $p+p_z-(p_m+p_z)=p-p_m$ times in the counter-clockwise direction. ■

Theorem 4 is proven using above lemmas.

Proof: The characteristic matrix of the system (1) is given by $I+G(s)C(s)$. If the Nyquist plot of $\det(I+G(s)C(s))$ encircles the origin $(0,0)$ $p+p_c$ times in the counter-clockwise direction, then the system (1) is robustly stable. Here, p_c means the number of poles of the controller $C(s)$ in the closed right half plane and p means the number of poles of the plant $G(s)$ in the closed right half plane. Determinant of characteristic polynomial is written as

$$\begin{aligned}
&\det\left(I+G(s)C(s)\right)\\
&\quad= \det\left\{I+\left(I+\Delta(s)\right)G_m(s)C(s)\right\}\\
&\quad= \det\left[\left\{I+\Delta(s)G_m(s)C(s)(I+G_m(s)C(s))^{-1}\right\}\right.\\
&\qquad \left.(I+G_m(s)C(s))\right]\\
&\quad= \det\left[(I+\Delta(s))\left\{I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right\}\right.\\
&\qquad \left.(I+G_m(s)C(s))\right]\\
&\quad= \det\left(I+\Delta(s)\right)\det\left\{I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right\}\\
&\qquad \det\left(I+G_m(s)C(s)\right).
\end{aligned} \tag{35}$$

From the assumption that $C(s)$ is a stabilizing controller of the nominal plant $G_m(s)$, the Nyquist plot of $\det(I+G_m(s)C(s))$ encircles the origin $(0,0)$ p_m+p_c times in the counter-clockwise direction, where p_m is the number of poles of the nominal plant $G_m(s)$ in the closed right half plane. Therefore, if the Nyquist plot of

$$\det\left\{(I+\Delta(s))\right\}\det\left\{\left(I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right)\right\}$$

for all $\Delta(s)\in\Omega$ encircles the origin $(0,0)$ $p-p_m$ times in the counter-clockwise direction, then $C(s)$ is a robust stabilizing controller for the set Ω. From Lemma 4, the Nyquist plot of $\det(I+\Delta(s))$ encircles the origin $p-p_m$ times in the counter-clockwise direction. Therefore, the necessary and sufficient condition that $C(s)$ is a robust stabilizing controller for the set Ω is that the Nyquist plot of

$$\det\left\{I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right\}$$

does not encircle the origin.

Finally, the necessary and sufficient condition that the Nyquist plot of

$$\det\left\{I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right\}$$

does not encircle the origin must be proven. The condition is expressed in the same form as (34). It is obvious that if (34) is satisfied, then the Nyquist plot of

$$\det\left\{I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right\}$$

does not encircle the origin any time. From Lemma 3, if (34) does not hold, then there exist $(I+\Delta(s))^{-1}\Delta(s)\in RH_\infty$ with

$$\left\|(I+\Delta(s))^{-1}\Delta(s)/W(s)\right\|_\infty<1$$

to let the Nyquist plot of

$$\det\left\{I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right\}$$

across on the origin. Therefore it is proved that the necessary and sufficient condition that the Nyquist plot of

$$\det\left\{I-(I+\Delta(s))^{-1}\Delta(s)S(s)\right\}$$

does not encircle the origin any times is equivalent to (34).

From the above discussion, Theorem 4 is proven. ■

This theorem is very interesting because the robust stability condition is identical whether or not the number of poles of the nominal plant in the closed right half plane is equal to that of the plant. A similar robust stabilizability problem having an uncertain number of right half plane poles was considered by Verma, Helton, and Jonckheere [5]. Since Verma, Helton, and Jonckheere [5] adopt additive uncertainty

$$G(s)=G_m(s)+\Delta(s) \tag{36}$$

and

$$\|\Delta(s)\|_\infty<1, \tag{37}$$

the robust stability condition is not similar to that shown in Theorem 4. In addition, the robust stability condition for the system having an uncertain number of right half plane poles is not the same as that having an invariant number of right half plane poles.

5. NUMERICAL EXAMPLE

In this section, we show a numerical example to illustrate the effectiveness of the proposed method.

Consider the problem to design a robustly stabilizing controller for the set Ω, where

$$G_m(s) = \begin{bmatrix} \frac{2}{(s+2)(s+1)} & \frac{s+5}{(s+2)(s+1)} \\ \frac{1}{(s+2)(s+1)} & \frac{s+4}{(s+2)(s+1)} \end{bmatrix} = \left[\begin{array}{cccc|cc} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ \hline 2 & 4 & -2 & -3 & 0 & 0 \\ 1 & 3 & -1 & -2 & 0 & 0 \end{array}\right] \quad (38)$$

and

$$W(s) = \frac{0.77s^2 + 27.5s + 187}{s^2 + 48s + 50}. \quad (39)$$

$C(s)$ satisfying (34) is designed using LMI control toolbox.

Next, we confirm that the designed controller $C(s)$ satisfies (34). The maximum singular value plot of $S(s) = (I + G_m(s)C(s))^{-1}$ and the gain plot of $1/W(s)$ are shown in Fig. 1. Here, the solid line shows maximum singular value plot of $S(s) = (I+G_m(s)C(s))^{-1}$ and the dotted line shows the gain plot of $1/W(s)$. Figure 1 shows that the designed controller $C(s)$ satisfies (34).

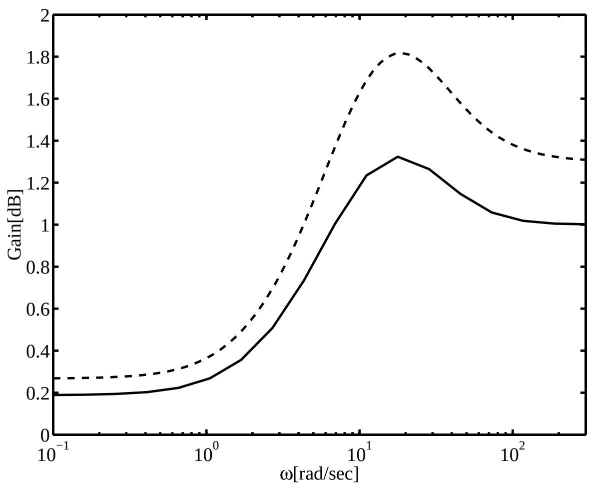

Fig.1: *Maximum singular value plot of $S(s)$ and the gain plot of $1/W(s)$*

Let $\Delta(s)$ be

$$\Delta(s) = \begin{bmatrix} \frac{s+5}{(s+3)(s-1)} & 0 \\ \frac{s+1}{(s+3)(s-1)} & \frac{2}{(s-1)} \end{bmatrix} = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \\ \hline 1.5 & 0 & -0.5 & 0 & 0 & 0 \\ 0.5 & 2 & 0.5 & 0 & 0 & 0 \end{array}\right], \quad (40)$$

that is,

$$G(s) = \begin{bmatrix} \frac{3}{(s+2)(s-1)} & \frac{s+5}{(s+2)(s-1)} \\ \frac{2}{(s+2)(s-1)} & \frac{s+4}{(s+2)(s-1)} \end{bmatrix} = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ \hline 1.0000 & 2.0000 & -1.0000 & -1.0000 & 0 & 0 \\ 0.6667 & 1.6667 & -0.6667 & -0.6667 & 0 & 0 \end{array}\right]. \quad (41)$$

From (38) and (41), the number of poles of the plant $G(s)$ in the closed right half plane is not equal to that of the nominal plant $G_m(s)$. The fact that $\Delta(s)$ in (40) satisfies (7) is confirmed by showing the maximum singular value plot of $(I+\Delta(s))^{-1}\Delta(s)$ and the gain plot of $W(s)$ as Fig. 2. Here, the solid line shows the maximum singular value plot of $(I+\Delta(s))^{-1}\Delta(s)$

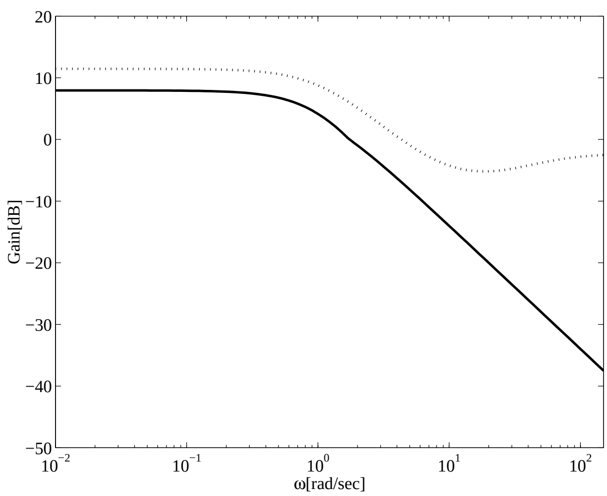

Fig.2: *Gain plot of $W(s)$ and maximum singular value plot of $(I + \Delta(s))^{-1}\Delta(s)$*

and the dotted line shows the gain plot of $W(s)$. Figure 2 shows that $\Delta(s)$ in (40) satisfies (7). From (38) and (41), the number of zeroes in the closed right half plane of the nominal plant $G_m(s)$ is equal to that of the plant $G(s)$. In addition, the relative degree of the nominal plant $G_m(s)$ is equivalent to that of the plant $G(s)$. Therefore, the plant $G(s)$ in (41) is an element of the set Ω.

Using the designed controller $C(s)$, the response of the output

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{42}$$

for the step reference input

$$r(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{43}$$

is shown in Fig. 3 Here, the solid line shows the

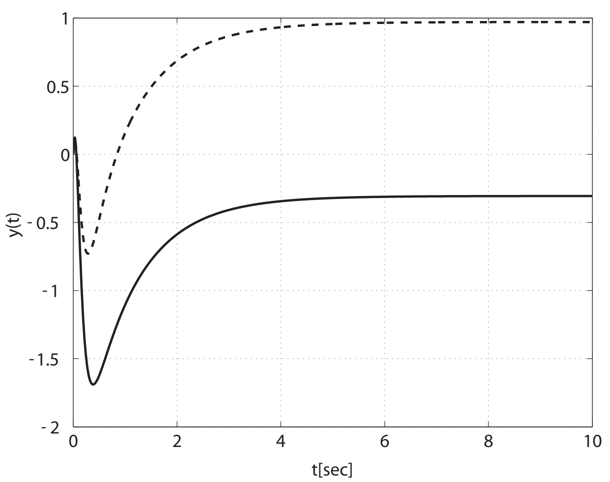

Fig.3: *Response of the output $y(t)$ for the step reference input $r(t)$*

response of $y_1(t)$ and the dotted line shows that of $y_2(t)$. Figure 3 shows that the controller $C(s)$ makes the control system in (1) stable, even if there exists uncertainty with uncertain number of poles in the closed right half plane.

6. CONCLUSION

In the present paper, the robust stability condition for multiple-input/multiple-output continuous time invariant systems having an uncertain number of right half plane poles was considered. The robust stability condition presented in the present paper was identical whether or not the number of poles of the nominal plant is equal to that of the plant in the closed right half plane. Under these conditions, a control system having low-sensitivity characteristics and robust stability can be constructed. When the robust control design method in the present paper applies to real plant, the relative degree of real plant is required. In some cases, exact relative degree of real plant cannot be found. Even if exact relative degree of real plant can not be obtained, using the idea of parallel feed forward compensator proposed by Iwai et al. [17], the method of the present paper can be applied to the real plant.

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