Uniformly Loaded Square Plate with Partially Simply Supported at the Middle Edges and Point-Column Supported at the Corners: I – Theoretical Formulation

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ABSTRACT

The objective of the present paper is to present an efficient theoretical method to solve the bending problem of uniformly loaded square plate in which the plate is point-column supported at all corners and partially simply supported at all central portion edges. The solution can first be set up by using the Lévy-Nádai's approach and the mixed boundary conditions are then written in the form of dual-series equations. By making use of the proper finite Hankel integral transform, the dual-series equations can further finally be reduced to an inhomogeneous Fredholm integral equation of the second kind. Importantly, the highlight of the problem is that the analytical formulation explicitly considers the moment singularities existed at the ends of partial simple supports.

Keywords: *Dual-series equations, Fredholm integral equation, Hankel integral transform, Mixed boundary conditions, Singularities, Square plate.*

1 INTRODUCTION

It seems that problems on the bending of thin plates have been investigated for almost all combinations of boundary edge conditions. Nevertheless, it is remarkable that much less research has been conducted for studying the static bending problem of plates supported by point column supports, although it commonly encountered in practice. However, some problems for free vibrations of plates with point supports were found in literature.

Azimi [1], who employed the receptance method for formulating the natural frequency and mode shape equations of elastically and rigidly point-supported at an arbitrary point in circular plates. LeClair [2] applied the Green's function to analytically derive the characteristic equation and mode shapes for a circular plate with a free edge and interior, simple supports where the effects of elasticity at the support or a concentrated mass was included in his investigation. Other vibration problems of plates having a concentrated mass rather than a point support were studied in [3]-[8]. Cortinez and Laura [9] presented the approximate determination for forced vibrations of a simply supported polygonal plate having a concentric, rigid, circular inclusion.

In the case of plates subjected to static loads, Lee and Ballesteros [10] proposed an approximate deflection function in the form of polynomials for a uniformly loaded rectangular plate supported at the corners. Their obtained results showed that the error is largest in the vicinity of the corners. Yu and Pan [11] derived the general expressions for determining the deflections of a uniformly loaded circular plate supported at several points equally spaced along the circumference of a concentric circle. Yuhong [12] applied the reciprocal theorem to solve the bending problem of rectangular plates with each edge arbitrary a point supported under concentrated load acting at any point of the plate.

With the best knowledge of the authors, none has considered the problem of square plate supported at the corners and partially simply supported at the edges found in technical journal, except for the seminar paper that presented by Sompornjaroensuk and Dy [13]. Thus, the principal concern of this paper is to analytically treat and present the method of solution for solving the problem of uniformly loaded square plate supported at the corners by point column support. In addition, the plate is also partially simply supported at all four middle edges while the length of partial simple supports can be varied symmetrically with respect to the center of the plate as illustrated in Figure 1.

It can clearly be seen that there are the mixed boundary conditions around the plate edges, which are of type simply supported and free. These conditions can be formulated as dual-series equations and analytically solved by using the Hankel integral transform method. Thus, with this method, the final equation governing the problem solution can be formulated in the form of an inhomogeneous Fredholm-type integral equation of the second kind. An important point to note here is that the inverse-square-root moment singularities [14], [15] due to the discontinuous supports have provided in the analysis.

Before proceeding further to analyze the problem, it is necessary to consider and review some literatures as listed in the followings below.

A related problem was made by Dempsey et al. [16], who analytically investigated the bending of uniformly loaded square plate supported by unilateral supports. The method of finite Fourier integral transforms was used to solve the dual-series equations that led to a Cauchy-type singular integral equation of the first kind.

The same problem as treated previously in [16] was reanalyzed by Sompornjaroensuk and Kiattikomol [17] and problem was analytically reduced to the form of an inhomogeneous Fredholm integral equation by using the finite Hankel integral transform techniques. Kongtong and Sukawat [18] extended the identical method [17] with considerable success to obtain the coupled integral equations for uniformly loaded rectangular plates resting on unilateral supports.

Considering the numerical analyses, Salamon et al. [19] modeled the unilateral supports with discrete elastic springs by means of finite element method. Another numerical method was done by Hu and Hartley [20], who used a direct boundary element method to model the problem. However, it can be observed in two latter numerical methods [19], [20] that they do not consider the singular behavior at the transition points between the plate and the supports.

Another problem was treated by Sompornjaroensuk [21] and Kongtong, Sompornjaroensuk and Sukawat [22] for the bending of uniformly loaded square plate supported by partial simple supports at all four middle edges. Furthermore, some numerical results have shown in Sompornjaroensuk and Dy [23]. It can be noted that since the length of partial simple supports is very small, the problem becomes the plate supported by finite narrow strip columns at all middle edges [24].

However, it is interesting to note that the problems as presented in [16]-[18] and [21]-[24], an inverse-square-root shear singularity is introduced at the ends of partial supports in order to insure the corners of the plate to be lifted up during bending. In the present problem under consideration, all corners of the plate are anchored by point column supports; therefore, an inverse-square-root moment singularity is proper at those ends of partial simple supports.

2 PLATE'S DIFFERENTIAL EQUATION

Consider the plate that demonstrated in Figure 1, the geometry of the plate has scaled by the factor $\pi/\bar{a}$ where $\bar{a}$ is the actual length of square plate.

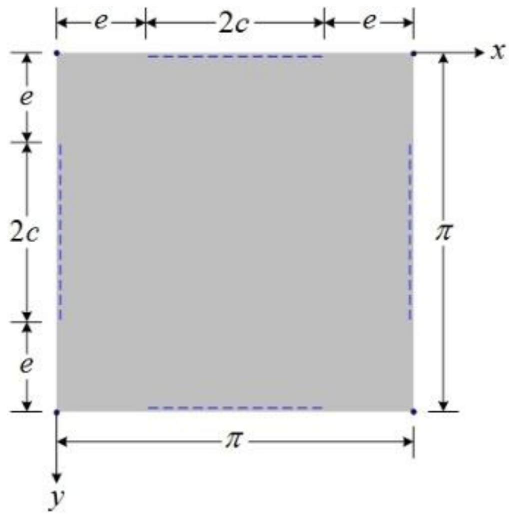

***Fig. 1** Corner-supported square plate with partially simply supported edges.*

Hence, the following relation can be used to obtain the actual coordinates $(\bar{x}, \bar{y})$ and dimensions $(\bar{c}, \bar{e})$,

$$(\bar{x}, \bar{y}, \bar{c}, \bar{e}) = \left(\frac{\bar{a}}{\pi}\right)(x, y, c, e). \tag{1}$$

Within the framework of theory of thin elastic plates [25], the deflection function $w(x,y)$ can be determined by solving the governing partial differential equation,

$$\nabla^4 w(x, y) = \frac{q\bar{a}^4}{\pi^4 D}, \tag{2}$$

with the biharmonic operator given by

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \tag{3}$$

and the bending rigidity of the plate defined as

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{4}$$

where q is a uniform load, h is the thickness of the plate and E, ν are the Young's modulus and Poisson's ratio, respectively.

The deflection function in Eq.(2) can be taken in the form of the Lévy-Nádai's solution. Therefore, the total deflection is the sum of the complementary solution $w_c(x,y)$ and particular solution $w_p(x,y)$,

$$w(x,y) = w_c(x,y) + w_p(x,y)\,, \tag{5}$$

where

$$\begin{aligned} w_c(x,y) = \frac{q\bar{a}^4}{2D}\sum_{m=1,3,5,...}^{\infty} \{ & [A_m\cosh(my) + B_m my\sinh(my) \\ & + C_m\sinh(my) + D_m my\cosh(my)]\sin(mx) \\ & + [A_m\cosh(mx) + B_m mx\sinh(mx) \\ & + C_m\sinh(mx) + D_m mx\cosh(mx)] \\ & \times\sin(my)\}\,, \end{aligned} \tag{6}$$

and

$$w_p(x,y) = \frac{q\bar{a}^4}{2D}\sum_{m=1,3,5,...}^{\infty}\frac{4}{\pi^5 m^5}\left[\sin(mx) + \sin(my)\right]. \tag{7}$$

3 BOUNDARY CONDITIONS

Due to the two-fold symmetry in deflection function, the unknown constants of integration; namely, A_m, B_m, C_m and D_m can be determined from the boundary conditions as follows:

$$\frac{\partial w}{\partial y} = 0:\ y = \frac{\pi}{2}\,;\ 0 \le x \le \frac{\pi}{2}\,, \tag{8}$$

$$V_y = 0:\ y = \frac{\pi}{2}\,;\ 0 \le x \le \frac{\pi}{2}\,, \tag{9}$$

$$M_y = 0:\ y = 0\,;\ 0 \le x \le \frac{\pi}{2}\,, \tag{10}$$

$$w = \frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2} = 0:\ y = 0\,;\ e < x \le \frac{\pi}{2}\,, \tag{11}$$

$$V_y = 0:\ y = 0\,;\ 0 \le x < e\,, \tag{12}$$

in which the bending moment M_y and the Kirchhoff's shearing force V_y are, respectively, defined by [25],

$$M_y = -D\left(\frac{\pi}{\bar{a}}\right)^2\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{13}$$

and

$$V_y = -D\left(\frac{\pi}{\bar{a}}\right)^3\left(\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2\partial y}\right). \tag{14}$$

4 DUAL-SERIES EQUATIONS

Application of the regular boundary conditions that presented in Eq.(8), Eq.(9), and Eq.(10) and together with utilizing Eq.(13) and Eq.(14) leads to the following relations:

$$A_m = \frac{4\nu\eta'}{\pi^5 m^5} + 2D_m\eta'\coth\beta\,, \tag{15}$$

$$B_m = -D_m\coth\beta\,, \tag{16}$$

$$C_m = -\frac{4\nu\eta'\tanh\beta}{\pi^5 m^5} - D_m\left[2\eta' + \beta\left(\tanh\beta - \coth\beta\right)\right], \tag{17}$$

with

$$\eta' = \frac{1}{1-\nu}\,, \tag{18}$$

and

$$\beta = \frac{m\pi}{2}\,. \tag{19}$$

The mixed boundary conditions given in Eq.(11) and Eq.(12) can be written as the dual-series equations, which are:

$$\sum_{m=1,3,5,...}^{\infty} m^2 P_m\sin(mx) = 0\,;\ 0 \le x \le \frac{\pi}{2}\,, \tag{20}$$

$$\begin{aligned} \sum_{m=1,3,5,...}^{\infty} \{ & m^3 P_m(1 + F_m^{(1)})\sin(mx) \\ & + m^3 P_m\left[F_m^{(2)}\sinh(mx) - 2\eta\cosh(mx)\right. \\ & \left. + F_m^{(3)} mx\cosh(mx) - \eta mx\sinh(mx)\right]\} \\ = \sum_{m=1,3,5,...}^{\infty} & \left[F_m^{(4)}\sin(mx) + F_m^{(5)} + F_m^{(6)}\sinh(mx)\right. \end{aligned}$$

$$-F_m^{(5)}\cosh(mx)+F_m^{(7)}mx\cosh(mx)$$
$$\left.-F_m^{(4)}mx\sinh(mx)\right];\ 0\le x<e\,, \qquad (21)$$

where

$$P_m=\frac{2}{\pi^5m^5}+D_m\coth\beta\,, \qquad (22)$$

$$F_m^{(1)}=\frac{(3+\nu)\sinh\beta\cosh\beta-(1-\nu)\beta}{(3+\nu)\cosh^2\beta}-1\,, \qquad (23)$$

$$F_m^{(2)}=\eta(2\tanh\beta+\beta\text{sech}^2\beta)\,, \qquad (24)$$

$$F_m^{(3)}=\eta\tanh\beta\,, \qquad (25)$$

$$F_m^{(4)}=\frac{2\left[(3-\nu)\tanh\beta-(1-\nu)\beta\text{sech}^2\beta)\right]}{(3+\nu)\pi^5m^2}\,, \qquad (26)$$

$$F_m^{(5)}=\frac{4}{(3+\nu)\pi^5m^2}\,, \qquad (27)$$

$$F_m^{(6)}=\frac{2\left[2\tanh\beta+(1-\nu)\beta\text{sech}^2\beta\right]}{(3+\nu)\pi^5m^2}\,, \qquad (28)$$

$$F_m^{(7)}=\frac{2\eta\tanh\beta}{\pi^5m^2}\,, \qquad (29)$$

$$F_m^{(8)}=\frac{2\eta}{\pi^5m^2}\,, \qquad (30)$$

and

$$\eta=\frac{1-\nu}{3+\nu}\,. \qquad (31)$$

5 FINITE HANKEL INTEGRAL TRANSFORM

It can be seen that the problem is now reduced to finding one unknown function P_m in the dual-series equations where this function can be determined by assuming the proper finite Hankel integral transforms.

Since the present problem seems likely to the previous problem [17], [21]-[23] except that the corners of the plate are now anchored in this paper and then, the dual-series equations can be reduced to the form of integral equation of Fredholm-type. However, it has been revealed previously that the proper form of P_m satisfying Eq.(20) and also exhibiting the inverse-square-root moment singularities [14], [15] at the points of discontinuous supports is taken in the following form [21], [22]

$$m^2P_m=\tilde{E}J_1(me)+\int_0^e t\phi(t)J_1(mt)dt\ ;\ m=1,3,5,...\,, \qquad (32)$$

where t is a dummy variable, $\phi(t)$ is an introduced unknown auxiliary function in the Hankel integral transforms, $J_n(u)$ is the Bessel function of the first kind and order n with argument u [26], [27], and the constant $\tilde{E}$ can be determined from the zero deflection condition (w = 0) as presented in Eq.(11) at only one point, i.e., $w(\pi/2,0)$.

Hence, integrating Eq.(20) twice with respect to x and substituting P_m from Eq.(32) with setting $x = \pi/2$ results in, after interchanging the order of summation and integration,

$$\tilde{E}=-\frac{\int_0^e t\phi(t)\sum_{m=1,3,5,...}^{\infty}m^{-2}J_1(mt)(-1)^{\frac{(m-1)}{2}}dt}{\sum_{m=1,3,5,...}^{\infty}m^{-2}J_1(me)(-1)^{\frac{(m-1)}{2}}}\,. \qquad (33)$$

With the use of identity given below [28]-[30],

$$\sum_{m=1,3,5,...}^{\infty}m^{-2}J_1(mt)\sin(mx)=\frac{\pi}{8}t : x\ge t\ ;\ x+t<\pi\,, \qquad (34)$$

and setting $x = \pi/2$ yields

$$\sum_{m=1,3,5,...}^{\infty}(-1)^{\frac{(m-1)}{2}}m^{-2}J_1(mt)=\frac{\pi}{8}t\ ;\ t<\frac{\pi}{2}\,. \qquad (35)$$

In view of Eq.(35), the previous Eq.(33) becomes

$$\tilde{E}=-\int_0^e\frac{t^2}{e}\phi(t)dt\,. \qquad (36)$$

Substituting $\tilde{E}$ into Eq.(32) results in

$$m^2P_m=\int_0^e t\phi(t)\left[J_1(mt)-\frac{t}{e}J_1(me)\right]dt\ ;\ m=1,3,5,... \qquad (37)$$

It is easy to verify that function P_m satisfies Eq.(20). Substituting P_m into Eq.(20) and considering the identity [28]-[30],

$$\sum_{m=1,3,5,...}^{\infty}J_1(mt)\sin(mx)=\frac{xH(t-x)}{2t(t^2-x^2)^{1/2}}\ ;\ x+t<\pi\,, \qquad (38)$$

where $H(\cdot)$ is the Heaviside unit step function [31], leads to the following equation

$$\int_0^e t\phi(t)\left[\frac{xH(t-x)}{2t(t^2-x^2)^{1/2}}-\frac{xtH(e-x)}{2e^2(e^2-x^2)^{1/2}}\right]dt = 0\ ;\ e<x\le\frac{\pi}{2}\ . \tag{39}$$

Since x is always larger than t and e, the Heaviside functions in Eq.(39) are all zero. Thus, Eq.(39) is automatically satisfied.

Similarly, the remaining zero slope condition ($\partial w/\partial x = 0$) that presented in Eq.(11) is also automatically satisfied by integrating Eq.(20) once with respect to x and using the identity as given by,

$$\sum_{m=1,3,5,\dots}^{\infty} m^{-1}J_1(mt)\cos(mx)=\frac{H(t-x)}{2t(t^2-x^2)^{1/2}}\ ;\ x+t<\pi\ , \tag{40}$$

which is derived from a direct integration of Eq.(38) between the limits of 0 and x.

According to the work of Williams [14], the correct singularity at the point of discontinuity is of $O(\varepsilon^{-1/2})$ in the moments or of $O(\varepsilon^{-3/2})$ in the shears where ε is an infinitesimal distance measured from the singular point. Verification is easily made by consideration of Eq.(21) in the following form

$$V_y(e<x\le\pi/2,0)=-2\pi^3q\bar{a}\left\{-\frac{d}{dx}\sum_{m=1,3,5,\dots}^{\infty} m^2P_m\cos(mx)+\sum_{m=1,3,5,\dots}^{\infty} m^3P_mF_m^{(1)}\sin(mx)+\sum_{m=1,3,5,\dots}^{\infty} m^3P_m\left[F_m^{(2)}\sinh(mx)-\dots\right]-\sum_{m=1,3,5,\dots}^{\infty}\left[F_m^{(4)}\sin(mx)+\dots\right]\right\}\ ;\ 0\le x<e\ . \tag{41}$$

Substitution of P_m defined by Eq.(32) into the first term on the right side of Eq.(41) and using the identity as given by [28]-[30],

$$\sum_{m=1,3,5,\dots}^{\infty} J_1(mt)\cos(mx)=\frac{1}{2t}-\frac{xH(x-t)}{2t(x^2-t^2)^{1/2}}+\int_0^{\infty}\frac{I_1(ts)\cosh(xs)}{\exp(\pi s)+1}ds\ ;\ x+t<\pi\ , \tag{42}$$

where $I_n(u)$ is the modified Bessel function of the first kind and order n with argument u [26], [27], Eq.(41) becomes

$$V_y(e+\varepsilon,0)=\pi^3q\bar{a}\tilde{E}\,\frac{e}{2}(2e\varepsilon)^{-3/2}+O(\varepsilon)^{-1/2}+\dots \tag{43}$$

It is obvious that the singularity is in the order of $O(\varepsilon^{-3/2})$ in shearing force. This indicates that there is an inverse-square-root singularity in moment [14], [15].

6 ABEL'S INTEGRAL EQUATION

Having shown that P_m as given in either Eq.(32) or Eq.(37) automatically satisfies all conditions shown in Eq.(11) and there is an inverse-square-root singularity in moment, it remains to reduce Eq.(21) to the form of an integral equation.

Integrating Eq.(21) once with respect to x and substituting P_m given by Eq.(37), after changing the order of integration and summation leads to

$$\int_0^e t\phi(t)\sum_{m=1,3,5,\dots}^{\infty}(1+F_m^{(1)})\left[J_1(mt)-\frac{t}{e}J_1(me)\right]\cos(mx)dt-\int_0^e t\phi(t)\sum_{m=1,3,5,\dots}^{\infty}\left[J_1(mt)-\frac{t}{e}J_1(me)\right]\{(F_m^{(2)}-F_m^{(3)})\times\cosh(mx)+F_m^{(3)}mx\sinh(mx)-\eta\sinh(mx)-\eta mx\cosh(mx)\}dt=-\sum_{m=1,3,5,\dots}^{\infty}F_m(x)\ ;\ 0\le x<e\ , \tag{44}$$

where

$$F_m(x)=-m^{-1}F_m^{(4)}\cos(mx)+F_m^{(5)}x+m^{-1}(F_m^{(6)}-F_m^{(7)})\times\cosh(mx)+F_m^{(7)}x\sinh(mx)+m^{-1}(F_m^{(8)}-F_m^{(5)})\times\sinh(mx)-F_m^{(8)}x\cosh(mx)\ . \tag{45}$$

In view of Eq.(42), Eq.(44) can be cast in the form of Abel's integral equation as

$$\int_0^x\frac{x\phi(t)}{\sqrt{x^2-t^2}}dt=h(x)\ ;\ 0\le x<e\ , \tag{46}$$

in which

$$h(x)=e\int_0^1\phi(er)\left\{1-r^2\right.$$

$$+2er\int_0^\infty \frac{\cosh(xs)\left[I_1(ser)-rI_1(se)\right]}{\exp(\pi s)+1}ds\Big\}dr$$
$$+2e^2\int_0^1 r\phi(er)\sum_{m=1,3,5,...}^{\infty}\Big[F_m^{(1)}\cos(mx)$$
$$-(F_m^{(2)}-F_m^{(3)})\cosh(mx)-F_m^{(3)}mx\sinh(mx)$$
$$+\eta mx\cosh(mx)+\eta\sinh(mx)\Big]$$
$$\times\left[J_1(mer)-rJ_1(me)\right]dr$$
$$+2\sum_{m=1,3,5,...}^{\infty}F_m(x)\,;\;0\le x<e\,. \tag{47}$$

Note that the change of variable $t = er$ is introduced in Eq.(47). Moreover, it can be remarked that Eq.(44) should include an arbitrary constant of integration resulting from the integration of Eq.(21) with respect to x. Nevertheless, it has no effect on the solution process of solving the Abel's integral equation and thus, the constant is excluded [32].

The solution of Eq.(46) can be taken in the form as

$$\phi(t)=\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{h(x)}{\sqrt{t^2-x^2}}dx\,;\;0\le t<e\,. \tag{48}$$

7 FREDHOLM INTEGRAL EQUATION

Substituting Eq.(47) into Eq.(48) and using the identities given below that already derived with the aid of identities found in Gradshteyn and Ryzhik [33],

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{1}{\sqrt{t^2-x^2}}dx=0\,, \tag{49}$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{\cos(mx)}{\sqrt{t^2-x^2}}dx=-mJ_1(mt)\,, \tag{50}$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{\sinh(mx)}{\sqrt{t^2-x^2}}dx=m\left[\frac{2}{\pi}+L_1(mt)\right], \tag{51}$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{x\sinh(mx)}{\sqrt{t^2-x^2}}dx=mtI_0(mt)\,, \tag{52}$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{\cosh(mx)}{\sqrt{t^2-x^2}}dx=mI_1(mt)\,, \tag{53}$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{x\cosh(mx)}{\sqrt{t^2-x^2}}dx=\frac{2}{\pi}+mtL_0(mt)\,, \tag{54}$$

and then, the final result can be manipulated as in the following equation, which is an inhomogeneous Fredholm integral equation,

$$\Phi(\rho)+\int_0^1 K(\rho,r)\Phi(r)dr=f(\rho)\,;\;0\le\rho\le 1\,, \tag{55}$$

with

$$\Phi(\rho)=\phi(e\rho)\,;\;\Phi(r)=\phi(er)\,, \tag{56}$$

and

$$K(\rho,r)=2e^2r\sum_{m=1,3,5,...}^{\infty}\Big[-4\eta m/\pi-\eta mL_1(me\rho)$$
$$-\eta m^2e\rho L_0(me\rho)+mF_m^{(1)}J_1(me\rho)$$
$$+m(F_m^{(2)}-F_m^{(3)})I_1(me\rho)+m^2F_m^{(3)}e\rho I_0(me\rho)\Big]$$
$$\times\left[J_1(mer)-rJ_1(me)\right]-2e^2r$$
$$\times\int_0^\infty\frac{sI_1(se\rho)\left[I_1(ser)-rI_1(se)\right]}{\exp(\pi s)+1}ds\,, \tag{57}$$

$$f(\rho)=2\sum_{m=1,3,5,...}^{\infty}\Big[F_m^{(4)}J_1(me\rho)+(F_m^{(6)}-F_m^{(7)})I_1(me\rho)$$
$$+mF_m^{(7)}e\rho I_0(me\rho)+(F_m^{(8)}-F_m^{(5)})L_1(me\rho)$$
$$-mF_m^{(8)}e\rho L_0(me\rho)\Big], \tag{58}$$

where $L_n(u)$ is the modified Struve function of order n with argument u [26].

8 DISCUSSION AND CONCLUSION

It is immediately seen that the analytical solution of Eq.(55) is very difficult to obtain due to the complexity of the kernel and inhomogeneous part of integral equation as shown in Eq.(57) and Eq.(58), respectively.

However, the unknown auxiliary function $\Phi(\rho)$ can be determined using the standard numerical method [34], which has been described previously [29].

Mathematically, since the length of free edges equals zero (e = 0), the problem becomes a limiting case of simply supported square plate as shown in Figure 2 and there is no existence of singularity in the problem. This can be seen in the function P_m that involved with singularity of the solution.

By considering Eq.(37) and setting e = 0, yields

$$m^2 P_m = \int_0^{e=0} t\phi(t)\left[J_1(mt) - \frac{t}{e}J_1(me)\right]dt = 0\ ;\ m = 1,3,5,\ldots\ , \tag{59}$$

and further consideration of Eq.(22) and also Eq.(15) to Eq.(17), leads to the relations:

$$D_m = -\frac{2}{\pi^5 m^5}\tanh\beta\ , \tag{60}$$

$$A_m = -\frac{4}{\pi^5 m^5}\ , \tag{61}$$

$$B_m = \frac{2}{\pi^5 m^5}\ , \tag{62}$$

$$C_m = \frac{2}{\pi^5 m^5}(2\tanh\beta - \beta\,\mathrm{sech}^2\,\beta)\ . \tag{63}$$

Applying Eq.(60) to Eq.(63) in Eq.(6), the deflection function can thus be obtained in closed-form of the Lévy-Nádai's solution [25] as

$$w(x,y) = \frac{q\bar{a}^4}{\pi^5 D}\sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{m^5}\Big[\{2[1-\cosh(my)] + (2\tanh\beta - \beta\mathrm{sech}^2\beta)\sinh(my) + my\sinh(my) - my\tanh\beta\cosh(my)\}\sin(mx) + \{2[1-\cosh(mx)] + (2\tanh\beta - \beta\mathrm{sech}^2\beta)\times\sinh(mx) + mx\sinh(mx) - mx\tanh\beta\times\cosh(mx)\}\sin(my)\Big];\ 0 \le x, y \le \pi\ . \tag{64}$$

However, the deflection function as presented in Eq.(64) can be written in the new compact form, when the x-axis is relocated in the middle line as shown in Figure 3 [25],

$$w(x,y) = \frac{4q\bar{a}^4}{\pi^5 D}\sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{m^5}\left[1 - \frac{\beta\tanh\beta + 2}{2\cosh\beta}\times\cosh\left(\frac{2\beta y}{\pi}\right) + \frac{\beta y}{\pi\cosh\beta}\sinh\left(\frac{2\beta y}{\pi}\right)\right]\times\sin(mx)\ : 0 \le x \le \pi\ ;\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}\ . \tag{65}$$

If the free edge lengths are equal to $\pi/2$ ($e = \pi/2$), the solution cannot be determined within the present method [17], [23], [24], [28]-[30]. Noted, however, that this case is corresponded with the problem of point-corner supported square plate having a point column support placed at all middle edges.

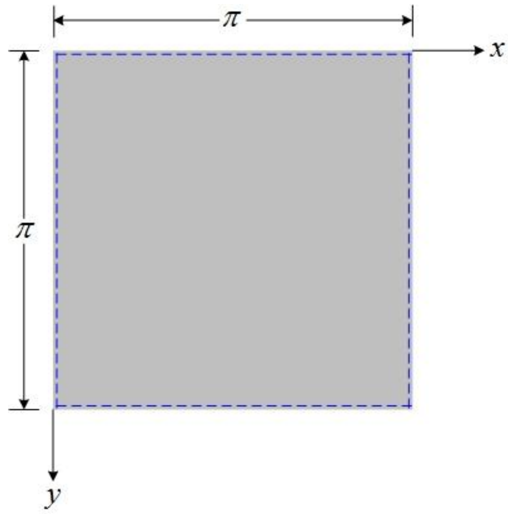

Fig. 2 *Simply supported square plate*.

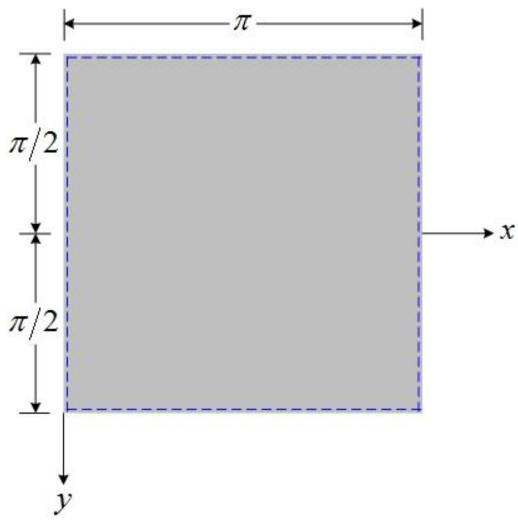

Fig. 3 *Simply supported square plate with the x-axis placed along the middle.*

For a limiting case of e that approaches $\pi/2$ such as $e = 0.495\pi$, the solution can still be determined as the limiting case as shown in Figure 4.

In closing, it can strongly be concluded that the finite Hankel integral transform method is considerable with success to analytically solve the problem mentioned herein, which has never been found in the past scientific or technical journals up to date.

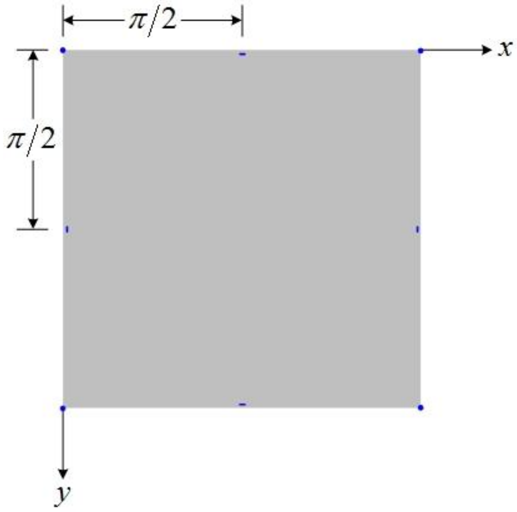

Fig. 4 *Corner-supported square plate with narrow strip column supported at the middle edges.*

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