

# Robust Iterative Learning Control for Linear Systems Subject to Time-Invariant Parametric Uncertainties and Repetitive Disturbances

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## ABSTRACT

This paper presents the design of a robust Iterative Learning Control (ILC) algorithm for linear systems in the presence of parametric uncertainties and repetitive disturbances. The robust ILC design is formulated as a min-max problem with a quadratic performance index subjected to constraints of the control input. Employing Lagrange duality, we can reformulate the robust ILC design as a convex optimization problem over linear matrix inequalities (LMIs). An LMI algorithm for the robust ILC design is then given. Finally, the effectiveness of the proposed robust ILC algorithm is demonstrated through a numerical example.

**Keywords:** Iterative Learning Control, Quadratic Performance, Min-Max Problem, Parametric Uncertainties, Repetitive Disturbances, Linear Matrix Inequalities

## 1. INTRODUCTION

Iterative Learning Control (ILC) is an advanced methodology introduced in 1984 by Arimoto *et al.* [1] that improves the output tracking performance of the systems operating repetitively such as rotary systems, robotic manipulators, batch processes, injection molding. Based on the way that human being experiences, ILC employs the knowledge of the control input and the system error in the past executions to modify the control input in the next run so that the system output gets closer to the target trajectory.

ILC is a promising control method for systems in the presence of uncertainties and disturbances. Some techniques in robust control such as  $\mu$ -synthesis [2],  $H_\infty$  approach [3,4], feedback-based approach [5-7]

were applied to design ILC algorithms for uncertain linear systems. However, they usually formulate the robust ILC design problems in either infinite continuous time domain or frequency domain. In addition, they implicitly assume that the time domain is infinite whilst in practice, the systems controlled by an ILC algorithm work on a finite time interval and the algorithm is implemented in discrete-time domain.

There are a number of articles analyzing the effect of disturbances in the system under ILC [8-12]. Norrlof [8] proposed an iteration-varying filter in order to deal with both repetitive and non-repetitive disturbances. In [9], Tomizuka studied the effect of periodic disturbances in mechanical systems where rotational elements such as motors and vibratory elements are usually sources of such periodic disturbances. Moreover, an ILC algorithm was proposed based on the solution of an optimal control problem in [10] to reduce the disturbance effect. Repetitive disturbances are also encountered in chemical industries, for example simulated moving bed processes which are processes with pressure or temperature swing [11]. Another source of periodic disturbances comes from load disturbances for instance in a continuous steel casting process [12]. To the best of our knowledge, there have not been any articles addressing the issue of repetitive disturbances in the design of robust ILC. Thus, it is reasonable to consider the effect of repetitive disturbances in the robust ILC design.

In this paper, we aim to design a novel robust ILC algorithm for linear systems containing parametric uncertainties and repetitive disturbances. The design problem is formulated as a min-max optimization problem with a quadratic cost function. Normally, soft constraints or no constraints of the control input are considered, and there are only a few articles investigate the optimization problem with hard constraints of the control input [13,14]. This paper extends the results in [14] to the case of multiple time-invariant parametric uncertainties and take into account of repetitive disturbances in the system. An upper-bound of the worst-case performance is proposed, then the initial min-max problem is relaxed to a minimization problem. Next, the Lagrange dual problem of the minimization problem is considered and an explicit formula of the control input update is derived. This dual problem is reformulated as a convex optimization problem over Linear Matrix In-

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equalities (LMIs) which can be efficiently solved.

The paper is organized as follows. Section 2 presents the system description and the robust ILC design problem. Section 3 introduces the main results including the worst-case performance analysis and the LMI-based ILC algorithm. The simulation results of a numerical example are shown in Section 4. Finally, Section 5 presents the conclusions.

## 2. PROBLEM FORMULATION

### 2.1 System description

Consider an uncertain discrete-time linear system described by the following state-space model

$$\begin{cases} x_k(t+1) = Ax_k(t) + Bu_k(t), \\ y_k(t) = Cx_k(t) + Ed(t) \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^q$ ,  $y \in \mathbb{R}^p$ ,  $d \in \mathbb{R}^l$  are state vector, control input, output, and disturbances, respectively,  $A, B, C, E$  are system matrices with appropriate dimensions,  $t \in \{0, 1, \dots, N\}$  is the time index,  $k$  is the iteration index.

Define

$$\begin{aligned} \mathbf{y}_k &= [y_k(1)^T \ y_k(2)^T \ \dots \ y_k(N)^T]^T \\ \mathbf{u}_k &= [u_k(0)^T \ u_k(1)^T \ \dots \ u_k(N-1)^T]^T \\ \mathbf{x}_k &= [x_k(1)^T \ x_k(2)^T \ \dots \ x_k(N)^T]^T \\ \mathbf{d} &= [d(1)^T \ d(2)^T \ \dots \ d(N)^T]^T \end{aligned} \quad (2)$$

where  $\mathbf{y}_k, \mathbf{u}_k, \mathbf{x}_k, \mathbf{d}$  are the corresponding super vectors including the system output, control input, state vector, and disturbances of all sample times over the time interval  $[0, N]$  in the  $k$ th iteration. Then, the system (1) now can be reformulated in the super-vector framework as follows.

$$\mathbf{y}_k = G\mathbf{u}_k + G_d\mathbf{d} \quad (3)$$

where  $G$  and  $G_d$  are described as follows.

$$G = \begin{bmatrix} CB & 0 & 0 & \dots & 0 \\ CAB & CB & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \dots & \dots & CB \end{bmatrix} \quad (4)$$

$$G_d = \begin{bmatrix} E & 0 & 0 & \dots & 0 \\ E & E & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ E & E & \dots & \dots & E \end{bmatrix} \quad (5)$$

Let  $e$  be the error between the output  $y$  and the reference input  $r$  defined as  $e(t) = r(t) - y(t)$ . Thus, we have

$$\mathbf{e}_k = \mathbf{r} - \mathbf{y}_k = \mathbf{r} - G\mathbf{u}_k - G_d\mathbf{d} \quad (6)$$

where

$$\begin{aligned} \mathbf{r} &= [r(1)^T \ r(2)^T \ \dots \ r(N)^T]^T \\ \mathbf{e}_k &= [e_k(1)^T \ e_k(2)^T \ \dots \ e_k(N)^T]^T \end{aligned}$$

Note that the reference input is invariant with respect to iterations, so the iteration index  $k$  is dropped out from the super-vector  $\mathbf{r}$ . Hence, from (6), we get the error update model of the system (3) as follows.

$$\mathbf{e}_{k+1} = \mathbf{e}_k - G\Delta\mathbf{u}_{k+1} \quad (7)$$

where  $\Delta\mathbf{u}_{k+1} = \mathbf{u}_{k+1} - \mathbf{u}_k$  is the difference of the control input between iterations which is defined as the control input update of the system.

Suppose that in the system (1), there are multiple time-invariant parametric uncertainties which are denoted by  $\theta_1, \theta_2, \dots, \theta_m$ , or simply denoted by a uncertainty vector  $\theta \in \mathbb{R}^m$ . Therefore, from now on, we use the notation  $G(\theta)$  instead of  $G$ . In our approach, we are interested in the systems satisfying the following conditions.

A1.  $\theta \in \Theta$  where  $\Theta$  is a set of bounded parametric uncertainties. Without loss of generality,

$$\Theta = \{\theta : \|\theta\|_\infty \leq 1\}.$$

A2. The Markov matrix  $G(\theta)$  of system (1) is an affine function of  $\theta$ .

Due to condition A2, the Markov matrix  $G(\theta)$  of system (1) has the following form

$$G(\theta) = G_0 + G_1\theta_1 + G_2\theta_2 + \dots + G_m\theta_m \quad (8)$$

where  $G_0$  represents the nominal system, and  $G_1, \dots, G_m$  represent uncertain dynamic matrices.

### 2.2 The robust ILC design problem

In real applications, there are restrictions on the control inputs which can be described by the following constraints.

C1. Bounded magnitude:  $\mathbf{u}_l \leq \mathbf{u}_{k+1} \leq \mathbf{u}_h$

C2. Bounded rate w.r.t. time index:

$$\delta\mathbf{u}_l \leq \delta\mathbf{u}_{k+1} \leq \delta\mathbf{u}_h$$

C3. Bounded rate w.r.t. iteration index:

$$\Delta\mathbf{u}_l \leq \Delta\mathbf{u}_{k+1} \leq \Delta\mathbf{u}_h, \Delta\mathbf{u}_l \leq 0, \Delta\mathbf{u}_h > 0$$

where  $\delta\mathbf{u}_{k+1}$  is the difference of control input with respect to time index, namely,

$$\delta\mathbf{u}_{k+1} = \begin{bmatrix} u_{k+1}(0) \\ u_{k+1}(1) - u_{k+1}(0) \\ \vdots \\ u_{k+1}(N-1) - u_{k+1}(N-2) \end{bmatrix}.$$

Alternatively, it can be written as  $\delta\mathbf{u}_{k+1} = J\mathbf{u}_{k+1}$  with

$$J = \begin{bmatrix} I & 0 & \dots & 0 & 0 \\ -I & I & \dots & 0 & 0 \\ 0 & -I & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & -I & I \end{bmatrix}.$$

Rewrite the constraints C1–C3 as

$$\Pi \Delta \mathbf{u}_{k+1} \leq \phi \quad (9)$$

$$\text{where } \Pi = \begin{bmatrix} -I \\ I \\ -J \\ J \end{bmatrix}, \phi = \begin{bmatrix} -\Delta \mathbf{u}_l \\ \Delta \mathbf{u}_h \\ -\delta \mathbf{u}_l + J \mathbf{u}_k \\ \delta \mathbf{u}_h - J \mathbf{u}_k \end{bmatrix},$$

$\Delta \mathbf{u}_l = \max \{ \mathbf{u}_l - \mathbf{u}_k, \Delta \mathbf{u}_l \}$ ,  $\Delta \mathbf{u}_h = \min \{ \mathbf{u}_h - \mathbf{u}_k, \Delta \mathbf{u}_h \}$ . Note that (9) is an affine inequality of  $\Delta \mathbf{u}_{k+1}$ .

To design the robust ILC algorithm, we use the quadratic performance criterion

$$J_{k+1} = \mathbf{e}_{k+1}^T Q \mathbf{e}_{k+1} + \Delta \mathbf{u}_{k+1}^T R \Delta \mathbf{u}_{k+1}$$

where  $Q = Q^T > 0, R = R^T > 0$ . The robust ILC design is formulated as a min-max problem

$$\min_{\Delta \mathbf{u}_{k+1} \in U_{k+1}} \max_{\theta \in \Theta} J_{k+1} \quad (10)$$

where  $U_{k+1}$  is a convex set defined by (9).

Now, substitute (7) into  $J_{k+1}$  and consider the cost function without  $\mathbf{e}_k^T Q \mathbf{e}_k$  since  $\mathbf{e}_k$  is known from current iteration.

$$J_{k+1} = \Delta \mathbf{u}_{k+1}^T (R + G(\theta)^T Q G(\theta)) \Delta \mathbf{u}_{k+1} - 2\mathbf{e}_k^T Q G(\theta) \Delta \mathbf{u}_{k+1} + \mathbf{e}_k^T Q \mathbf{e}_k \quad (11)$$

### 3. ROBUST ITERATIVE LEARNING CONTROL ALGORITHM

#### 3.1 The worst-case performance analysis

Consider the worst-case performance defined by

$$\max_{\theta \in \Theta} J_{k+1} \quad (12)$$

Utilizing the error update model (7) and the system model (8), we can rewrite the cost function as

$$J_{k+1} = \theta^T P \theta + 2\theta^T a + b$$

where  $P \in \mathbb{R}^{m \times m}$ ,  $a \in \mathbb{R}^m$ ,  $b \in \mathbb{R}$ ,

$$\begin{aligned} P_{ij} &= \Delta \mathbf{u}_{k+1}^T G_i^T Q G_j \Delta \mathbf{u}_{k+1}, \\ a_i &= \Delta \mathbf{u}_{k+1}^T \left( \frac{G_i^T Q G_0 + G_0^T Q G_i}{2} \right) \Delta \mathbf{u}_{k+1} - \mathbf{e}_k^T Q G_i \Delta \mathbf{u}_{k+1}, \\ b &= \Delta \mathbf{u}_{k+1}^T (G_0^T Q G_0 + R) \Delta \mathbf{u}_{k+1} - 2\mathbf{e}_k^T Q G_0 \Delta \mathbf{u}_{k+1}. \end{aligned}$$

In addition,  $J_{k+1}$  can be rewritten as a quadratic function

$$J_{k+1} = z^T H z \quad (13)$$

where  $z = \begin{bmatrix} \theta \\ 1 \end{bmatrix}$ ,  $H = \begin{bmatrix} P & a \\ a & b \end{bmatrix}$ ,  $Z = \left\{ \begin{bmatrix} \theta \\ 1 \end{bmatrix} \mid \theta \in \Theta \right\}$ .

Since  $\|\theta\|_\infty \leq 1$ , we have  $\|z\|_\infty \leq 1$ .

Now, we establish the following theorem to find an upper bound of the worst-case performance (12).

**Theorem 1.** *If there exists  $T$  such that  $H \leq T$  where  $T$  is diagonal, an upper bound of the worst-case performance (12) can be determined by solving the following optimization problem.*

$$\begin{aligned} \min \quad & \text{trace}(T) \\ \text{s.t.} \quad & H \leq T \\ & T \text{ is diagonal} \end{aligned} \quad (14)$$

*Proof:* Suppose there exists  $T$  such that  $H \leq T$  where  $T$  is diagonal. Then

$$z^T H z \leq z^T T z = \sum_{i=1}^m t_i z_i^2 + t_{m+1} \leq \sum_{i=1}^{m+1} t_i = \text{trace}(T) \quad (15)$$

where  $t_i, i = 1, 2, \dots, m+1$  are elements on the diagonal of matrix  $T$ . Note that the second inequality in (15) holds due to  $\|z\|_\infty \leq 1$  or equivalently,  $z_i^2 \leq 1 \forall i = 1, 2, \dots, m$ . Therefore,

$$\max_{z \in Z} z^T H z \leq \text{trace}(T) \quad (16)$$

Consequently, an upper bound of the worst-case performance (12) can be found by solving (14).

#### 3.2 The robust ILC algorithm

Replacing (14) into (10) and consolidating two minimization problems, the iterative input update  $\Delta \mathbf{u}_{k+1}$  can be calculated by solving the following minimization problem.

$$\begin{aligned} \min \quad & \text{trace}(T) \\ \text{s.t.} \quad & H \leq T \\ & T \text{ is diagonal} \\ & \Delta \mathbf{u}_{k+1} \in U_{k+1} \end{aligned} \quad (17)$$

This is a minimization problem with variables  $T$  and  $\Delta \mathbf{u}_{k+1}$ . To solve (17), we consider its dual problem. First, let us reformulate (17) as follows. Since the matrix  $T$  is diagonal, we can rewrite it as

$$T = \sum_{i=1}^{m+1} t_i F_i \quad (18)$$

where  $F_i$  is a matrix with zero for all elements except 1 is at the  $i^{\text{th}}$  position on the diagonal. Then, (17) becomes

$$\begin{aligned} \min \quad & \mathbf{1}^T t \\ \text{s.t.} \quad & H - \sum_{i=1}^{m+1} t_i F_i \leq 0 \\ & \Pi \Delta \mathbf{u}_{k+1} \leq \phi \end{aligned} \quad (19)$$

where  $t = [t_1 \ t_2 \ \dots \ t_{m+1}]^T$ ,  $\mathbf{1}$  is a  $(m+1) \times 1$  vector whose elements are all equal to 1. Define a Lagrangian

$$\begin{aligned} L(t, \Delta \mathbf{u}_{k+1}, W, \nu) &= \mathbf{1}^T t + \nu^T (\Pi \Delta \mathbf{u}_{k+1} - \phi) \\ &\quad + \text{trace} \left( \left( H - \sum_{i=1}^{m+1} t_i F_i \right) W \right) \end{aligned} \quad (20)$$

where  $W = W^T \geq 0$  with dimension  $(m+1) \times (m+1)$ ,  $\nu$  is a vector with dimension  $4qN$ . In order to obtain the iterative input update, we consider the dual problem of (19) which is generated from finding the minimum of the Lagrangian with respect to  $t$  and  $\Delta \mathbf{u}_{k+1}$ .

After some mathematical manipulation, we receive

$$\begin{aligned} & \inf_{t, \Delta \mathbf{u}_{k+1}} L(t, \Delta \mathbf{u}_{k+1}, W, \nu) \\ &= -\nu^T \phi + \inf_{\Delta \mathbf{u}_{k+1}} \{ \text{trace}(HW) + \nu^T \Pi \Delta \mathbf{u}_{k+1} \} \end{aligned} \quad (21)$$

Note that (21) holds when

$$\text{trace}(F_i W) = 1 \Leftrightarrow w_{ii} = 1 \quad \forall i = 1, 2, \dots, m+1$$

Substituting the expression of the elements of matrix  $H$  into  $\text{trace}(HW)$ , we obtain

$$\text{trace}(HW) + \nu^T \Pi \Delta \mathbf{u}_{k+1} = \Delta \mathbf{u}_{k+1}^T \hat{G} \Delta \mathbf{u}_{k+1} + \beta^T \Delta \mathbf{u}_{k+1} \quad (22)$$

where

$$\begin{aligned} \hat{G} &= \sum_{i=1}^m \sum_{j=1}^m G_i^T Q G_j w_{ij} + G_0^T Q G_0 + R \\ &\quad + \sum_{i=1}^m (G_i^T Q G_0 + G_0^T Q G_i) w_{i,m+1} \\ \beta &= \Pi^T \nu - 2 \sum_{i=1}^m G_i^T Q \mathbf{e}_k w_{i,m+1} - 2 G_0^T Q \mathbf{e}_k \end{aligned} \quad (23)$$

Next, an explicit formula of the control input update  $\Delta \mathbf{u}_{k+1}$  is given in the following theorem.

**Theorem 2.** *If there exist  $W$  and  $\nu$  obtained from the solution of the following LMI problem*

$$\begin{aligned} \min \quad & \rho \\ \text{s.t.} \quad & \begin{bmatrix} \hat{G} & \beta \\ \beta^T & \rho - 4\nu^T \phi \end{bmatrix} \geq 0, \quad \nu \geq 0, \\ & W \in \mathbf{S}_+^{m+1}, \quad w_{ii} = 1, \end{aligned} \quad (24)$$

then, the iterative input update  $\Delta \mathbf{u}_{k+1}$  is determined as follows.

$$\Delta \mathbf{u}_{k+1} = -\frac{1}{2} \hat{G}^{-1} \beta. \quad (25)$$

*Proof:* First, we note that the right hand side of (22) is a quadratic function of  $\Delta \mathbf{u}_{k+1}$ . Hence, it is easy to see that if  $\hat{G} > 0$ , then

$$\inf_{\Delta \mathbf{u}_{k+1}} \left\{ \Delta \mathbf{u}_{k+1}^T \hat{G} \Delta \mathbf{u}_{k+1} + \beta^T \Delta \mathbf{u}_{k+1} \right\} = -\frac{1}{4} \beta^T \hat{G}^{-1} \beta$$

with the optimal solution of  $\Delta \mathbf{u}_{k+1}$  given by (25). Moreover, the dual function is as follows.

$$\inf_{t, \Delta \mathbf{u}_{k+1}} L(t, \Delta \mathbf{u}_{k+1}, W, \nu) = -\nu^T \phi - \frac{1}{4} \beta^T \hat{G}^{-1} \beta \quad (26)$$

Thus, by finding the dual function, we obtain the ILC update law as described in (25). In addition, the iterative input update is constructed from  $\hat{G}$  and  $\beta$  which depend on the Lagrange multipliers  $W$  and  $\nu$ . Therefore, we need to solve the dual problem to obtain the Lagrange multipliers, then calculate the

iterative input update. We obtain the dual function as in (26), then the dual problem of (17) is

$$\begin{aligned} \max \quad & \left( -\nu^T \phi - \frac{1}{4} \beta^T \hat{G}^{-1} \beta \right) \\ \text{s.t.} \quad & \hat{G} > 0, \quad \nu \geq 0, \\ & W \in \mathbf{S}_+^{m+1}, \quad w_{ii} = 1. \end{aligned} \quad (27)$$

which is equivalent to

$$\begin{aligned} \min \quad & \left( \nu^T \phi + \frac{1}{4} \beta^T \hat{G}^{-1} \beta \right) \\ \text{s.t.} \quad & \hat{G} > 0, \quad \nu \geq 0, \\ & W \in \mathbf{S}_+^{m+1}, \quad w_{ii} = 1. \end{aligned} \quad (28)$$

Suppose that  $\rho$  is an upper bound of  $(4\nu^T \phi + \beta^T \hat{G}^{-1} \beta)$ , it can be proved that (28) is equivalent to the following optimization problem

$$\begin{aligned} \min \quad & \rho \\ \text{s.t.} \quad & 4\nu^T \phi + \beta^T \hat{G}^{-1} \beta \leq \rho, \\ & \hat{G} > 0, \quad \nu \geq 0, \\ & W \in \mathbf{S}_+^{m+1}, \quad w_{ii} = 1. \end{aligned} \quad (29)$$

Using Schur complement [15], we can rewrite (29) as the LMI problem (24).

**Remark 1.** The LMI problem (24) can be solved using available software containing convex optimization solvers such as `cvx` [?, ]. The control input update  $\Delta \mathbf{u}_{k+1}$  in (25) is calculated after solving (24). Obviously,  $\Delta \mathbf{u}_{k+1}$  depends on the feasibility of the LMI problem (24). Therefore, we can adjust the constraints of the control input, namely, varying the values of  $\Pi$  and  $\phi$ , until (24) is feasible.

Now, to summarize the proposed methodology, we introduce the following algorithm for the robust ILC design. The stopping criteria for the robust ILC algorithm are  $\|\mathbf{e}_k\| \leq \epsilon$  and  $k = \text{iter\_max}$  where  $\epsilon$  is a tolerance chosen by the designer and  $\text{iter\_max}$  is the maximum number of iterations.

**Algorithm 1.** An LMI-based ILC algorithm for linear systems with multiple time-invariant parametric uncertainties and repetitive disturbances

1. Set  $k := 0$ ,  $\mathbf{u}_k := 0$ , and measure  $\mathbf{e}_k$ .
2. Solve the LMI problem according to (24).
3. Calculate  $\Delta \mathbf{u}_{k+1}$  according to (25).
4. Apply  $\mathbf{u}_{k+1}$  to the system and measure  $\mathbf{e}_{k+1}$ .
5. If one of the stopping criteria is satisfied, then stop the iteration, else, set  $k := k + 1$ , return to step 2.

**Remark 2.** The proposed algorithm in this paper can be considered as an extension of the one in [17] since the repetitive disturbances are not considered there.

**Remark 3.** There might be a conservatism in the proposed robust ILC design since an upper bound of

the worst-case performance is used in the maximization problem (13). However, the algorithm appears to work well as demonstrated in the numerical examples.

### 3.3 Convergence properties

Next, we will prove the convergence of the control input  $\mathbf{u}_k$  and the error  $\mathbf{e}_k$ .

**Theorem 3.** *Under assumptions A1-A2 and constraints C1-C3, the control input  $\mathbf{u}_k$  of system (3) converges.*

*Proof:* Let  $V(\mathbf{e}_k)$  be defined as

$$V(\mathbf{e}_k) = \min_{\Delta \mathbf{u}_{k+1} \in U_{k+1}} \max_{\theta \in \Theta} J_{k+1} \quad (30)$$

with  $J_{k+1}$  is in (11). Then,  $V(\mathbf{e}_k) \geq 0 \forall k$  since  $J_{k+1} \geq 0 \forall k$ . We have

$$V(\mathbf{e}_k) \leq J_{k+1}|_{\Delta \mathbf{u}_{k+1}=0} = \mathbf{e}_k^T Q \mathbf{e}_k$$

Suppose that  $\theta_k^*$  is the optimizer of the maximization problem at the  $k$ th iteration. Hence,

$$\mathbf{e}_k^T Q \mathbf{e}_k \leq \mathbf{e}_k^T Q(\theta_k^*) \mathbf{e}_k = V(\mathbf{e}_{k-1}) - \Delta \mathbf{u}_k^T R \Delta \mathbf{u}_k$$

Therefore,

$$V(\mathbf{e}_k) \leq V(\mathbf{e}_{k-1}) - \Delta \mathbf{u}_k^T R \Delta \mathbf{u}_k \quad (31)$$

Inequality (31) leads to

$$V(\mathbf{e}_k) + \sum_{i=1}^k \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i \leq V(\mathbf{e}_0) \quad (32)$$

Since  $V(\mathbf{e}_k) \geq 0$ , we get

$$\sum_{i=1}^k \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i \leq V(\mathbf{e}_0) < \infty \quad (33)$$

Moreover, since  $R$  is positive definite,  $\Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i \geq 0 \forall i$  and the sequence  $\left\{ \sum_{i=1}^k \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i \right\}$  is non-decreasing. Combining with (33), it deduces that  $\left\{ \sum_{i=1}^k \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i \right\}$  converges. Accordingly,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \Delta \mathbf{u}_k^T R \Delta \mathbf{u}_k \\ &= \lim_{k \rightarrow \infty} \left( \sum_{i=1}^k \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i - \sum_{i=1}^{k-1} \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i \right) \\ &= \lim_{k \rightarrow \infty} \sum_{i=1}^k \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i - \lim_{k \rightarrow \infty} \sum_{i=1}^{k-1} \Delta \mathbf{u}_i^T R \Delta \mathbf{u}_i \\ &= 0. \end{aligned}$$

It implies that  $\Delta \mathbf{u}_k \rightarrow 0$  as  $k \rightarrow \infty$ . Thus,  $\{\mathbf{u}_k\}$  converges.

**Theorem 4.** *Under assumptions A1-A2 and constraints C1-C3, the error  $\mathbf{e}_k$  of system (3) converges.*

*Proof:* We have

$$\begin{aligned} \|G(\theta)\| &= \|G_0 + G_1\theta_1 + G_2\theta_2 + \cdots + G_m\theta_m\| \\ &\leq \|G_0\| + \|G_1\theta_1\| + \cdots + \|G_m\theta_m\| \\ &\leq \|G_0\| + \|G_1\| + \cdots + \|G_m\| \end{aligned}$$

Hence,  $\|G(\theta)\|$  is bounded. It leads to  $G(\theta)\Delta \mathbf{u}_{k+1} \rightarrow 0$  as  $k \rightarrow \infty$ . Equivalently,  $(\mathbf{e}_k - \mathbf{e}_{k+1}) \rightarrow 0$  as  $k \rightarrow \infty$ . This results in the convergence of  $\{\mathbf{e}_k\}$ .

## 4. NUMERICAL EXAMPLE

We consider the following system with transfer function

$$G(s) = \frac{1}{15s^2 + 8s + 1} + \theta_1 \frac{0.8e^{-s}}{5s + 1} + \theta_2 \frac{0.5e^{-2s}}{2s + 1} \quad (34)$$

where  $\theta_1, \theta_2$  are the uncertain parameters and  $\theta_1, \theta_2 \in [-1, 1]$ . The system (34) is subjected to disturbances which are randomly generated with a bound of 0.05 and repeated over the iterations. The sampling time is chosen to be 1 second and the number of samples is 101. The tracking target is a half-period sinusoidal signal, namely,

$$r(t) = \sin\left(\frac{\pi t}{100}\right), t \in [0, 100].$$

The constraints of control inputs are specified by

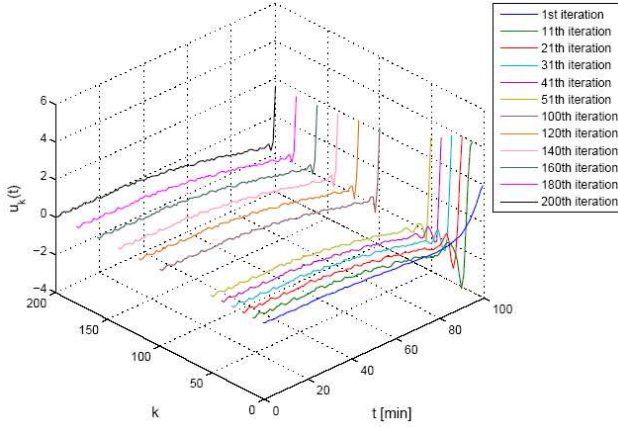
$$\begin{aligned} \mathbf{u}_l &= -4 \times \mathbf{1}_N, & \mathbf{u}_h &= 4 \times \mathbf{1}_N, \\ \delta \mathbf{u}_l &= -3 \times \mathbf{1}_N, & \delta \mathbf{u}_h &= 3 \times \mathbf{1}_N, \\ \Delta \mathbf{u}_l &= -4 \times \mathbf{1}_N, & \Delta \mathbf{u}_h &= 4 \times \mathbf{1}_N. \end{aligned} \quad (35)$$

where  $\mathbf{1}_N$  is a vector in  $\mathbb{R}^N$  with all elements equal to 1. The weighting matrices are chosen as follows:  $Q = I_1, R = 0.02I_2$  where  $I_1, I_2$  are identity matrices with appropriate dimension. For the stopping criteria, we choose  $\epsilon = 0.01$  and `iter_max` = 200.

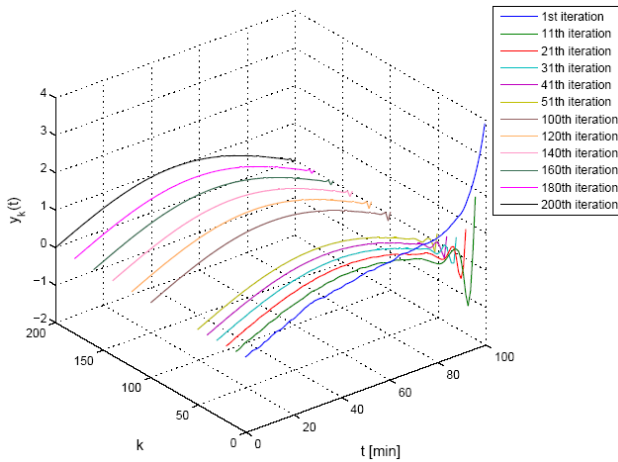
The simulation results are shown as follows. Figure 1 displays that the control input converges and satisfies constraint C1. In addition, the rates of control input with respect to time index and iteration index are bounded and satisfy constraints C2 and C3. The tracking of the system output to the target trajectory is exhibited in Figure 2. We can see that the system output tracks the sinusoidal reference input very well even though there are uncertainties and disturbance in the system. This proves the ability of the proposed ILC algorithm to attenuate the effects of disturbance and uncertainties as pointed out in the previous section. The ILC performance is verified as we observe a monotonic convergence of the system error shown in Figure 3.

To further explore the proposed ILC algorithm, we test the system (34) with an increased bound of disturbance, namely, 0.1. The control input and system output are depicted in Figures 4–5. It can be seen that the system output has larger deviation from the

reference input at first iterations but effectively converges to the reference. We also observe that the control input contains more ripples in all iterations. In addition, the  $Q$ -norm of error shown in Figure 6 converges more slowly comparing to the error in Figure 3. The results reveal that as the bound of disturbance is increased, the tracking performance becomes worse. Therefore, it is suggested to design an output filter together with an ILC controller when the system is heavily perturbed by disturbances. This is an ongoing research topic.



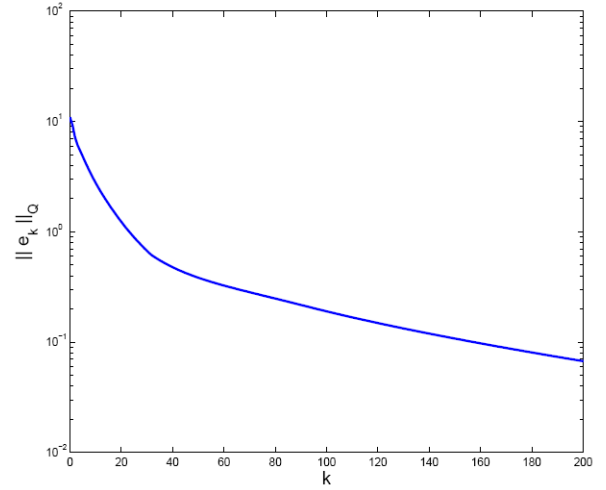
**Fig.1:** Control input of system with uncertainties and disturbance bounded by 0.05.



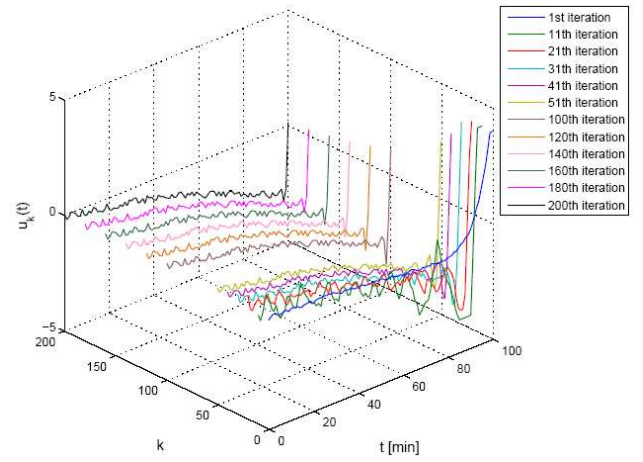
**Fig.2:** Output response of system with uncertainties and disturbance bounded by 0.05.

## 5. CONCLUSIONS

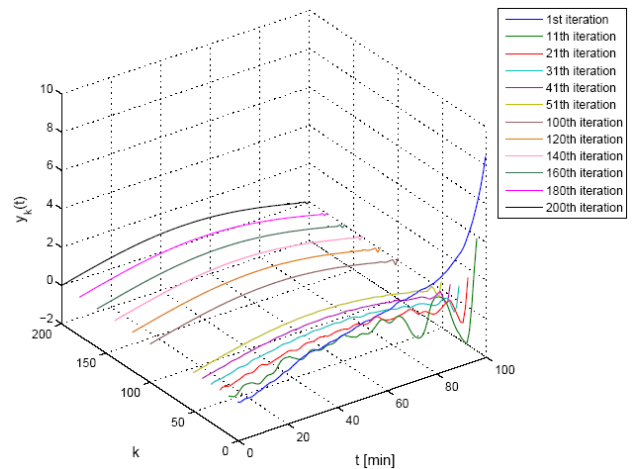
This paper has proposed a robust ILC algorithm for linear systems in the presence of time-invariant parametric uncertainties and repetitive disturbances. We first determine an upper bound of the worst-case performance, then solve the minimization problem to get the update of iterative input. Using Lagrange duality, we cast the robust ILC design as a series of LMI



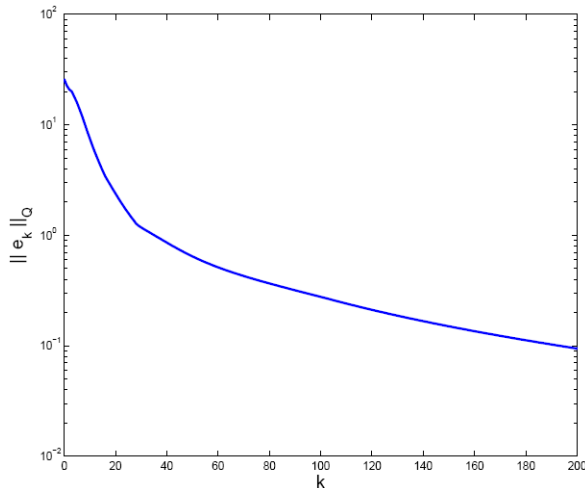
**Fig.3:**  $Q$ -norm of error of system with uncertainties and disturbance bounded by 0.05.



**Fig.4:** Control input of system with uncertainties and disturbance bounded by 0.1.



**Fig.5:** Output response of system with uncertainties and disturbance bounded by 0.1.



**Fig. 6:**  $Q$ -norm of error of system with uncertainties and disturbance bounded by 0.1.

problems which can be efficiently solved. The proposed algorithm is applied to a SISO system with two uncertainties and repetitive disturbances. The results show that the designed ILC is capable of tracking the reference input and attenuating the disturbance effect. There might be conservatism in the design due to the utilization of an upper bound of the worst-case quadratic performance.

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