Construction and Characterization of Optical Orthogonal Codes $(n,w,1)$ for Fast Frequency Hopping-Optical Code Division Multiple Access

Samuel Nlend and **Theo G. Swart**†, Non-members

ABSTRACT

This paper investigates a new combinatorial construction approach based on cyclic difference packings for the construction of an optical orthogonal code that is suitable as a time spreading sequence for fast frequency hopping optical code division multiple access (OCDM) systems. The proposed construction provides a solution to the electrical decoding delay problem when compared to its optical counterpart, as well as other optimization issues. The criteria for optimizing the performance of such an OCDMA system are provided, namely: the flexibility and simplicity in constructing the optimal code for any length and weight, a reduction in the encoding/decoding complexity that complies with changes in the fast frequency hopping system, and the provision of system transfer transparency, a decoding rule that exploits the embedded asymmetric error correcting capability of the code. Our neighbor difference approach refers to the partition of different solutions into sub-classes, given in the correlation matrix form. It takes into account both direct and recursive combinatorial code construction methods, and the resulting characteristics are evaluated accordingly. A performance analysis based on the OCDMA channel is given, and a comparison with other difference family solutions is performed.

Keywords: Asymmetric Error Correcting Code, Bragg Gratings, Combinatorial, Difference Family, Fast Frequency Hopping, Optical Code Division Multiple Access, FFH-OCDMA, Optical Orthogonal Code, OOC

1. INTRODUCTION

Interest in optical orthogonal codes (OOCs) comes from the fact that as a unipolar code, it can be utilized for the generation of optical signature sequences for users, time spreading sequences, and asymmetric error correcting codes for incoherent optical code division multiple access (OCDMA) systems. However, using such an optical system in combination with end electrical equipment causes delay issues, on which this paper focuses.

An OCDMA channel is an asymmetric Z-channel [1]. Today, asymmetric error correction is critical in optical bit error rate (BER) transmission. Studies like [2] proved that at high transmission speed and a large number of users, the consideration of asymmetric errors improves channel performance. The Reed-Solomon code as a forward error correcting (FEC) code was subsequently used in combination with low complexity soft decoding, while [3] proposed an embedded asymmetric error correcting (AEC) code in an OCDMA sequence.

Exploiting this embedded capability, an OOC construction for time spreading (TS) sequences is suggested in this paper to provide a decoding rule equivalent to minimum distance decoding.

Access wise, most OCDMA studies focus on OOC because it gives up cardinality for better correlation properties in comparison to the congruence codes [4]. This cardinality is optimal if, according to Johnson's bound [5, 6], it is equal to $\lfloor (n-1)/w(w-1) \rfloor$ when the correlation is equal to one, for a code of length n and Hamming weight w. Studies, such as [7], showed that the optimal condition is reached for $w = 3$ if and only if $n \neq 6l + 2$, where l is such that $l \cong 2$ or $3 \pmod 4$. Due to difficulties in the tractability of the OOC and despite the amount of work done, much still needs to be done for $w \geq 4$ [7]. In that regard, [8] and [9] suggested solutions for $(n, 4, 1)$ and $(9n, 4, 1)$ based on incomplete difference matrices and prime congruences, respectively. Approaches classified as recursive are derived from an existing solution or direct method.

In this study, we make use of the direct and recursive methods to achieve greater flexibility in the code construction for any given n and $w \geq 3$. Yielding optimal OOC, this flexible solution is the neighbor difference introduced by [10]. The neighbor difference approach is a subset of the difference of position (DoP) which is extended by [11] using the click set approach to reduce code construction complexity from $O(w^4)$ to $O(w^3)$.

Exploiting these approaches, the bounds of the code length in this paper allow the optimization of any given pair (n, w). We emphasize the congruence class $n \cong 1 \pmod{w(w-1)}$, an incomplete difference family solution given by [12] derived from the Wilson structure defined according to the Galois field (GF(n)) [4] and Johnson's bound [6]. We expand the work of [10] on the neighbor solution based on the code length bounds derived from

Manuscript received on April 13, 2021; revised on September 11, 2021; accepted on September 27, 2021. This paper was recommended by Associate Editor Kampol Woradit.

The authors are with the Center for Telecommunications, University of Johannesburg, South Africa.

†Corresponding author: tgswart@uj.ac.za

Digital Object Identifier: 10.37936/ecti-eec.2022201.246105

[2] and the difference family of [4].

Djordjevic and Vasic [13] suggested three classes of OOC for both asynchronous and synchronous OCDMA: Mutually Orthogonal Latin Squares/Rectangles (MOLSs/MOLRs) and Spectral Amplitude Coding (SAC), which provide major advantages such as flexibility in the number of users, but these require a multiuser interference cancellation device, making it costly in terms of detection delay. As a controller, it has three parameters, resulting in a complexity of at least $O(w^3)$, but most importantly, it requires a larger code length than Johnson's bounds [5, 6].

As a contribution, this paper suggests an improved approach to code construction, significantly reducing the complexity and yielding a decoding rule that allows the electrical decoder to match its optical counterpart. This is advantageous considering the requirement of fast encoding/decoding imposed by the Fast Frequency Hopping OCDMA (FFH-OCDMA) system and, in general, the requirements of access, line, and error correction code. These characteristics are evaluated accordingly.

The rest of the paper is outlined as follows: Section 2 provides the background and notation used, Section 3 follows with cyclic and difference packing analysis from where we derive the neighborhood solution, which is given in Section 4. In Section 5 we propose the construction of the $(n, w, 1)$-OOC, while in Section 6, we evaluate the required characteristics. Section 7 provides the performance of our scheme and a comparison with other difference families.

2. BACKGROUND AND NOTATION

2.1 Optical Orthogonal Code and Cardinality

An optical orthogonal code C, denoted by $(n, w, \lambda_a, \lambda_c)$-OOC, is a family of (0,1)-sequences, each representing a code word of length n, Hamming weight w, auto- and cross-correlation indices λ_a and λ_c respectively, satisfying the following correlation properties [14, 15]:

- Auto-correlation: $\sum_{i=1}^{n} X_i X_{i+\tau} \leq \lambda_a$ for any $X \in C$ and any integer $\tau \neq 0, 0 < \tau < n$;
- Cross-correlation: $\sum_{i=1}^{n} X_i Y_{i+\tau} \leq \lambda_c$ for any $X \neq Y \in C$ and any integer τ;

where the subscripts take modulo n.

In [16], it is found that some combinatorial structures yield optimal OOC. Hence, we focus on a combinatorial solution where $\lambda_a = \lambda_c = \lambda = 1$. Two methods are utilized in OOC combinatorial construction, namely, the direct method and the recursive method, both of which have the same goal: achieving optimal (n, w, λ)-OOC in terms of the number of simultaneous users.

2.2 Direct Method

An optimal (n, w, λ)-OOC is equivalent to a combinatorial structure called cyclic l-packing [17], which can also be presented using the difference family.

2.2.1 Packing Design and Block Design

Cyclic l-packing consists of a pair (N, B), where N is an n-set of points and B is a collection of w-subsets of N called blocks, such that every l-subset of distinct points of N occurs in most λ blocks of B. The idea of cyclic l-packing is clearly presented in [17] for optimal (n, w, λ)-OOC, in [15] for a cyclic l-(n, w, λ) packing design which is optimal according to [18] if $l = \lambda + 1$ and $\lambda < w$, a condition that is tractable only for $\lambda = 1$, or equivalently $l = 2$, as studied in [19], hence yielding a packing design with order n denoted by $P(w, 1; n)$, called a pairwise balanced packing design. In [15], a cyclic l-(n, w, λ) packing design admitted through mapping, while the bijection in B from a bijection in N is an automorphism of the packing $P(w; 1; n)$. Furthermore, if a $P(w; 1; n)$ admits an automorphism consisting of a single cycle of length n, then $P(w; 1; n)$ is cyclic such that set N can be identified by Z_n, the residue ring of the integer modulo n and denoted as $CP(w, 1{:}\, n)$. From the perspective of these theoretical cyclic sets, the correlation properties mentioned above can be reformulated as [7, 15]:

- Auto-correlation: $|(X + a) \cap (X + b)| \leq \lambda_a$, for any $X \in C$ and $a \neq b \pmod{n}$;
- Cross-correlation: $|(X + a) \cap (Y + b)| \leq \lambda_c$, for any $X \neq Y$ and any a, b.

Note that $X + a = \{x + a : x \in C\}$ and all integers under consideration take modulo n.

Accordingly, an optimal $CP(w, 1; n)$ exists whenever a cyclic block (CB), denoted by $CB(w, 1; n)$, occurs. The conditions of existence and full orbit for a cyclic pairwise balanced (PB) design or $B(w, 1; n)$ involve a particular variant called the balanced incomplete block design (BIBD), denoted by $CB(w, 1; n)$, and provided in [20] and [21]. As with OOCs, a $CP(w, 1; n)$ with $\lfloor (n-1)/w(w-1) \rfloor$ block orbits is said to be optimal [16, Th. 1].

2.2.2 Packing Design and Difference Family Design

A difference family (DF) called scarce DF (SDF) denoted as $(n, w, 1)$-SDF is a collection F of subsets Z_n called starter or base blocks B, such that for every non-zero x element of Z_n the congruence $x_i - x_j \cong x \pmod{v}$ has one solution pair at most (x_i, x_j), where $x_i, x_j \in B$. If B_i, $\{B_i + j : j \in Z_n\} \in B$ are of full orbit, then the development of B yields a $CP(w, 1; n)$ [16]. Therefore, an $(n, w, 1)$-SDF with $\lfloor (n-1)/w(w-1) \rfloor$ starter blocks forms an optimal $CP(w, 1; n)$, and vice versa. Such a $CP(w, 1; n)$ denotes a cyclic difference packing (CDP) with order n, block size w and index 1. See [16] for further details. From this cyclic difference perspective, the correlation properties above can be written as follows [15]:

- Auto-correlation: for each $X \in C$, any integer $\delta_{i,j} \neq 0$ can be represented as the difference $x_i - x_j$, with $x_i, x_j \in X$, in at most λ_a ways.
- Cross-correlation: similarly, for each $X \neq Y \in C$, any integer $\delta_{i,j} \neq 0$ can be represented as the difference

$x_i - y_j$, with $x_i \in X$, $y_j \in Y$, in at most λ_c ways.

2.3 Recursive Method

The general construction of OOC using recursive methods combines two or more BIBDs to yield another BIBD with a different order, as proposed by Colbourn and Colbourn [22, 23]. Another recursive method makes use of the notion of h-regular, mostly derived from the SDF and CB, where h is the order of the set $L \cup \{0\}$, L is the difference leave (DL), and is the non-zero elements of Z_n not covered by the differences in F. The results of both approaches are given in [16]. The Colbourn recursive method was generalized by [24], while [25] suggested recursive construction using the difference matrix. A difference matrix (DM) denoted by (w, n)-DM is a $w \times n$ matrix $D = (d_{i,j})$, $1 \le i \le w$, $1 \le j \le n$ with entries from Z_n, such that for any two distinct rows s and t, the difference $d_{s,j} - d_{t,j}$, $j = 1, 2, \ldots, n$ contains each integer of Z_n exactly once. D is of interest to us since we associate the orthogonal array as defined in [24], and the proposition on row difference scheme as proposed in [26] for our OOC construction.

2.4 The OOC Representation

An (n, w, λ)-OOC can be represented as a collection of w-sets of the 1-bit modulo n to express a code word. This representation is called the weighted position representation (WPR) [27]. A fixed bit position yields the fixed weighted position representation (FWPR). To construct a unique OOC in the FWPR, the difference of position (DoP) proposed by [15] was further developed in [11] to yield the extended DoP (EDoP). The DoP is the difference between the 1-bit positions of a code vector. The OOC representation is illustrated by the following example.

Example 1: Let $X = \{1101000\}$ be a $(7, 3, 1)$-OOC with one code word. In a set notation, $X = \{1, 2, 4\}$ (mod 7) and its difference of position is $DoP(X) = (2 - 1, 4 - 2, 4 - 1) = (1, 2, 3)$.

2.5 Asymmetric Error Correction Properties

By definition, code C is said to be a t-asymmetric error correcting (t-AEC) code if it can correct up to t asymmetric errors, and where a rule exists such that for a code vector $X \in C$, if Y is obtained from X by changing at most the t ones of X into zeros, X will be recovered from Y.

For any vectors X and Y of $\{0, 1\}^n$, a set of asymmetric codes of length n and define $N(X, Y) := |\{i/x_i = 0 \text{ and } y_i = 1\}|$, we make use of the following properties given in [28]:

- $\Delta(X, Y) = \max\{N(X, Y), N(Y, X)\}$, asymmetric distance between X and Y;
- $d(X, Y) = N(X, Y) + N(Y, X)$, Hamming distance.

In [29], a code C is shown as a t-AEC code if and only if for any two code vectors $X \ne Y \in C$, $\Delta(X, Y) > t$. Similarly, for X and Y of a code C, define the set of vectors by changing t or reducing the 1 bits in X into 0 bits by $S_t(X) = \{V \in \{0, 1\}^n | V \le X, N(V, X) \le t\}$. If $S_t(X) \cap S_t(Y) = \varnothing$, then C is a t-AEC code.

In the following sections, we investigate how to derive a new approach to constructing the OOC with index correlation $\lambda = 1$ and discuss its characteristics.

3. NEIGHBOR DIFFERENCE APPROACH

In this approach, we make use of both the recursive DM and direct methods by relying on the DoP [15] taken from the EDoP [11] to suggest a new neighbor difference method for constructing an $(n, w, 1)$-OOC. Introduced by [10], the neighbor difference aims to manipulate the sequence to yield the required pattern.

According to the definition of the difference matrix D, whereby the difference between points of different rows can be represented exactly once ($\lambda = 1$), if the condition of automorphism, the correlation and Hamming correlation properties [30], condition of optimal CP consequently of the CB with DL $L = \varnothing$, are achieved, then D can be defined as the difference correlation matrix by considering only the differences between 1-bit positions in the CB. Based on weighted positions, a 1-bit position is denoted by b_i at position i while b_i^x denotes a 1 bit at position i of vector X. According to the properties of the difference correlation operation:

- Auto-correlation: for each X, a code vector taken in a weighted representation of the DoP, any integer $d_{i,j}^x = b_i^x - b_j^x \ : \ i > j = 1, 2, \ldots, w$ can be represented as difference in at most $\lambda = 1$ block.
- Cross-correlation: similarly, for every pair (X, Y), $X \ne Y$, code vectors considered in a weighted representation of the DoP, any integers $d_{i,j}^x = b_i^x - b_j^x$, and $d_{i,j}^y = b_i^y - b_j^y \ : \ i > j = 1, 2, \ldots, w$, with $d_{i,j}^x \ne \delta_{i,j}^y \ne 0$, can be represented as the difference in at most $\lambda = 1$ block.

In these conditions, the difference correlation matrix for a code vector X can be given by

$$D^x = \begin{bmatrix} 0 & d_{1,2}^x & d_{1,3}^x & d_{1,4}^x & \cdots & d_{1,w}^x \\ d_{2,1}^x & 0 & d_{2,3}^x & d_{2,4}^x & \cdots & d_{2,w}^x \\ d_{3,1}^x & d_{3,2}^x & 0 & d_{3,4}^x & \cdots & d_{3,w}^x \\ d_{4,1}^x & d_{4,2}^x & d_{4,3}^x & 0 & \cdots & d_{4,w}^x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{w,1}^x & d_{w,2}^x & d_{w,3}^x & d_{w,4}^x & \cdots & 0 \end{bmatrix} \quad (1)$$

where D^x represents the CDP and is equivalent to an EDoP. It can be decomposed into two triangular matrices. The two triangular matrices have a complementary cyclic shift operation. Exploiting this symmetry, each element of the upper triangular matrix (UTM) and its complementary element from the lower triangular matrix (LTM), i.e., $\delta_{i,j} = n - \delta_{i,j}$ modulo n operation constitute the same element for the block. We make use of the UTM, consisting of DoP elements as defined in the Eq. (2), from where we can derive the neighbor difference matrix.

$$D^x_{UTM} = \begin{bmatrix} 0 & d^x_{1,2} & d^x_{1,3} & d^x_{1,4} & \cdots & d^x_{1,w} \\ 0 & 0 & d^x_{2,3} & d^x_{2,4} & \cdots & d^x_{2,w} \\ 0 & 0 & 0 & d^x_{3,4} & \cdots & d^x_{3,w} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d^x_{w-1,w} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (2)$$

3.1 Neighbor Difference Packing Description

3.1.1 Definition

For a given code vector X, the i-th neighbor set is the $X_i = \{\delta^x_{i,1}, \delta^x_{i,2}, \ldots, \delta^x_{i,w-1}\}$ where $\delta^x_{i,j}$ represents the difference in position between the $(j+i)$-th 1 bit and the j-th 1 bit of code vector X and is given by $\delta^x_{i,j} = \left|b^x_{j+i} - b^x_j\right|$ [10] where b^x_j is the position of the j-th 1 bit in X and $i = j = 1, 2, \ldots, w$.

This neighbor difference approach is illustrated by the following example and compared with the DoP.

Example 2: For the neighbor difference, let $X = \{1101000\}$ be a $(7, 3, 1)$ code with one code word. $X_1 = \{1, 2\}$ is the first neighbor set, $X_2 = \{3\}$ is the second neighbor set and the union is $X_1 \cup X_2 = \{1, 2, 3\}$, which is the same as the DoP in Example 1.

According to Example 2 $\delta^x_{i,j} = |b^x_{j+i} - b^x_j|$, the i-th 1-bit neighbor of the j-th 1 bit can be interpreted as the difference of position between the $(j+i)$-th 1 bit and the j-th 1 bit of vector X in a weighted representation. It makes sense to compute the DoP to derive the neighbor difference by further letting the position $p = j + i$.

Moreover, as a block code represented only by 1 bits, it is convenient to replace X by $B = \{b_j\}$, and if there are several blocks, they will be differentiated by B_k, where the k species of the k-th block with the i-th neighbor of the j-th 1-bit block B_k being $\delta^k_{i,j} = |b^k_{j+i} - b^k_j|$.

Definition: The neighbor difference of an (n, w, λ)-OOC can be considered in this case as a family F of $w-1$ subsets of i-th neighbor positions $\{\delta_{i,j} : 1 \leq i, j \leq w\}$, such that the difference of position in the 1-bit block B_k based on the i-th neighbor $\delta^k_{i,j} = |b^k_{j+i} - b^k_j|$ is represented in, at most, $\lambda = 1$ block. Hence, the following neighbor correlation properties:

- Auto-correlation: for each $b^k_j \in B_k$, where b^k_j is the j-th 1 bit of a block B_k, any integer $\delta^k_{i,j} \neq 0$ called i-th neighbor of j-th 1 bit can be represented as the difference $\{\delta^k_{i,j} = |b^k_{j+i} - b^k_j| : i, j = 1, 2, \ldots, w\}$ in at most $\lambda = 1$ block.
- Cross-correlation: for each $b^k_j \in B_k$ and $b^l_u \in B_l$, two blocks of the neighbor difference, any integer $\delta_{i,j} \neq \delta_{i,u} \neq 0$ can be represented as the difference $\{\delta^k_{i,j} = |b^k_{j+i} - b^k_j| : i = 1, 2, \ldots, w\}$ and $\{\delta^l_{i,u} = |b^l_{u+i} - b^l_u| : i = 1, 2, \ldots, w\}$, $j, u = 1, 2, \ldots, w$, in at most $\lambda = 1$ block.

Accordingly, if the $j + i$-th 1 bit is at position $p = i + j$, then its i-th neighbor will necessarily be at position $p + i$. Consequently, the DoP $d_{p-i,p} = \left|b^{(\cdot)}_p - b^{(\cdot)}_{p-i}\right|$ can be reduced to the neighbor difference $\delta_{i,j} = \left|b^{(\cdot)}_{j+i} - b^{(\cdot)}_j\right|$. Based on this change of variable, the UTM, which is the DoP, yields the neighbor difference matrix, given by:

$$\begin{bmatrix} \delta^k_{1,1} & \delta^k_{2,1} & \cdots & \delta^k_{w,1} \\ 0 & \delta^k_{1,2} & \cdots & \delta^k_{w,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta^k_{1,w-2} \end{bmatrix}. \quad (3)$$

The specificity of Eq. (3) is that its diagonal and first row elements yield the same 1-bit position in the code vector. The diagonal of Eq. (3) yields the first neighbors, while the first row provides the required delay, each containing the entire sequence information. This result aligns the position of 1 bits with the timing of different pulses, and hence, a time-space linearization.

4. CONSTRUCTION OF (n,w,1)-OOC

4.1 Considerations

The construction of the OOC using the neighbor difference approach is motivated by its simplicity and a reduction in the encoding/decoding complexity. First, we need to consider certain aspects.

1. Cycle length: from the theoretical CB, the cycle length is the number of blocks in the orbit. If the order of the cycle length is equal to the code length, then the blocks are of full orbit, otherwise, they are of short orbit [23]. In this construction, we are interested in the CB of the full orbit.
2. Neighbor matrix and its complementary elements: if B_k is a CB modulo n, a cyclic shift operation of $\delta^k_{i,j}$ yields its complementary element, given by $\delta^k_{j,i} = n - \delta^k_{i,j}$, both constitute the same element of the block. Hence, the pair $(\delta^k_{i,j}, \delta^k_{j,i})$ is determined by finding one exclusive element or another. Accordingly, the diagonal elements of the Eq. (3) constitute the first neighbor elements of each 1-bit position of the vector and can be put in the set representation and ultimately in vector form. This set is very important since it provides us with all the information on the sequence, i.e. $\text{Diag} = \{\delta^k_{1,1}, \delta^k_{1,2}, \ldots, \delta^k_{1,w-1}\}$. As illustrated in the following example.

Example 3: Let C be the code of one vector $X = \{1101000\}$ assumed to be a full orbit block. Its neighbor differences are given in the form of pairs $(\delta^1_{i,j}, \delta^1_{j,i})$, as $(1, 6), (2, 5), (3, 4)$ (each pair with its complementary element). These are put into a difference correlation Eq. (1). Considering, for instance, that in the pair $(1, 6)$, 1 is equivalent to having 6 in a full orbit bock, Eq. (1) can be reduced to LHS of matrix in Eq. (3) which is called the neighbor difference matrix. The neighbor difference can be obtained from the CDP in Eq. (3) as in the following matrices:

$$\begin{bmatrix}0&1&3\\6&0&2\\4&5&0\end{bmatrix}=\begin{bmatrix}0&0&0\\6&0&0\\4&5&0\end{bmatrix}+\begin{bmatrix}0&1&3\\0&0&2\\0&0&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&3\\0&2\end{bmatrix}.$$

Diag $= \{1, 2\}$ and the code can be represented assuming a fixed 1 bit at 0 as $\{0, 1, 2 + 1\} = \{0, 1, 3\}$.

3. Code length bounds: two bounds are investigated for a given n and w, of which a unique solution is possible.
   - Length of upper bounds: according to Johnson's bound $n = w(w-1)|C| + 1$, the maximal code length achievable for the same cardinality $|C| = l$ blocks is given by $n_{\max} = w(w-1)l + n'$ where $1 \le n' \le w(w-1)$.
   - Length of lower bounds: for a code vector X, the first and the i-th neighbors of the j-th 1 bit are $\delta^x_{1,j} = |b^x_{j+1} - b^x_j|$ and $\delta^x_{i,j} = |b^x_{j+i} - b^x_j|$. If $\delta^x_{i,1}$ is the i-th farthest neighbor from the j-th 1 bit which is assumed position 1 ($j = 1$) and the sequence of 1 bit is such that $\delta^x_{i,j+1} = \delta^x_{i,j} + 1$ for any $i, j = 1, 2, \dots, w$, then the neighbor difference of the sequence given by $\delta^x_{1,1}\delta^x_{1,2}\dots\delta^x_{1,w-1}$ achieves the minimum code length, given as $n_{\min} = 2\delta^x_{i,1} + 1$.

Therefore, the following propositions are derived, where the *Proposition 2* from [23].

Proposition 1: For any family of block $F = \{B_1, B_2, \dots, B_l\}$ of an $(n, w, 1)$-OOC, defines the farthest i-th neighbor from the first position of the 0 bit of the starter block B_k of F by $\delta^k_{i,0}$. The code length is then bounded as $2\delta^k_{i,1} + 1 \le n \le w(w-1)l + n'$, where n' is such that $1 \le n' < w(w-1)$ and l is the number of blocks or code words in family F.

Proof: On the one hand, since $1 \le n' \le w(w-1)$, if $n' = 1$, then $n = w(w-1)l+1$, which is Johnson's bounds. If $n' = w(w-1)+1$, then $n = w(w-1)(l+1)+1$, yielding another cardinality $l' = l + 1$. Thus the maximum code length achievable for $|C| = l$ is for $n' = w(w-1)$.

On the other hand, let i-th be the farthest neighbor from the 1 bit which is at position 1, and if the sequence of 1 bit is such that $\delta^x_{i,j+1} = \delta^x_{i,j} + 1$ for any $i, j = 1, 2, \dots, w$, then $\delta_{i,1} = \delta_{i,2} - 1 = (i+1) - 1 = i$, which is the farthest neighbor of the 1 bit at position 1, as is the 1 bit at position 0, which means $\delta_{i,0} = i$. For a full orbit operation with a correlation index of 1 where i and $n - i$ are the same element of a cyclic block, it requires $i \le (n-1)/2$, that is, $\delta_{i,0} \le (n-1)/2$.

Since $(n-1)/2 = w(w-1)/2$, that is $2\delta_{i,0} = n - 1$ or $2\delta_{i,0} = w(w-1)$, consequently $2\delta_{i,0} + 1 = w(w-1) + 1$. Therefore, $w(w-1) + 1 \le n \le w(w-1) + 1 + n'$, or generally $w(w-1)l + 1 \le n \le w(w-1)l + 1 + n'$ are the bounds of the code length required to achieve F for a given $(n, w, 1)$-OOC.

One can say for $l = 1$ that, when $n = w(w-1)+1$, the construction yields only one block or codeword with a unique representation. However, when $n = w(w-1)+1+n'$, the construction still achieves one block or codeword, but with several or different representations. This allows the flexibility to match the number of users observed in the literature. ■

It is important therefore to mention that for a given w and l block, the $(n, w, 1)$-OOC is not achievable for a code length shorter than the lower bound, and for a code length larger than the upper bound above, the number of blocks l' is such that $l' > l$. In this paper, we focus on the bounds as defined above.

Proposition 2: Let the starter block B_k be a $CB(n, w, 1)$, is based on the neighbor differences for any positive integers j, p, as shown in the following block set $B_i = \{b_j + p \pmod{n} : j = 0, 1, \dots, (n-1)/2; p = n - 2j\}$ is also a $CB(n, w, 1)$.

Proof: b_j is a 1 bit at position j, which means $b_j + p = b_{j+p}$ is a 1 bit at position $j+p$. If $p = n-2j$, then $j+p = n-j$, consequently, B_i is made up of $\delta^k_{j,i}$ elements. Since $\delta^k_{i,j}$ and $\delta^k_{j,i}$ are elements of B, then B_i is also a $CB(n, w, 1)$. ■

One can say that the obtained blocks are of full orbit.

4.2 Code Construction

The principle of the neighbor difference approach is based on the difference family construction, which in itself is a difference family. A family is a set of blocks. However, instead of having $n-1$ differences, the number of these operations is limited to $(n-1)/2$ for each block according to Eq. (3). The blocks are all of full orbit with respect to the neighbor difference correlation properties and are derived from the neighbor difference set N_d, where for each block B of the family derived from N_d, any element $a \in N_d$ obtained by the neighbor difference operation is also an element of B if and only if its complementary element $n - a$ is such that $n - a \notin B$, since $n - a$ is obtained through the cyclic shift operation in B. In principle, for a cyclic block construction, it is necessary to make exclusive use of one or another since having a as a neighbor yields $n - a$ by the cyclic shift in B and vice versa. Such a pair of elements is denoted by $(a, n - a)$ and the neighbor difference set to $N_d = \{a\}$. Note that a is the difference between a 1-bit reference and its i-th neighbor, which means $N_d = \cup N_{d_i}$, the union set of all i-the neighbor difference subsets.

To construct the code, it is necessary to find the $(n-1)/2$ elements of Eq. (3). To avoid any confusion between starting position 1 and neighbor difference 1, we will consider 0 as the first position of the 1 bit in the vector for the upcoming notations when constructing the code. We determine the eventual neighbor difference pairs given by: $\Delta = \{(1, n-1), (2, n-2), \dots, (n-2, 2), (n-1, 1)\}$, which by cyclic shift can be decomposed as $\Delta = \{(1, n-1),\ (2, n-2),\ \dots,\ ((n-1)/2, (n+1)/2)\} \cup \{((n+1)/2, (n-1)/2),\ \dots,\ (n-2, 2),\ (n-1, 1)\}$. The different elements of the N_d are selected from Δ such that N_d is the union of the actual i-th neighbor subsets. The block or codeword is derived from the first neighbors of N_d, which correspond to the fixed 1-bit position and

$$\begin{array}{cccccc}
0\to & a\to & b\to & c\to & \cdots\to & d\\
 & & \downarrow & \downarrow & \downarrow & \downarrow\\
 & & b+a & c+b & \cdots & d+\cdots\\
 & & & \downarrow & \downarrow & \downarrow\\
 & & & c+b+a & \cdots & d+\cdots+c\\
 & & & & \downarrow & \downarrow\\
 & & & & \ddots & \vdots\\
 & & & & & \downarrow\\
 & & & & & d+\cdots+c+b+a
\end{array}$$

Fig. 1: *Neighbor difference algorithm for $l = 1$*

$$\begin{array}{ccccc}
0\to a^{(l)}\to & b^{(l)}\to & c^{(l)}\to & \cdots\to & d^{(l)}\\
 & \downarrow & \downarrow & \downarrow & \downarrow\\
 & b^{(l)}+a^{(l)} & c^{(l)}+b^{(l)} & \cdots & d^{(l)}+c^{(l)}\\
 & & \downarrow & \downarrow & \downarrow\\
 & & c^{(l)}+b^{(l)}+a^{(l)} & \cdots & d^{(l)}+\cdots\\
 & & & \downarrow & \downarrow\\
 & & & \ddots & \vdots\\
 & & & & \downarrow
\end{array}$$

Fig. 2: *Neighbor difference algorithm for $l > 1$*

the $w - 1$ first neighbor difference element positions of N_d in $\boldsymbol{B}$, with the requirement that if $\delta_{ij} \in N_d$, then $\delta_{ij} \in \boldsymbol{B}$ if and only if $n - \delta_{ij} \notin \boldsymbol{B}$.

In general, for a given family $\boldsymbol{F} = \{\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_l\}$ of l OOC blocks denoted by $(n, w, 1)$-OOC, there are l neighbor difference sets N_{d_l}, each with neighbor difference elements selected from $\Delta = \{(1, n-1), (2, n-2), \ldots, (d, n-d)\}$ pairs of elements, from where the l blocks are derived. In that case, $n = w(w-1)l + 1$.

These l blocks can be obtained by computing the diagonal elements of RHS of matrix in Eq. (3) for each block known as first neighbors $[\delta^{(l)}_{1,1}\delta^{(l)}_{1,2} \cdots \delta^{(l)}_{1,w-1}]$. The resulting block or code sequence can be written as $\{0,\ \delta^{(l)}_{1,1},\ \delta^{(l)}_{1,1} + \delta^{(l)}_{1,2} + \cdots + \delta^{(l)}_{1,1} + \delta^{(l)}_{1,2} + \cdots + \delta^{(l)}_{1,w-2}\}$ in a set form (fixed 1 bit at position 0). Its vector form is cyclic modulo n and can be constructed by the following solution.

For a code length $n = w(w-1)l+1$, if $a = \delta^{k}_{i,j}$, then its complementary element is $\overline{a} = n - \delta^{k}_{i,j} = n - a$, hence the neighbor difference set of pairs $\Delta_g = \{(a, n-a), (b, n-b), (c, n-c), \ldots, (d, n-d)\}$ where $d = (n-1)/2$. The code is entirely constructed by determining one (exclusive) or another element of each first neighbor pair of the neighbor difference set of pairs and their positions in the vector as shown in Fig. 1.

Therefore, if $a \in N_d$, then $a \in \boldsymbol{B}$ and $\overline{a} \notin \boldsymbol{B}$ and vice-versa, as generalized in Fig. 1 where the rows are the first neighbor elements from which we can derive the positions (diagonal elements) in the code vector.

Keeping in mind that there is always a 1 bit at position 0, by finding $\{a, b, c, \ldots, d\}$ as the first neighbors, the code set representation, which corresponds to the required delay lines for Bragg gratings, is given by $\{0, a, b+a, c+b+a, \ldots, d+\cdots+c+b+a\}$.

To achieve this, two major cases are to be taken into account.

• Case $l = 1$,

If $n > w(w-1)+1$, n odd, there exists $a \neq b \neq c \neq, \ldots, \neq d$ such that $d + \cdots + c + b + a \leq (n-1)/2$.

If $n = w(w-1)+1$ and $w \geq 3$, for a block $\boldsymbol{B}$, obtained from the neighbor difference set N_d, there exists $a, b, c, \ldots, d \in [1, (n-1)/2]$ and $\in \boldsymbol{B}$ such that: $a \neq b \neq c \neq \ldots \neq d$ and $\overline{a}, \overline{b}, \overline{c}, \ldots, \overline{d} \notin \boldsymbol{B}$.

The correlation properties and cyclic block operation impose the following theorem:

Theorem 1: For any element $a, b \in N_d$ with $a \neq b$, $a, b \in \boldsymbol{B}$ if and only if $\overline{a}$ and $\overline{b} \notin \boldsymbol{B}$. That is:

- If $a, b \leq (n-1)/2$, then for any $c, d \in N_d$, $c, d \in \boldsymbol{B}$ if and only if $c \neq a \neq b \neq d$ and $c \neq a + [2[((n-1)/2) - a] + 1]$; similarly, $d \neq a \neq b \neq c$ and $d \neq b + [2[((n-1)/2) - b] + 1]$.
- If $a, b \geq (n-1)/2$, then for any $c, d \in N_d$, $c, d \in \boldsymbol{B}$ if and only if $c \neq a \neq b \neq d$ and $c \neq a - [2[a - ((n+1)/2)] + 1]$; similarly, $d \neq a \neq b \neq c$ and $d \neq b - [2[b - ((n+1)/2)] + 1]$.

Proof: Assume $a \in N_d$ and $a \in \boldsymbol{B}$, if for any c such that $a \neq c \leq (n-1)/2$, then $c = a + [2[((n-1)/2) - a] + 1] \in \boldsymbol{B}$. However, $c = a + [2[((n-1)/2) - a] + 1] = a + (n-1) - 2a + 1 = n - a$ which is the same element as a by the cyclic shift operation in $\boldsymbol{B}$. Hence, $c \neq a + [2[((n-1)/2) - a] + 1]$.

If for any c such that $a \neq c \geq (n-1)/2$, $a \in \boldsymbol{B}$, then $c = a - [2[a - ((n+1)/2)] + 1] \in \boldsymbol{B}$. However, $c = a - [2[a - ((n+1)/2)] + 1] = a - 2a + n + 1 - 1 = n - a$ which is the same element as a by the cyclic shift operation in $\boldsymbol{B}$. Hence, $c \neq a - [2[a - ((n+1)/2)] + 1]$.

One can say that if a is a position padded with 1 bit in the vector $\boldsymbol{B}$, then the $a + [2[((n-1)/2) - a] + 1]$ position must be padded with 0 bit. ■

• Case $l > 1$, there are more than 1 block, or better, exactly l blocks to construct. The neighbor differences from a to d are distributed among different l blocks labeled $a^{(l)}$ to $d^{(l)}$ as in Fig. 2 where $a^{(l)}$ is element a of block l.

If $n > w(w-1)l + 1$, n odd, there are always $\boldsymbol{B}_1 \cdots \boldsymbol{B}_l$ blocks made of elements $a^{(l)} \neq b^{(l)} \neq c^{(l)} \neq \ldots \neq d^{(l)} \ldots \neq a^{(l)} \neq b^{(l)} \neq c^{(l)} \ldots \neq d^{(l)} \in N_d, \ldots, N_{d_l}$ such that $\overline{a}^{(l)} \neq \overline{b}^{(l)} \neq \overline{c}^{(l)} \neq \ldots \neq \overline{d}^{(l)} \ldots \neq \overline{a}^{(l)} \neq \overline{b}^{(l)} \neq \overline{c}^{(l)} \ldots \neq \overline{d}^{(l)} \notin \boldsymbol{B}_1, \ldots, \boldsymbol{B}_l$.

If $n = w(w-1)l + 1$ and $w \geq 4$, the $a^{(l)}$ to $d^{(l)}$ first neighbor differences are found and distributed according to the neighbor difference set with at least one neighbor at a position $> (n-1)/2$.

$$\begin{aligned}\Delta_{l>1} = \{&(a^{(1)}, n-a^{(1)}), (b^{(1)}, n-b^{(1)}), \ldots,\\ &(d^{(1)}, n-d^{(1)}), \ldots, (a^{(l)}, n-a^{(l)}), (b^{(l)}, n-b^{(l)}),\\ &(c^{(l)}, n-c^{(l)}), \ldots, (d^{(l)}, n-d^{(l)})\},\end{aligned}$$

such that as case $l = 1$ under the conditions above, one can add the following cyclic block correlation properties.

Theorem 2: For this case:

- If $a^{(k)} \leq (n-1)/2 \in N_{d_k}$ and B_k respectively, and $b^{(l)} \leq (n-1)/2 \in N_{d_l}$ and B_l respectively, $l \neq k$, then for any $c^{(k)}, d^{(l)} \in N_{d_k}$ and N_{d_l} respectively, $c^{(k)} \in B_k$, $d^{(l)} \in B_l$ if and only if $c^{(k)} \neq a^{(k)} \neq b^{(l)} \neq d^{(l)}$ and $c^{(k)} \neq a^{(k)} + [2[((n-1)/2) - a^{(k)}] + 1]$; similarly, $d^{(l)} \neq a^{(k)} \neq b^{(l)} \neq c^{(k)}$ and $d^{(l)} \neq b^{(l)} + [2[((n-1)/2) - b^{(l)}] + 1]$.
- If $a^{(k)} \geq (n-1)/2 \in N_{d_k}$ and B_k, $b^{(l)} \geq (n-1)/2 \in N_{d_l}$ and B_l, then for any $c^{(k)}, d^{(l)} \in N_{d_k}$ and N_{d_l} respectively, $c^{(k)} \in B_k$, $d^{(l)} \in B_l$ if and only if $c^{(k)} \neq a^{(k)} \neq b^{(l)} \neq d^{(l)}$ and $c^{(k)} \neq a^{(k)} - [2[a^{(k)} - ((n+1)/2)] + 1]$; similarly, $d^{(l)} \neq a^{(k)} \neq b^{(l)} \neq c^{(k)}$ and $d \neq b^{(l)} - [2[b^{(l)} - ((n+1)/2)] + 1]$.

Proof: see Theorem 1 ■

Example 4: Let us construct an $(n, 3, 1)$-OOC. The construction of this code is given here within the bounds of its length: $2\delta_{i,1} + 1 \leq n \leq w(w-1)l + n'$.

For $l = 1$, $w = 3$, we get $n = 7$. The code has one block with the neighbor difference set to $\Delta = \{(1,6),(2,5),(3,4)\}$. If $a = 1$, then $b \neq 1$, $b \neq n-1 = 7-1 = 6$, $1+b \neq 1$, $1+b \neq 6$. If $b = 2$, then $b+1 \neq 5$ or alternatively, if $b = 3$, $b+1 \neq 4$ which are complementary $(3,4)$, the only solution is $b = 2$.

The elements of the sequence are determined according to the following diagram:

0 →	a→	b	0 →	1→	2
		↓			↓
		$b+a$			3

The solution in the set form is $\{0,1,3\}$ or in the vector form 1101000.

Example 5: For $l = 2$, $n = 13$, $w = 3$, the code allows two blocks which can be written according to the following two-block diagram:

Block	$B_{(l=1,2)}$	$w = 3$
0 →	$a^{(l=1,2)}$ →	$b^{(l=1,2)}$
		↓
		$a^{(l=1,2)} + b^{(l=1,2)}$

resulting in

Block	B_1	$w = 3$	Block	B_2	$(w = 3)$
0 →	1 →	3	0 →	2 →	5
		↓			↓
		4			7

The neighbor difference set is given by the pair elements $\Delta_{l=2} = \{(1,12),(2,11),(3,10),(4,9),(5,8),(6,7)\}$, and are distributed as follows for both blocks having bit 1 at position 0. $a^{(1)} = 1$ and $a^{(2)} = 2$, therefore we must find $b^{(l)}$ and $b^{(2)}$ such that:

- $b^{(1)} \neq 1$ and $b^{(1)} \neq 2$, $b^{(2)} \neq 1$ and $b^{(2)} \neq 2$;
- $b^{(1)} + 1 \neq 1$ or $b^{(1)} + 1 \neq 12$ and $b^{(1)} + 1 \neq 2$ or $b^{(1)} + 1 \neq 11$;
- $b^{(2)} + 2 \neq 1$ or 12 and $b^{(2)} + 2 \neq 2$ or 11;
- $b^{(1)} + 1 \neq b^{(2)} + 2$ or $n - (b^{(2)} + 2)$;
- $b^{(2)} + 2 \neq b^{(1)} + 1$ or $n - (b^{(1)} + 1)$.

The solution that satisfies the condition above requires $b^{(1)} = 3$ and $b^{(2)} = 5$, taking into account that the "or" in this example is exclusive, yielding the set $\{0,1,4\}$ for the starter block B_1 corresponding to the vector 1100100000000 and B_2 set $\{0,2,7\}$ corresponding to the second vector 1010000100000.

Therefore, a $(13,3,1)$-OOC yields two blocks for the family $F = \{B_1, B_2\}$ with $B_1 = \{0,1,4\}$ and $B_2 = \{0,2,7\}$. Section 5.4 demonstrates how this family can be used to construct user signatures for an OCDMA system.

Example 6: For $l = 3$, $n = 19$, $w = 3$, the code allows three blocks which can be written according to the following two-block diagram:

Block	$B_{i=1,2,3}$	$w = 3$	Block	B_1	$(w = 3)$
0 →	$a^{(i)}$ →	$b^{(i)}$	0 →	1 →	4
		↓			↓
		$a^{(i)} + b^{(i)}$			5
Block	B_2	$w = 3$	Block	B_3	$(w = 3)$
0 →	2 →	6	0 →	3 →	7
		↓			↓
		8			10

which corresponds to $F = \{B_1, B_2, B_3\}$ with $B_1 = \{0,1,5\}$, $B_2 = \{0,2,8\}$ and $B_3 = \{0,3,10\}$.

Example 7: In the same way as $n = 21$, $w = 5$, the code allows 1 block as in the following diagram derived from Fig. 1:

0 →	a→	b →	c →	d
		↓	↓	↓
		$b+a$	$c+b$	$d+c$
			↓	↓
			$c+b+a$	$d+c+b$
				↓
				$d+c+b+a$

which corresponds to the set $\{0,1,4,14,16\}$ or in a vector form 110010000000001010000.

Table 1 provides some of the resulting $(n, w, 1)$-OOC constructions where we revert back to position 1 in a vector, along with the number of users such a code will support in an OCDMA system.

5. OOC CHARACTERIZATION FOR OCDMA EVALUATION

5.1 Complexity Reduction

5.1.1 Coding Complexity

For $l = 1$ and according to Eq. (3), the construction of the OOC using the neighborhood approach necessitates $w(w-1)/2$ operations. This complexity is reduced to the diagonal of the RHS of matrix in Eq. (3), allowing the linearization of pulses in time and space. This time-space plane linearization is required for an FFH-OCDMA using the Bragg gratings encoder.

Table 1: *Some constructed OOC.*

n	w	l	Blocks	2-D Users
7	3	1	$\{1,2,4\}$	7
13	3	2	$\{1,2,5\},\{1,3,8\}$	26
21	3	3	$\{1,2,6\},\{1,3,9\},\{1,4,15\}$	63
13	4	1	$\{1,2,4,10\}$	13
15	4	1	$\{1,2,4,8\}$	15
25	4	2	$\{1,2,5,11\},\{1,3,10,15\}$	50
31	5	1	$\{1,2,5,15,17\}$	31
41	5	2	$\{1,2,5,10,21\},\{1,3,9,16,26\}$	82
133	4	11	$\{1,2,14,44\},\{1,3,17,49\},\{1,4,19,54\},\{1,5,22,60\},\{1,6,25,58\},\{1,7,27,63\},\{1,8,30,79\},\{1,9,32,64\},\{1,10,35,80\},\{1,11,38,96\},\{1,12,40,105\}$	1462

For $l > 1$, we have $w(w-1)/2$ differences due to the full orbit block consideration. A further $w(w-1)/4$ differences are carried out with $w(w-1)/4$ permutations added, yielding $w(w-1)$ operations in total. Since half of the differences are sufficient for the neighbor approach, it means $w(w-1)/2$ neighbor differences are required to construct a starter block. Therefore, the computational complexity for l blocks is $lw(w-1)/2$.

5.1.2 Decoding Complexity

The first neighbors are sufficient to reconstruct the information. Therefore, as a good correlation code, the detection of an $(n, w, 1)$-OOC can be achieved by considering the diagonal elements of matrix in Eq. (3) given by the set $B_k = \{1, \delta^k_{1,1}, \delta^k_{1,2}, \ldots, \delta^k_{1,w-1}\}$. The detection in vector form is $[b_1\ b_2\delta^k_{1,1}\ b_3(\delta^k_{1,2}+\delta^k_{1,1}) \cdots b_w(\delta^k_{1,1}+\delta^k_{1,2}+\cdots+\delta^k_{1,w-1})]$, for any given $\{b_1, b_2, \ldots, b_w\}$ symbol set, yielding a decoding complexity of $2(w-1)$.

Table 2 compares the encoding/decoding complexities of the theory and the proposed new method.

5.1.3 Processing Gain and Transfer Transparency

During each signaling interval characterized by the bit duration T_b and the chip duration T_c, the processing gain is defined as $G_p = T_b/T_c$. It is $2w(w-1) = 12$ for the DOP, and $2(w-1) = 4$ for the neighbor difference. As a result, since the data rate is proportional to the chip rate and inversely proportional to the code length, one gets three times the data rate of the DoP.

Having a processing gain as a linear function of the Hamming weight, and since the signal power is proportional to the energy per bit E_b and inversely proportional to G_p, as long as $G_p = f(w)$ is linear for the OOC, one will always have the appropriate sent information power-wise, and hence transfer transparency capability.

5.2 OOC as Asymmetric Error Correcting Code

As a time spreading code of an OCDMA system with embedded AEC capabilities, we can now outline

Table 2: *Computational complexity comparison.*

Construction scheme	Blocks	Coding	Decoding
Prime sequences [17]	1	$O(w^2)$	$O(w^2)$
Quasi Prime sequences [17]	1	$O(w^2)$	$O(w^2)$
MOLSs/MOLRs [13]	1	$O(w^3)$	$O(w^3)$
EDoP [17]	1	$O(w^3)$	$O(w^3)$
DoP [18]	1	$O(w^4)$	$O(w^4)$
Neighbor difference	1	$O(w^2)$	$O(w)$
Neighbor difference	l	$O(w^2 l)$	$O(wl)$
FFH-OCDMA application	1	$O(w)$	$O(w)$

Table 3: *Vectors compared against their erroneous versions.*

Code word	Error vectors
1101000	1100000, 1001000, 0101000
0110100	0110000, 0100100, 0010100
1100100000000	0100100000000, 1000100000000, 1100000000000
1010000100000	0010000100000, 1000000100000, 1010000000000

the related OOC AEC metrics and subsequently give its decoding rule. From here, the following properties can be derived for an $(n, w, 1)$-OOC:

1. Property 1: the asymmetric distance is such that $w-1 \le \Delta(X,Y) \le w$;
2. Property 2: the Hamming distance is such that $2(w-1) \le d(X,Y) \le 2w$;
3. Property 3: the minimum Hamming distance is $d_{\min} = 2(w-1)$.

From Property 1, we can write $\Delta(X,Y) > (w-1)-1$. According to the AEC property, C is a t-AEC code if the condition $\Delta(X,Y) > t$ holds. Therefore $t \ge (w-1)-1$, hence the following proposition.

Proposition 3: An $(n, w, 1)$-OOC can correct $t \ge (w-1)-1$ asymmetric errors. In general, an (n, w, λ)-OOC can correct $t \ge (w-1)-\lambda$ asymmetric errors.

From Property 1, $w-1 \le \Delta(X,Y)$, that is $\Delta(X,Y) \ge (w-1)$, which means $\Delta(X,Y) > (w-1)-1$. By letting $t = (w-1)-1$, one gets $\Delta(X,Y) > t$, Therefore $t \ge (w-1)-1$.

Example 8: For an $(7,3,1)$-OOC, $t = (3-1)-1 = 1$. Accordingly, Table 3 shows two different code vectors and their respective asymmetric error vectors. One can realize that it is always possible to correct $t = 1$ error.

From Properties 2 and 3, the Hamming distance, in general, is $d \ge 2(w-\lambda)$ depending on the length of the code with the minimum distance being $d_{\min} = 2(w-\lambda)$. As such, in general, the code as constructed is the largest size and can detect and correct any t errors, where t is such that $d_{\min} = 2t+1$. This is in line with the literature based on algebraic coding theory. As we know, a fundamental issue in algebraic coding is finding, in our case (n, w, λ)-OOC, the largest error correcting code with

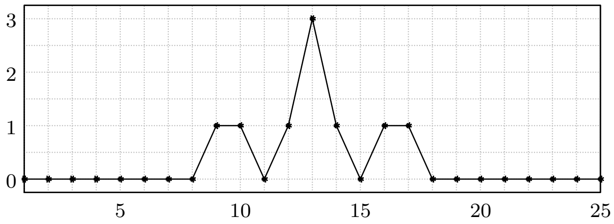

Fig. 3: *Auto-correlation of (13,3,1)-OOC with block {1,2,5}.*

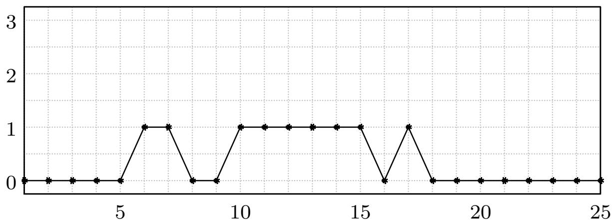

Fig. 4: *Cross-correlation of (13,3,1)-OOC with blocks {1,2,5} and {1,3,8}.*

length n, and the minimum distance $d_{\min}$ between every pair of codewords. To show that our result is in line with the previous results, we refer to the work of [31] and the [15, Th. 1].

According to [31], if $A(n, d_{\min}, w)$ is the maximum size for an error correcting code (n, w, λ)-OOC, then $A(n, d_{\min}, w) \leq n(n-1)\cdots(n-w+\eta)$ where $\eta = d_{\min}/2$.

Furthermore, according to [15, Th. 1], (n, w, λ)-OOC $\leq A(n, d_{\min}, w)/n \leq (n-1)(n-2)\cdots(n-\lambda)/w(w-1)\cdots(w-\lambda)$.

Taking into account this theorem and considering that $\lambda = 1$, one gets: $n(n-1)/nw(w-1) \leq (n-1)/w(w-1)$ which is true. Therefore, as in the literature, the number of t errors the code can correct for $\lambda = 1$ is such that $2(w-1) = 2t+1$, or $t = \lfloor w - 1 - (1/2) \rfloor$, which corresponds to our result $t \geq (w-1)-1$.

5.3 Error Correction: Majority rule

Relying on the AEC with the correcting capability of $w - 1 - \lambda$ errors, the detection of $w - \lambda$ 1 bits contains enough information to determine which sequence was sent.

5.4 OOC as an Access and Time Spreading Code

Figs. 3 and 4 demonstrate the testing of a specific code's correlation, namely the $(13, 3, 1)$-OOC constructed in Example 5, aligning with the correlation properties, and yielding the same results as [32]. These properties allow the discrimination of users and detect signals in the network.

The number of blocks l or cardinality is proportional to the code length, reducing the flexibility for choosing the number of users, contrary to [13].

To match the flexibility for choosing the number of users in [13], a 2-D code was suggested. which we adapted to yield ln number of users when combined with a 2-D gratings system. The number of users is a linear function of the number of blocks. If l is the number of blocks or code words, then the total number of users is ln as illustrated in Tables 4 and 5. Two blocks of a $(13, 3, 1)$-OOC are presented, giving 26 users when the encoder is considered to be non-tunable. Each user is assigned a particular sequence that depicts the time-frequency pattern in a (λ_i, T_i) WH/TS, whereby each user transmits and receives his allocated frequency sequence.

Table 4: *2D Construction with (13,3,1)-OOC.*

Blocks	Cyclic Blocks	Users	2D sequence
$\{1,2,5\}$	1100100000000	1	$\lambda_1\lambda_2 00\lambda_5 00000000$
	0110010000000	2	$00\lambda_3 0\lambda_5 0000\lambda_{10} 000$
	0011001000000	3	$00\lambda_3\lambda_0 0\lambda_7 000000$
	0001100100000	4	$000\lambda_4\lambda_5 00\lambda_8 00000$
	0000110010000	5	$0000\lambda_5\lambda_6 00\lambda_9 0000$
	0000011001000	6	$00000\lambda_6\lambda_7 00\lambda_{10} 000$
	0000001100100	7	$000000\lambda_7\lambda_8 00\lambda_{11} 00$
	0000000110010	8	$0000000\lambda_8\lambda_9 00\lambda_{12} 0$
	0000000011001	9	$00000000\lambda_9\lambda_{10} 00\lambda_{13}$
	1000000001100	10	$\lambda_1 00000000\lambda_{10}\lambda_{11} 00$
	0100000000110	11	$0\lambda_2 00000000\lambda_{11}\lambda_{12} 0$
	0010000000011	12	$00\lambda_3 00000000\lambda_{12}\lambda_{13}$
	1001000000001	13	$\lambda_1 00\lambda_4 00000000\lambda_{13}$
$\{1,3,8\}$	1010000100000	14	$\lambda_1 0\lambda_3 0000\lambda_8 00000$
	0101000010000	15	$0\lambda_2 0\lambda_4 0000\lambda_9 0000$
	0010100001000	16	$0\lambda_2\lambda_2 00\lambda_6 0000000$
	0001010000100	17	$000\lambda_4 0\lambda_6 0000\lambda_{11} 00$
	0000101000010	18	$0000\lambda_5 0\lambda_7 0000\lambda_{12} 0$
	0000010100001	19	$00000\lambda_6 0\lambda_8 0000\lambda_{13}$
	1000001010000	20	$\lambda_1 00000\lambda_7 0\lambda_9 0000$
	0100000101000	21	$0\lambda_2 00000\lambda_8 0\lambda_{10} 000$
	0010000010100	22	$00\lambda_3 00000\lambda_9 0\lambda_{11} 00$
	0001000001010	23	$000\lambda_4 00000\lambda_{10} 0\lambda_{12} 0$
	0000100100101	24	$0000\lambda_5 00000\lambda_{11} 0\lambda_{13}$
	1000010000010	25	$\lambda_1 0000\lambda_6 00000\lambda_{12} 0$
	0100001000001	26	$0\lambda_2 0000\lambda_7 00000\lambda_{13}$

Table 5: *Cardinality in terms of users.*

Code length	Blocks	2D Maximum Users
$n = w(w-1)l + 1$	$l = 1$	$\leq ln$
$w \geq 4$	$l > 1$	$\leq ln$
$n > w(w-1)l + 1$	$l = 1$	$\leq ln$
$w \geq 3$	$l > 1$	$\leq ln$

6. PERFORMANCE ANALYSIS

6.1 Decoding Complexity and Processing Gain

According to the neighbor correlation matrix in Eq. (3), the diagonal elements, which are the first neighbors of the sequence, are sufficient for retrieving the required information. Therefore, the associated difference decorrelation matrix is given by:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \delta_{1,1}^k & 0 & \cdots & 0 \\ 0 & 0 & \delta_{1,1}^k + \delta_{1,2}^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \delta_{1,1}^k + \cdots + \delta_{1,w}^k \end{bmatrix}. \quad (4)$$

The received sequence $R = \{r_1, r_2, \dots, r_w\}$, where r_i represents the position of the i-th bit 1 received, is correlated with the decorrelator matrix in Eq. (4), which results in the vector:

$$Y = \left[r_1\delta_{1,1} \quad r_2(\delta_{1,1} + \delta_{1,2}) \quad \cdots \quad r_w(\delta_{1,1} + \cdots + \delta_{1,w}) \right].$$

Example 9: Assume two received code vectors $C_1 = 1101000$ and $C_2 = 1010001$ to be discriminatory. The decorrelation matrix is given by:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5)$$

The decorrelation process yields $C_1M = 3 = w$ and $C_2M = 1$. C_1 is the code vector to select.

Due to this time-space linearization and the correlation and decorrelation processes, there are $2w(w-1)$ operations for the DoP family while our approach necessitates $2(w-1)$. This is the appropriate electrical decoding complexity to match the required frequency change speed of an FFH-OCDMA system using the Bragg gratings encoder.

The processing gain obtained from the operation is a linear function of the Hamming weight. Since the autocorrelation value is also proportional to the same value, this exhibits a constant energy per bit, allowing service transparency.

6.2 Error Correction: Majority Rule

Considering Proposition 4 and the majority rule, the detection is reduced to $w-1$ correlations, followed by $w-\lambda$ decorrelation differences. That is $2w-1-\lambda$ decoding operations. In the case of $\lambda = 1$, this number becomes $2(w-1)$ which is the minimum distance of the code.

6.3 Error Probability of the Sequence

Our performance analysis is based on the OCDMA a Z-channel and we assume the correlation process at the photo detector to be Gaussian with power density $N_0/2$. The transmission is on-off keying with NRZ signaling, hence direct detection is used. With the Z-channel, the optimum threshold level for signal detection using Leibniz's approximation is given by $(w/n)\sqrt{E_b/T_b}$. The average error probability of symbol 0 being detected when symbol 1 in a binary scheme has been sent is therefore given by

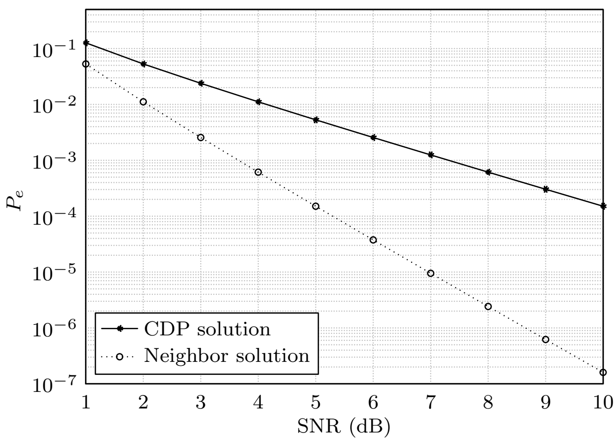

Fig. 5: *Error probability comparison of the proposed solution and DoP for $n = 7$.*

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{n-w}{n}\sqrt{\frac{E_b}{N_0}}\right). \quad (6)$$

The detection of the transmitted signal impaired by MAI and beat noise when using our construction has the probability of bit error

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{n-w}{n}\sqrt{\frac{wE_b}{N_0}}\right). \quad (7)$$

When decoding with the majority rule, employing the embedded AEC capability of the code, this probability becomes

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{n-w}{n}\sqrt{\frac{2(w-1)E_b}{N_0}}\right). \quad (8)$$

Note that in the same way, the probability of bit error in the case without a complexity reduction in CP and EDoP is given by:

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{n-w}{n}\sqrt{\frac{(w-1)E_b}{N_0}}\right). \quad (9)$$

6.4 Analysis Results

For a given signal-to-noise ratio as shown in Figs. 5 and 6, one can observe that our approach will experience a lower probability of error than the other difference family, and a further improved BER is experienced when complexity is reduced in detecting user 1.

According to Table 3, $S_{(w-1)-1}(X) \cap S_{(w-1)-1}(Y) = \varnothing$ shows that there is always a possibility to recover the sequence if asymmetric error occurs.

Moreover, the decision to use majority rule decoding for $w-\lambda$ 1-bit detection is equivalent to minimum distance decoding, thereby optimizing the error correction capability of the code.

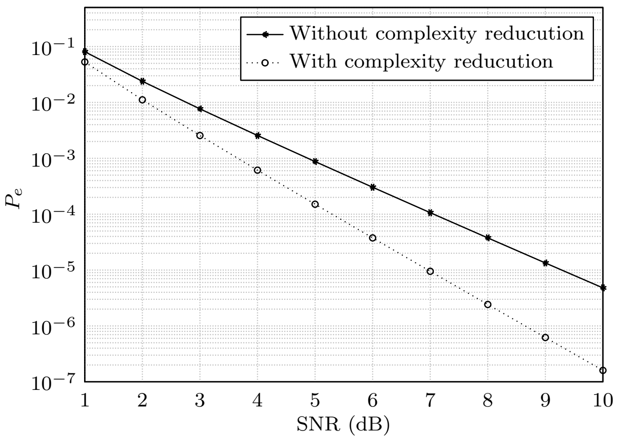

Fig. 6: *Error probability of the proposed solution with and without a decoding complexity reduction for* $n = 7$.

Furthermore, flexibility and simplicity lead to a general solution of $(n, w, 1)$-OOC, for all n and w, henceforth allowing the construction of OOC even for $w \geq 4$ as restricted in the literature. According to Table 5, we can construct optimal OOC yielding ln number of users, except for the incomplete difference family $n \cong 1 \pmod{w(w-1)l}$ for which we will have at most ln number of users.

Finally, the computational complexity is reduced to the order of half $O(w^2)$ compared to $O(w^2)$ or $O(w^3)$ in other schemes or $O(w^4)$ of DoP, while the processing gain has an increase of $\log(w)$ in a log scale. The decoding complexity is reduced to $O(w)$, which is very important in the FFH-OCDMA system since the correlation for the detection necessitates at most w operations during the opto-electric conversion.

7. CONCLUSION

In this paper, we present a simple solution based on neighbor differences to construct the OOC from which a set of criteria is derived to characterize the code for FFH-OCDMA, enhancing the performance of the system. The approach yields a flexibility that allows the construction of an optimal OOC for any value of n within the bounds of the code length and any w. The solution also provides a significant reduction in coding complexity of around half $O(w^2)$. This further becomes $O(w)$ through linearization for the FFH system, while a decoding complexity of around $O(w)$ provides us with a majority rule that equals the embedded minimum distance decoding for the code's AEC capability. This decoding complexity complies with the speed and structure of the FFH-OCDMA system since there are w frequencies to transmit in w time slots within the transmission time interval, reinforcing the system transparency.

However, some limitations need to be addressed:

- The code construction is possible only for a code length larger or equal to $n \geq w(w-1)l + 1$, and in our case, $n = w(w-1)l + 1$ is chosen.
- $n = w(w-1)l + 1$ does not allow the same flexibility in terms of the number of users offered by MOLS, MOLR, and SAC construction techniques. However, since these techniques are feasible for larger n, in this paper we provide the upper bounds of the code length for a given w, whereby we can achieve this flexibility for the number of users.
- The cardinality is sacrificed to reduce interference issues. However, associating the OOC as constructed herein with the BER gratings allow the achievement of FFH in the form of a 2-D solution, enabling the required cardinality to be matched.

The Bragg gratings encoder/decoder yielding the required 2-D code, the improved recursive method, and the mathematical model for short orbit remain topics for further study.

ACKNOWLEDGMENTS

The authors would like to thank B. Twala (previously with the Institute for Intelligent Systems, University of Johannesburg), C. Mbhowa (previously with the Faculty of Engineering and Built Environment, University of Johannesburg), and S. Motala (with the Postgraduate School, University of Johannesburg) for their financial support.

REFERENCES

[1] R. J. McEliece and E. R. Rodemich, "The Constantin-Rao construction for binary asymmetric error-correcting codes," *Information and Control*, vol. 44, no. 2, pp. 187–196, Feb. 1980.

[2] J.-G. Zhang, "Performance improvement of fibre-optic code-division multiple-access systems by using error-correction codes," *IEE Proceedings - Communications*, vol. 144, no. 5, pp. 316–321, Oct. 1997.

[3] K. Kamakura and I. Sasase, "A new modulation scheme using asymmetric error-correcting codes embedded in optical orthogonal codes for optical CDMA," *Journal of Lightwave Technology*, vol. 19, no. 12, pp. 1839–1850, Dec. 2001.

[4] R. M. Wilson, "Cyclotomy and difference families in elementary abelian groups," *Journal of Number Theory*, vol. 4, no. 1, pp. 17–47, Feb. 1972.

[5] S. Johnson, "A new upper bound for error-correcting codes," *IEEE Transactions on Information Theory*, vol. 8, no. 3, pp. 203–207, Apr. 1962.

[6] R. Omrani and P. V. Kumar, "Codes for optical CDMA," in *Sequences and Their Applications – SETA 2006*, Lecture Notes in Computer Science, vol. 4086, G. Gong, T. Helleseth, H.-Y. Song, and K. Yang, Eds. Berlin, Germany: Springer, 2006, pp. 34–46.

[7] Y. Yang and X. Lin, *Code and Cryptography.* Beijing, China: Renmin Youdian Press (in Chinese), 1992.

[8] G. Ge and J. Yin, "Constructions for optimal (v, 4, 1) optical orthogonal codes," *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 2998–3004, Nov. 2001.

[9] R. Fuji-Hara, Y. Miao, and J. Yin, "Optimal ($9v$, 4, 1)

optical orthogonal codes," *SIAM Journal on Discrete Mathematics*, vol. 14, no. 2, pp. 256–266, Feb. 2001.

[10] S. Martirosyan and A. J. H. Vinck, "A construction for optical orthogonal codes with correlation 1," *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, vol. E85-A, no. 1, pp. 269–272, Jan. 2002.

[11] R. C. S. Chauhan, "Design of unipolar (optical) orthogonal codes and their maximal clique sets," Ph.D. dissertation, Faculty of Engineering and Technology, Uttar Pradesh Technical University, Lucknow, India, 2015.

[12] H. Chung and P. V. Kumar, "Optical orthogonal codes-new bounds and an optimal construction," *IEEE Transactions on Information Theory*, vol. 36, no. 4, pp. 866–873, Jul. 1990.

[13] I. B. Djordjevic and B. Vasic, "Novel combinatorial constructions of optical orthogonal codes for incoherent optical cdma systems," *Journal of Lightwave Technology*, vol. 21, no. 9, pp. 1869–1875, Sep. 2003.

[14] E. F. Brickell and V. K. Wei, "Optical orthogonal codes and cyclic block designs," *Congressus Numerantium*, vol. 58, pp. 175–192, 1987.

[15] F. R. K. Chung, J. A. Salehi, and V. K. Wei, "Optical orthogonal codes: design, analysis and applications," *IEEE Transactions on Information Theory*, vol. 35, no. 3, pp. 595–604, May 1989.

[16] J. Yin, "Some combinatorial constructions for optical orthogonal codes," *Discrete Mathematics*, vol. 185, pp. 201–219, Apr. 1998.

[17] R. Fuji-Hara and Y. Miao, "Optical orthogonal codes: their bounds and new optimal constructions," *IEEE Transactions on Information Theory*, vol. 46, no. 7, pp. 2396–2406, Nov. 2000.

[18] M. Buratti, "Cyclic designs with block size 4 and related optimal optical orthogonal codes," *Designs, Codes and Cryptography*, vol. 26, no. 1-3, pp. 111–125, Jun. 2002.

[19] Y. Chang and Y. Miao, "Constructions for optimal optical orthogonal codes," *Discrete Mathematics*, vol. 261, pp. 127–139, 2003.

[20] M. J. Colbourn and R. A. Mathon, "On cyclic Steiner 2-designs," *Annals of Discrete Mathematics*, vol. 7, pp. 215–253, 1980.

[21] C. R. Rao, "Finite geometries and certain derived results in theory of numbers," *Proceedings of the Indian National Science Academy*, vol. 11, pp. 136–149, 1945.

[22] M. J. Colbourn and C. J. Colbourn, "On cyclic block designs," *Mathematical Reports of the Academy of Science*, vol. 2, pp. 95–98, 1980.

[23] M. J. Colbourn and C. J. Colbourn, "Recursive constructions for cyclic block designs," *Journal of Statistical Planning and Inference*, vol. 10, no. 1, pp. 97–103, Jul. 1984.

[24] M. Jimbo and S. Kuriki, "On a composition of cyclic 2-designs," *Discrete Mathematics*, vol. 46, no. 3, pp. 249–255, Dec. 1983.

[25] M. Jimbo, "Recursive constructions for cyclic BIB designs and their generalizations," *Discrete Mathematics*, vol. 116, no. 1-3, pp. 79–95, Jun. 1993.

[26] R. C. Bose and K. A. Bush, "Orthogonal arrays of strength two and three," *The Annals of Mathematical Statistics*, vol. 23, no. 4, pp. 508–524, Dec. 1952.

[27] R. C. S. Chandran, R. Asthana, and Y. N. Singh, "Uni-polar orthogonal codes: design, analysis and applications," in *International Conference on High Performance Computing*, Goa, India, 2010.

[28] T. Kløve, "Error correcting codes for the asymmetric channel," Report, Department of Informatics, University of Bergen, Norway, 1995.

[29] W. Kim and C. Freiman, "Single error-correcting codes for asymmetric binary channels," *IRE Transactions on Information Theory*, vol. 5, no. 2, pp. 62–66, Jun. 1959.

[30] H. Fathallah, L. A. Rusch, and S. LaRochelle, "Optical frequency-hop multiple access communications system," in *1998 IEEE International Conference on Communications (ICC '98)*, vol. 3. Atlanta, GA, USA, 1998, pp. 1269–1273.

[31] F. J. MacWilliams and N. J. A. Sloane, Eds., *The Theory of Error-Correcting Codes.* Amsterdam, The Netherlands: North-Holland Pub., 1977.

[32] H. Yin and D. J. Richardson, *Optical Code Division Multiple Access Communication Networks: Theory and Applications.* Berlin, Germany: Springer, 2009.

Samuel Nlend received the B.Sc. (Physics) degree in 1994 and the M.Phil. (Electrical and Electronic) degree in 2013. Currently he is a Ph.D. candidate and assistant lecturer in the Department of Electrical and Electronic Engineering Science at the University of Johannesburg, South Africa. He worked from 1996 to 2010 respectively as the Regional Head of Department of Statistics and QoS and of Technical Department at Cameroon Telecommunications, Cameroon, and as a Technical Engineer at Phonelines International, South Africa. Mr. Nlend is a member of the Engineering Council of South Africa since 2017 and a student member of IEEE since 2019.

Theo G. Swart received the B.Eng. and M.Eng. degrees (both cum laude) in electrical and electronic engineering from the Rand Afrikaans University, South Africa, in 1999 and 2001, respectively, and the D.Eng. degree from the University of Johannesburg, South Africa, in 2006. He is an associate professor in the Department of Electrical and Electronic Engineering Science and a member of the UJ Center for Telecommunications. His research interests include digital communications, error-correction coding, constrained coding and powerline communications. Dr. Swart is a senior member of the IEEE, and was previously the chair of the IEEE South Africa Chapter on Information Theory. He is a specialist editor for the SAIEE Africa Research Journal.