

Modified Pole Re-position Technique for Optimal IIR Multiple Notch Filter Design

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ABSTRACT

This paper presents a novel and useful technique based on pole optimal re-position techniques and Groebner basis theory for solving the gain non-uniformity problem in the frequency band between notch frequencies, normally emerged when cascading several single notch filters to form a multiple notch filter. The previously proposed pole-reposition technique can apply only to the two notch case and does not consider the minimization of the passband gain deviation. The optimality considered in this paper is based on the minimization of resulting filter magnitude response from the ideal notch magnitude response, measured in term of numerical sum of error square. Furthermore, by using Gröbner basis, the symbolic solution and design method can be generalized to the higher number of notch frequencies case. The design algorithm can be implemented directly on Matlab platform and optimal filter coefficients can be found via numerical optimization/search. The proposed method can 1) limit the maximum passband gain 2) maximize the passband gain flatness and 3) generalize to the higher order of multi-notch filter. The searching space of the possible filters are maximized by increasing the number of free-variables, namely the gains at critical frequencies. In addition, the proposed technique allows the inclusion of stability margin of the IIR (infinite impulse response) notch filter as one of optimization objectives, resulting in a robustly stable filter.

Keywords: Digital IIR Multiple Notch Filters, Groebner Basis, Pole Re-position, Stability Margin

1. INTRODUCTION

Typically, notch filters are used for removing, eliminating, or cancelling a single frequency or a narrowband sinusoidal interference. Thus, the multiple notch filters can be used for the removal of multiple narrowband or multiple frequency interferences such as the removal of single-tone interference and its

harmonics and the removal of neighboring cell interferences in the mobile phone cellular system. There also are many applications of notch filters in the field of signal processing such as removing the power line interferences in electrocardiograms (ECG), cutting noise in broadcast TV, extending the spectrum analyzer range, rejecting the interference in ultrawideband (UWB) radio system, howling control in speakerphone systems, eliminating hum in audio systems.

Since FIR (finite impulse response) notch filter of a reasonable filter length cannot be designed to have a narrow-enough stopband, the IIR notch filter of lower order and better frequency response is preferred. Hence, the practical digital notch filters are normally IIR, which can be designed in several ways such as transforming analog notch filter [7], implementing based on all-pass filters [4], [5] and the pole-zero placement [2], [3] which is considered in this paper.

Another classification of notch filter is given as fixed notch filter and adaptive notch filter. The fixed notch filters can be applied in the case that the unwanted frequencies are known, otherwise the adaptive ones will be employed. In this paper, we will focus only on fixed IIR notch filter based on pole-placement technique.

The organization of this paper is as follows. First the background and problem statement are described in Section 2. The proposed design technique and the derivation of the optimization process are explained in Section 3. The design example for the triple notch case ($N = 3$) and the quadruple-notch case ($N = 4$) are then shown in Section 4. followed by the conclusion and suggestion for future research in the last section.

2. BACKGROUND

Digital IIR single notch filter can be obtained by bilinear transformation of the analog single notch filter. The transfer function of the IIR single notch filter is based on that of the all-pass filter, except that the gain at notch frequency is exactly zero. The following subsections will explain each of the design techniques and their revolution.

2.1 Conventional IIR Single Notch Filter Design Technique

The frequency response of an IIR single notch filter can be shown in Eq.(1) and its transfer function $H(z)$

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can be described in Eq.(2) [8], [9].

$$H(e^{j\omega}) = \begin{cases} 0, & \omega = \omega_0 \\ 1, & \text{otherwise,} \end{cases} \quad (1)$$

$$H(z) = b_0 \cdot \left(\frac{1 - 2 \cos \omega_0 z^{-1} + z^{-2}}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}} \right), \quad (2)$$

where ω_0 represents the notch frequency or the pole-zero angle on z-plane, b_0 denotes the constant coefficient of an IIR notch filter, and r is the distance between the pole and the origin which is real constant.

From Eq.(2), let $\omega_0 = 0.2\pi$ rad/sec and $r = 0.6, 0.7, 0.8, 0.9, 0.99$. Then, Fig. 1 shows the angle of poles and zeros at the same positions and the magnitude response is shown in Fig. 2. According to the parameters above, the results show the asymmetric uncontrollable passband gains. The trade-off for larger value of r is that 1) the frequency response is closer to the ideal case and 2) the stability margin (defined as shortest distance of the largest pole to the unit circle) is narrower.

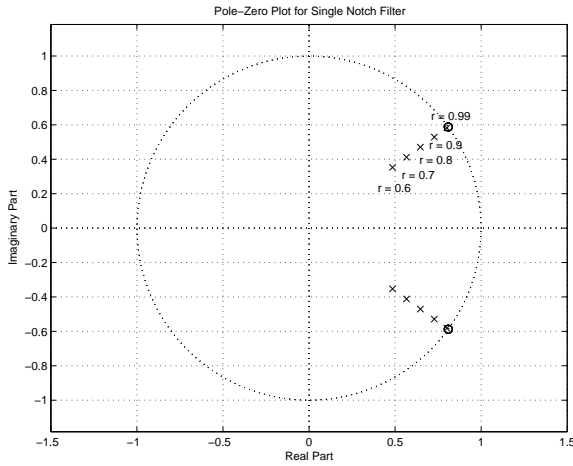


Fig.1: Pole-zero plot of IIR notch filters before changing the positions of poles

2.2 Pole Re-position Technique[2]

Since the pole-zero placement has constraint on the asymmetric uncontrollable passband gains, then the solution for solving this problem is to find the suitable pole positions to make the symmetric gains.

Therefore, the positions of the poles are adjusted to the appropriate positions. To make the passband gains to be at the same level, the DC (direct current) gain and gain at π frequency are paired up. Hence, the transfer function of this technique can be expressed as

$$H_P(z) = b_0 \cdot \left(\frac{1 - 2 \cos \omega_0 z^{-1} + z^{-2}}{1 - 2r \cos \hat{\omega}_0 z^{-1} + r^2 z^{-2}} \right), \quad (3)$$

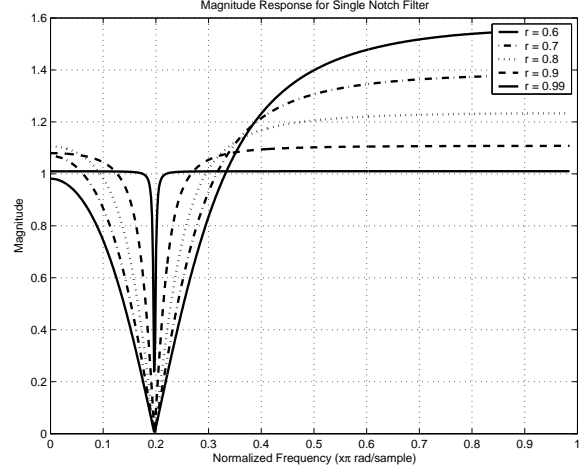


Fig.2: Magnitude responses of the conventional IIR notch filters before applying the pole re-position technique

where $\hat{\omega}_0$ represents the adjusted notch frequency [2] which can be represented as

$$\hat{\omega}_0 = \cos^{-1} \left(\frac{1 + r^2}{2r} \cos \omega_0 \right). \quad (4)$$

The magnitude responses after adjusting the positions of the poles are shown in Fig. 3. While the pole-zero plot of this approach is shown in Fig. 4.

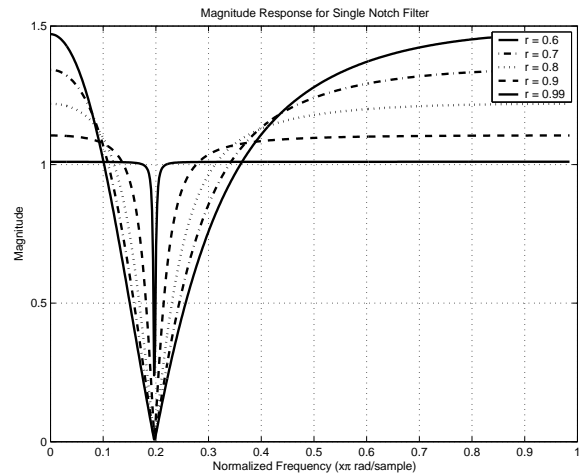


Fig.3: Magnitude responses of the conventional IIR notch filters after applying the pole re-position technique

2.3 Construction of IIR Multiple Notch Filter by Cascading

Multiple notch filters are used for removing multiple frequency interferences, to construct the IIR multiple notch, cascading of IIR single notch filters is employed, where the frequency response of IIR multiple

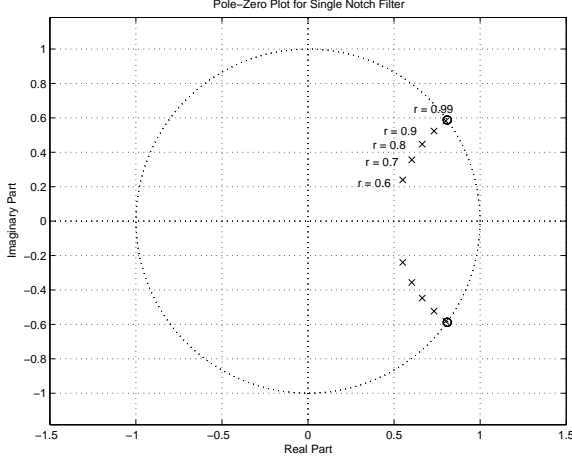


Fig.4: Pole-zero plot of IIR notch filters after changing the positions of the poles

notch filter is shown in Eq.(5)

$$H_M(e^{j\omega}) = \begin{cases} 0, & \omega = \omega_0, \omega_1, \dots, \omega_n \\ 1, & \text{otherwise} \end{cases} \quad (5)$$

Advantages of using cascading structure, in contrast with parallel structure, include 1) notch frequencies can be designed independently, 2) the passband gain can be maintained unity while the notch frequencies' attenuation remains unchanged.

On the other hand, the cascade structure has an intrinsic problem of non-unity passband gain when two notch frequencies are close to each other which can be solved by pole-reposition technique [3] or other techniques, i.e. [5].

The transfer function of the IIR multiple notch filter for the conventional scheme (before adjusting the positions of the poles) $H_M(z)$ can be expressed as

$$H_M(z) = \prod_{n=1}^N \left(b_n \cdot \frac{1 - 2 \cos \omega_n z^{-1} + z^{-2}}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}} \right), \quad (6)$$

where N denotes the number of notch frequencies.

Fig. 5 shows the magnitude response of the IIR multiple notch filter, before changing the positions of the poles (pole re-position) with $\omega_1 = 0.2\pi$, $\omega_2 = 0.3\pi$, and $r = 0.7$.

Fig. 5 shows the non-unity passband gains at π frequency and the frequency between the notch frequencies (ω_c). Therefore, the transfer function $\tilde{H}(z)$ which the pole re-position technique is applied to make the symmetric passband gains can be expressed as

$$\tilde{H}(z) = \prod_{n=1}^N \left(b_n \cdot \frac{1 - 2 \cos \omega_n z^{-1} + z^{-2}}{1 - 2r_n \cos \hat{\omega}_n z^{-1} + r_n^2 z^{-2}} \right). \quad (7)$$

After the positions of the poles are changed, the passband gains at DC and π frequencies can be controlled at the same level while frequency between the

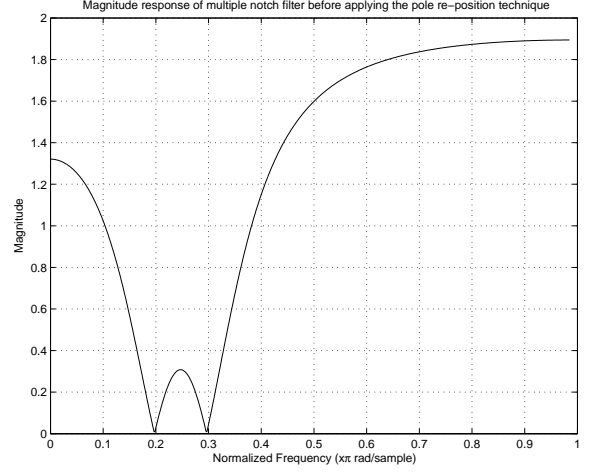


Fig.5: Magnitude response of IIR notch filters before changing the positions of the poles

notch frequencies, ω_c , still cannot be controlled as shown in Fig. 6. The problem becomes more severe when two notch frequencies are closer to each other.

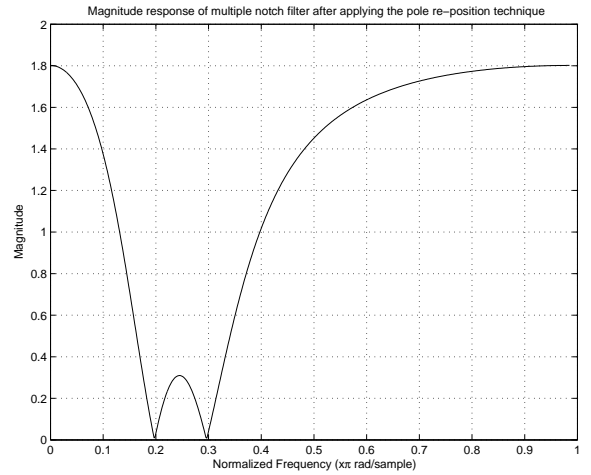


Fig.6: Magnitude response of IIR notch filters after changing the positions of the poles

Therefore, the idea to solve this problem is called "Optimum pole position" which was proposed by S. Yimman and K. Dejhan in [3]. This technique used to introduce the design of the IIR multiple notch filters only for $N = 2$, where N is the number of notch frequencies and it will be briefly explain in section 2.4

2.4 Previously Proposed Improved Pole Re-Position Technique [3]

The transfer function of this technique [3] can be shown as

$$\tilde{H}_M(z) = \prod_{n=1}^N \left(b_n \cdot \frac{1 - 2 \cos \omega_n z^{-1} + z^{-2}}{1 - 2r_n \cos \hat{\omega}_n z^{-1} + r_n^2 z^{-2}} \right). \quad (8)$$

The difference is that each IIR single notch filter will have different gains at DC and π frequencies and

there will also be a gain at frequency between the notch frequencies, ω_c , as shown in Table 1.

Defined that ω_1 and ω_2 are the notch frequencies of notch filter 1 and 2, respectively. An ω_c is the frequency between frequencies of notch filter 1 and notch filter 2 which can be defined as

$$\omega_1 < \omega_c < \omega_2.$$

In [3], ω_c is fixed to be $(\omega_1 + \omega_2)/2$ for simplicity.

Table 1: Controlled Gains for Pole Position Technique with $N = 2$, proposed in [3]

Frequency	DC	ω_1	ω_c	ω_2	π
Notch 1	k_1	0	k_1	-	$\frac{k_1}{a_1}$
Notch 2	k_2	-	k_2	0	$a_1 \cdot k_2$
Multiple	$k_1 \cdot k_2$	0	$k_1 \cdot k_2$	0	$k_1 \cdot k_2$

For notch filter 1 with gain (at π) = $\frac{k_1}{a_1}$, the adjusted notch frequency, $\hat{\omega}_1$, can be expressed as

$$\hat{\omega}_1 = \cos^{-1} \left[\frac{(r_1^2 + 1) \cdot (a_1 - 1 + \cos \omega_1 + a_1 \cos \omega_1)}{2r_1 \cdot (a_1 + 1 - \cos \omega_1 + a_1 \cos \omega_1)} \right]. \quad (9)$$

While gain (at π) = $a_1 k_2$ for notch filter 2, where the adjusted notch frequency, $\hat{\omega}_2$, can be expressed as

$$\hat{\omega}_2 = \cos^{-1} \left[\frac{(r_2^2 + 1) \cdot (-a_1 + 1 + \cos \omega_2 + a_1 \cos \omega_2)}{2r_2 \cdot (a_1 + 1 + \cos \omega_2 - a_1 \cos \omega_2)} \right]. \quad (10)$$

The resulting frequency response designed by this technique is shown in Fig. 7 with $\omega_1 = 0.2\pi$, $\omega_c = 0.25\pi$, and $\omega_2 = 0.3\pi$. The increasing value of “ a_1 ” will directly effect to the notch filter bandwidths as shown in Fig. 7.

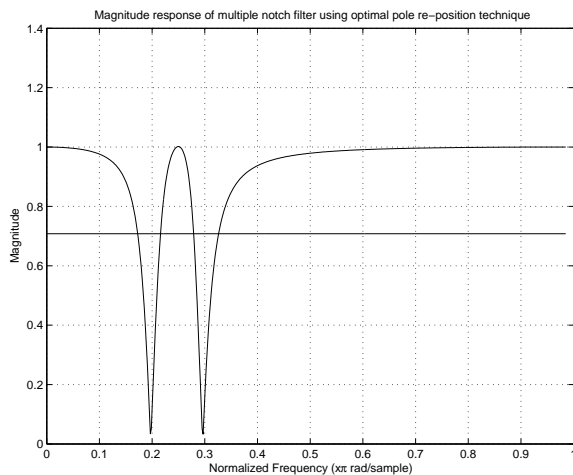


Fig.7: Cascading of two single notch filters with largest pole magnitude of $a = 0.95$ (i.e. stability margin = 0.05).

The gains k_1 and k_2 are normally set to one. The magnitude responses of notch filter 1 and notch filter 2 are shown in Fig. 8.

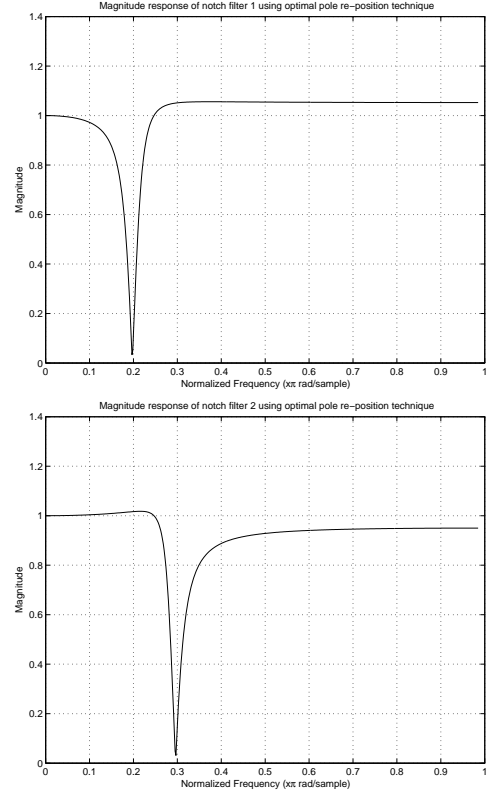


Fig.8: Frequency responses of notch filter 1 (top) and notch filter 2 (bottom)

To ensure the stability of the system, the positions of the poles must be within a unit circle, i.e. $r_n < 1$. The pole-zero plot of the IIR multiple notch filter is shown in Fig. 9. The approach proposed by [3] can provide good result but has not been optimized with respect to any cost function. Furthermore, the extension to higher order case is not provided.

2.5 Problems for the case of $N > 2$

For $N = 3$ case, assumed that ω_1 , ω_2 , and ω_3 are the notch frequencies of notch filter 1, 2, and 3, respectively. Thus,

$$\omega_1 < \omega_{c1} < \omega_2 < \omega_{c2} < \omega_3.$$

Since there is no information about how to find the suitable ω_{ci} in [3], then we first assumed that

$$\omega_{ci} = \frac{\omega_i + \omega_{i+1}}{2}, \quad \text{for } i = 1, \dots, N, \quad (11)$$

where ω_{c1} and ω_{c2} represent the frequency between notch filters 1-2, and the frequency between notch filters 2-3, respectively. Note that from the experiment, the choice of value of ω_{ci} also has some effect to the performance of the final filter.

For example, the designed filter in [1] is used to observe trend of the errors as the values of ω_c decrease where the notch frequencies ω_1 and ω_2 are defined as

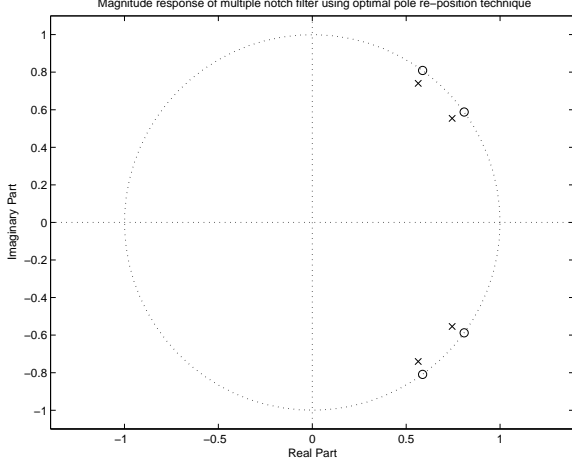


Fig.9: Pole-zero plot of multiple notch filter

0.2π and 0.3π , respectively. Since ω_c is the frequency between ω_1 and ω_2 , then $0.2\pi < \omega_c < 0.3\pi$. In this example, the value of ω_c is varied from 0.21π to 0.29π .

The errors, defined as summation of the deviation squares, are computed according to the method shown in Eq.(18) and shown in Table 2.

Table 2: Error (defined by deviation from the ideal notch filter) when ω_c is varied

ω_c	Error
0.29π	0.0229
0.28π	0.0188
0.27π	0.0160
0.26π	0.0141
0.25π	0.0132
0.24π	0.0130
0.23π	0.0136
0.22π	0.0147
0.21π	0.0164

From Table 2, the error function forms a U-like shape where the minimum occurs at $\omega_c = 0.24\pi$ rad/sec. The optimal value of ω_c has to be determined by optimization technique.

According to Eq.(21) in [3], for $N = 2$ case, the designers do not have to concentrate on the position of ω_{ci} because it has only one value. Unlike $N = 3$ case, the designers have to choose the value of ω_{ci} that the notch filter will be forced to i.e. if one controls the gains at both ω_{c1} and ω_{c2} , the equation system will be over-specified and no solution exists. Since we have two choices of ω_{ci} , i.e. ω_{c1} and ω_{c2} , then we can separate into totally 8 cases as shown in Table 3.

In this paper, only case no.1 is observed where each single notch filter will be forced to ω_{c1} . Referring to Eq.(9), Eq.(10), and Table 4, thus the magnitude response can be shown in Fig. 10 while the notch frequencies, ω_1 , ω_2 , and ω_3 are 0.2π , 0.3π , and 0.4π , respectively.

Table 3: Table of Forced Frequencies

Case No.	Notch Filter 1	Notch Filter 2	Notch Filter 3
1	ω_{c1}	ω_{c1}	ω_{c1}
2	ω_{c2}	ω_{c2}	ω_{c2}
3	ω_{c1}	ω_{c2}	ω_{c1}
4	ω_{c1}	ω_{c1}	ω_{c2}
5	ω_{c1}	ω_{c2}	ω_{c2}
6	ω_{c2}	ω_{c2}	ω_{c1}
7	ω_{c2}	ω_{c1}	ω_{c1}
8	ω_{c2}	ω_{c1}	ω_{c2}

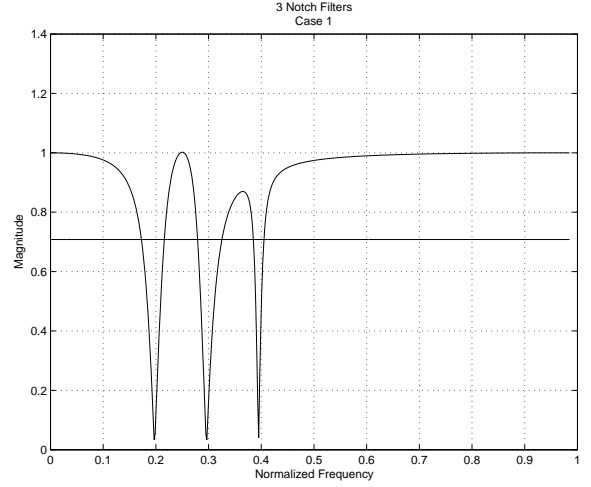


Fig.10: Cascading of three single notch filters when $A = 0.95$ and $B = 1.05$

Fig. 10 shows that the gains at DC and π frequencies can be controlled as usual while ω_{c1} and ω_{c2} can be controlled at most only one frequency. Therefore, our idea is proposed to solve this problem.

3. MAIN RESULT

For simplicity, we start with $N = 2$ case. The idea to solve this problem is that we are trying to make the system more flexible by adding more parameters named B and C as shown in Table 5, where k_1 and k_2 are initial assigned gains for notch filter 1 and notch filter 2 respectively, which will be eventually multiplied together to obtain the gain of double notch filter. In this case, we enforce them to be one on behalf of unity gain requirement at DC, ω_c , and π positions. The additional parameter A, B, C are real and positive values.

As mention in section 2.5 that the position of ω_c has the effect to the performance of the filter. Therefore, we defined ω_{ci} , frequency between ω_i and ω_{i+1} , to be

$$\omega_{ci} = \alpha_i \omega_i + (1 - \alpha_i) \omega_{i+1}, \quad (12)$$

for $i = 1, \dots, N$ where N is the number of notch frequencies, and α_i is a real value ($0 \leq \alpha_i \leq 1$) which

Table 4: Controlled Gains for Pole Re-Position Technique with $N = 3$, proposed in [3]

Frequency	DC	ω_1	ω_{c1}	ω_2	ω_{c2}	ω_3	π
Notch 1	k_1	0	k_1	-	k_1	-	$\frac{k_1}{A}$
Notch 2	k_2	-	k_2	0	k_2	-	$\frac{k_2}{B}$
Notch 3	k_3	-	k_3	-	k_3	0	$A \cdot B \cdot k_3$
Multiple	$k_1 \cdot k_2 \cdot k_3$	0	$k_1 \cdot k_2 \cdot k_3$	0	$k_1 \cdot k_2 \cdot k_3$	0	$k_1 \cdot k_2 \cdot k_3$

is not necessary to be exactly in the midst of ω_i and ω_{i+1} as described in section 2.5.

Next step is to determine the optimal modified notch frequency, $\cos \tilde{\omega}_n$ where $n = 1, 2$ for IIR double notch filter by using the relationship between DC gain and π gain as shown in Table 5.

Table 5: Desired gains for double notch filter.

Frequency	DC	ω_1	ω_c	ω_2	π
Notch Filter 1	$\frac{k_1}{C}$	0	$\frac{k_1}{B}$	-	$\frac{k_1}{A}$
Notch Filter 2	Ck_2	-	Bk_2	0	Ak_2
Double Notch Filter	k_1k_2	0	k_1k_2	0	k_1k_2

For notch filter 1, an optimal pole-zero placement, which depend on value of ω_1 and r_1 , will be considered.

Notch filter 1: (π gain = k_1/A)

According to Eq.(8), the transfer function of notch filter 1 can be expressed as

$$H_1(z) = b_1 \cdot \left(\frac{1 - 2 \cos \omega_1 z^{-1} + z^{-2}}{1 - 2r_1 \cos \tilde{\omega}_1 z^{-1} + r_1^2 z^{-2}} \right). \quad (13)$$

At DC frequency ($\omega=0$), $z = e^{j0} = 1$, then Eq.(13) becomes

$$H_1(e^{j0}) = \frac{k_1}{C} = b_1 \cdot \left(\frac{2 - 2 \cos \omega_1}{1 - 2r_1 \cos \tilde{\omega}_1 + r_1^2} \right). \quad (14)$$

At π frequency ($\omega=\pi$), $z = e^{j\pi} = -1$, then Eq.(13) becomes

$$H_1(e^{j\pi}) = \frac{k_1}{A} = b_1 \cdot \left(\frac{2 + 2 \cos \omega_1}{1 + 2r_1 \cos \tilde{\omega}_1 + r_1^2} \right). \quad (15)$$

After pairing up Eq.(14) and Eq.(15), the result of the modified notch frequency is represented as

$$\cos \tilde{\omega}_1 = \left(\frac{1 + r_1^2}{2r_1} \right) \frac{A(1 + \cos \omega_1) - C(1 - \cos \omega_1)}{A(1 + \cos \omega_1) + C(1 - \cos \omega_1)} \\ \left(\frac{1 + r_1^2}{2r_1} \right) \frac{A - C \tan^2(\omega_1/2)}{A + C \tan^2(\omega_1/2)}. \quad (16)$$

At ω_c frequency ($\omega=\omega_c$), $z = e^{j\omega_c}$, gain k_1 becomes

$$|H_1(e^{j\omega_{ci}})| = \frac{k_1}{B} \\ = b_1 \left| \frac{1 - 2 \cos \omega_1 e^{-j\omega_{ci}} + e^{-2j\omega_{ci}}}{1 - 2r_1 \cos \tilde{\omega}_1 e^{-j\omega_{ci}} + r_1^2 e^{-2j\omega_{ci}}} \right|.$$

Notch filter 2: (π gain = Ak_2)

The derivation is similar to that of notch filter 1, then the modified notch frequency can be represented as

$$\cos \tilde{\omega}_2 = \left(\frac{1 + r_2^2}{2r_2} \right) \frac{C - A \tan^2(\omega_2/2)}{C + A \tan^2(\omega_2/2)}. \quad (17)$$

At ω_c frequency ($\omega=\omega_c$), $z = e^{j\omega_c}$, gain k_2 becomes

$$|H_2(e^{j\omega_{ci}})| = Bk_2 \\ = b_2 \left| \frac{1 - 2 \cos \omega_2 e^{-j\omega_{ci}} + e^{-2j\omega_{ci}}}{1 - 2r_2 \cos \tilde{\omega}_2 e^{-j\omega_{ci}} + r_2^2 e^{-2j\omega_{ci}}} \right|.$$

And the cost function is employed to find the error of the designed filter which can be expressed as

$$\text{Error} = \sum_{i=1}^k (|H(\omega_i)| - |H_d(\omega_i)|)^2, \quad (18)$$

where ω_i is the sampled frequencies for $i = 1, \dots, k$ and k is the number of sampling frequency, $H_d(\omega_i)$ is the frequency response of the designed filter, and $H(\omega_i)$ is the frequency response of the ideal multiple notch filter expressed in Eq.(5)

4. DESIGN EXAMPLES

Example 1: For a multiple notch filter with $N = 3$, the frequency response is given in Eq.(19)

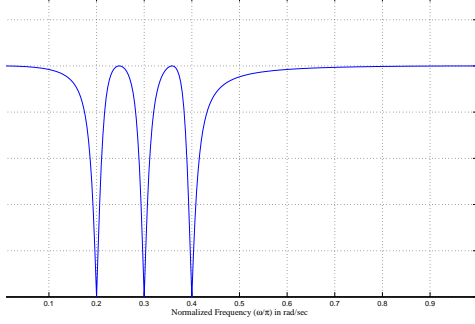
$$H_d(e^{j\omega}) = \begin{cases} 0, & \omega = 0.2\pi, 0.3\pi, 0.4\pi \\ 1, & \text{otherwise} \end{cases} \quad (19)$$

As described in section 2.5 each single notch filter is forced to ω_{c1} , then $\omega_{c1} = \alpha_1 \omega_1 + (1 - \alpha_1) \omega_2$. The values of r_1 , r_2 , and r_3 are equally set to be 0.95. Since this is $N = 3$ case, then the Table 5 has to be re-derived, i.e. using two of Eq.(16) for notch filter 1 and 2, and one of Eq.(17) for notch filter 3. Next, replace A , B , and C with a_1 , b_1 , and c_1 , respectively for notch filter 1. Similarly, a_2 , b_2 , and c_2 are substituted for notch filter 2, then $a_1 a_2$, $b_1 b_2$, and $c_1 c_2$ are respectively replaced for notch filter 3 as shown in Table 6.

The magnitude response of this design is shown in Fig. 11 where the resulting parameters are shown in Table 7. It is also noted that the tuning parameters for all examples are found by using the serial search with increment of 0.01π start from 0 to 0.5π , 0 to 0.6π , and 0 to 0.5π for example 1, 2, and 3 respectively.

Table 6: Desired gains for $N = 3$ case.

Frequency	DC	ω_{c1}	π
Notch 1	k_1/c_1	k_1/b_1	k_1/a_1
Notch 2	k_2/c_2	k_2/b_2	k_2/a_2
Notch 3	$c_1c_2k_3$	$b_1b_2k_3$	$a_1a_2k_3$
Triple	$k_1k_2k_3$	$k_1k_2k_3$	$k_1k_2k_3$

**Fig.11:** $N = 3$ case by using the proposed design

Example 2: For a multiple notch filter with $N = 4$, the frequency response can be assigned as

$$H_d(e^{j\omega}) = \begin{cases} 0, & \omega = 0.2\pi, 0.3\pi, 0.4\pi, 0.5\pi \\ 1, & \text{otherwise,} \end{cases} \quad (20)$$

where r_1, r_2, r_3 , and r_4 are equally given as 0.97. The table of gains has to be re-derived in the similar fashion as described in Example 1. Finally, the magnitude response and the resulting parameters are shown in Fig. 12 and Table 8, respectively.

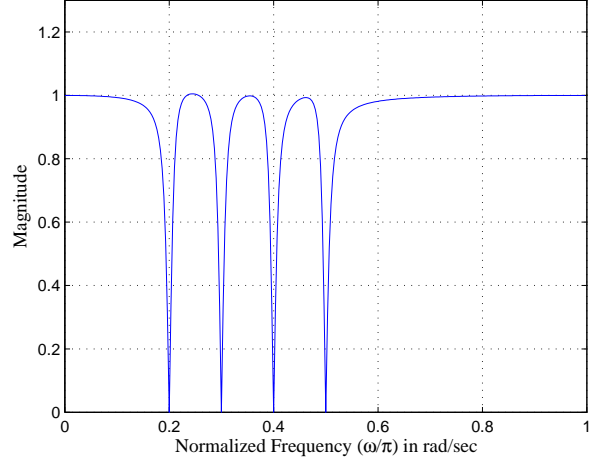
Example 3: This example needs to show the minimum allowable stability margin (r_n), by using our proposed. For convenience, a multiple notch filter with $N = 2$ is used and the frequency response can be expressed as

$$H_d(e^{j\omega}) = \begin{cases} 0, & \omega = 0.2\pi, 0.3\pi \\ 1, & \text{otherwise} \end{cases} \quad (21)$$

In this case, the minimum allowable stability margin means the possible value of r_n that maintains the stability of the multiple notch filter while the DC, π ,

Table 7: Tuning Parameters for $N = 3$ case

Parameter	Value
α_1	0.5000
a_1	0.9660
b_1	1.0000
c_1	1.0000
a_2	1.0000
b_2	1.0000
c_2	0.9955
Error	1.7588

**Fig.12:** $N = 4$ case by using the proposed design**Table 8:** Tuning Parameters for $N = 4$ case

Parameter	Value
α_1	0.5000
a_1	1.0335
b_1	1.0000
c_1	1.0495
a_2	0.9999
b_2	1.0000
c_2	1.0000
a_3	1.0040
b_3	1.0000
c_3	0.9990
Error	5.5467

and ω_c gains still can be controlled at the equal level. But the constraint is that the values of r_n must be **approximately the same**, therefore, for $N = 2$ case, it means that $r_1 \simeq r_2$ only be accepted.

Then, Table 5 can be used directly. The magnitude response of the designed filter is shown in Fig. 13 where the resulting parameters is shown in Table 9. After observe from the pole-zero plot in Fig. 14, we found that the minimum allowable values for r_1 and r_2 cannot be less than 0.85.

Table 9: Tuning Parameters for $N = 2$ case

Parameter	Value
α	0.4000
A	0.6000
B	0.6000
C	0.9240
Error	6.2272

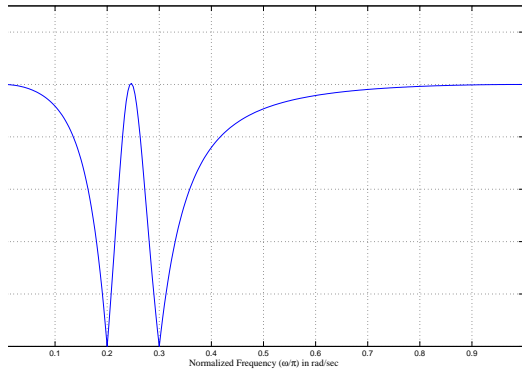


Fig.13: $N = 2$ case by using the proposed design with constraint of stability margin ≥ 0.15 i.e. $r_1, r_2 \leq 0.85$

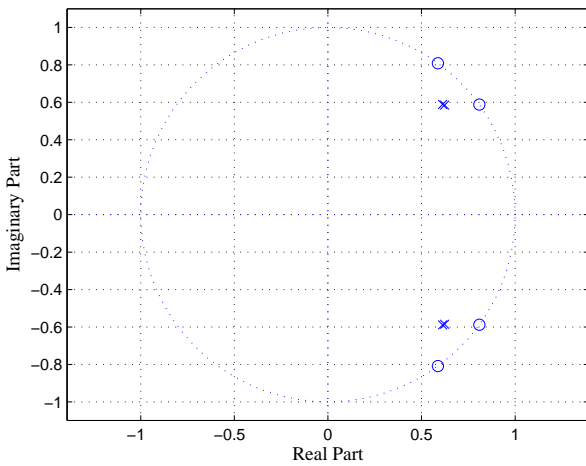


Fig.14: $N = 2$ case by using the proposed design, $r_1 \simeq r_2 \simeq 0.85$

5. CONCLUSION

The proposed IIR multiple notch filter design can guarantee that the magnitude response at DC, π , and ω_{ci} frequencies can be controlled to have uniformly flat passband gains. It has been generalized for the higher order of multiple notch filter, i.e. for case of $N \geq 2$. The selection of gain-controlled frequencies ω_{ci} has the effect to the filter performance. Another advantage of the proposed technique is that the designers are allowed to choose the filters based on their stability margin in the range of $0.85 \leq r_n < 1$ in order to fit their robustness requirement.

However, the disadvantage of this approach is its increasing complexity with the number of notch frequencies, N since the higher order of multiple notch the more tuning variables. A double notch filter requires a total four tuning parameters, i.e. α , A , B , and C , and the total number of tuning parameters can increase to $4(N - 1)$. As shown in section 2, the value of r_n is not only effect on the stability of a notch

filter but also its bandwidth (BW). Hence, another disadvantage is the bandwidth consideration because the limitation for the algorithms based on the conventional scheme is that the designers are not allowed to specify the exact bandwidth even though $r_n \propto \frac{1}{BW}$.

For the future research direction, since the complexity of the optimal IIR multiple notch filter search is not analyzed, the proposed technique can be improved further if the better search algorithm is considered e.g. the gradient-based method or the convex-optimization search.

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APPENDIX A

The magnitude square of the transfer function of a single notch filter can be expressed in Eq.(A1) as

$$|H_i(e^{j\omega_{ci}})|^2 = \left| \frac{N_i(e^{j\omega_{ci}})}{D_i(e^{j\omega_{ci}})} \right|^2, \quad (\text{A1})$$

where the numerator part is

$$|N_i(e^{j\omega_{ci}})|^2 = 4b_i^2(\cos \omega_{ci} - \cos \omega_i)^2, \quad (\text{A2})$$

and denominator part is

$$\begin{aligned} |D_i(e^{j\omega_{ci}})|^2 &= (1 + r_i^2)^2 + 2r_i^2 \cos 2\omega_{ci} \\ &\quad - 4(r_i + r_i^3) \cos \omega_{ci} \cos \tilde{\omega}_i \quad (\text{A3}) \\ &\quad + 2r_i^2 \cos 2\tilde{\omega}_i, \end{aligned}$$

for $i = 1, \dots, N$, where N denotes the number of notch frequencies.



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