Some Aspects on the Nature of Singularity Fields Allowed in Two Different Plate Problems

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ABSTRACT

The aim of this paper is to analytically examine the nature of singularity fields that can be allowed to exist in two different problems of uniformly loaded square plate in which both plates have partial simple supports symmetrically placed at the central portion of all edges. In the first case, the corners of the plate are unconstrained and can be bent away upon loading, but not for the second case where each corner is anchored by a point column support to prevent a deflection vertically. Significantly, both problems can be analyzed and treated within the same formulation with slight modifications by means of the proper finite Hankel integral transforms. Additionally, some limiting cases of the plates are also able to be determined within the frame of these problem formulations.

Keywords: *Singularity, Square plate, Partial simple support, Point support, Hankel integral transform.*

1 INTRODUCTION

Following the Kirchhoff's assumptions for theory of thin elastic plates, when the plates having the right-angle corners such as the square and rectangular plates loaded laterally, the corners in general have a tendency to rise, and this is prevented by the concentrated reactions (called the corner forces) at the corners. Therefore, it is clear that these mentioned plates supported in some way along the edges will usually produce not only reactions distributed along the boundary but also concentrated reactions at the corners. The magnitudes of these corner forces are equal to the magnitudes of the twisting couple at the corresponding corners of the plate [1]. If the plate is not constrained bilaterally in the vicinity of the corners, then parts of the plate near and including the corners will be bent away from the supports upon loading. For instance, this situation leads to mixed boundary value problem of plates.

Mathematically, the use of integral transforms has been applied with considerable success to solve many mixed boundary value problems of plates that involved with the problems of bending, vibration, and buckling. The methods usually lead to dual-series or dual integral equations. Stahl and Keer [2] analytically investigated the vibration and buckling of rectangular plates with an internal line support by means of the finite Hankel integral transforms. Further analyses were extended to treat the problems of rectangular plates with cracks [3] and rectangular plates with various mixed boundary conditions [4]. All problems were finally reduced to determining the solutions, which are the frequency parameters or the load factors, of the homogeneous Fredholm integral equations of the second kind. It is worth noting that the inverse-square-root moment singularities [5] have considered and also taken into account in their analyses.

Additionally, the bending problems of uniformly loaded rectangular plates with a partial internal line support were solved [6, 7] by using the same method as presented previously [2-4].

Consider the problems involving advancing contact that are more complicated in the sense, Dundurs, Kiattikomol, and Keer [8] examined two closely related contact problems between rectangular plate and unilateral sagged supports. The results showed that the extent of contact depends on the level of loading and the reactions are not proportional to the applied load [9]. Sompornjaroensuk and Kiattikomol [10], [11] further analytically investigated the advancing contact between the rectangular plates and an internal line sagged support by making use of finite Hankel integral transforms.

Another contact problem of the opposite type, or one involving receding contact, which is the problem of laterally loaded quarter infinite plate with no anchoring at the corner was analytically solved by Keer and Mak [12] for finding the loss of contact between the plate and the supports. The method used is the Fourier integral transforms and problem considered can then be reduced to the coupled pair of singular integral equations. After that Dempsey et al. [13] used a finite Fourier integral transform method for analyzing the problem of uniformly loaded square plate. The governing equation was expressed as the Cauchy-type singular integral equation of the first kind together with constraint condition for zero corner forces. The mentioned problem has numerically and analytically treated by Salamon, Pawlak, and Mahmoud [14] using finite element method and Sompornjaroensuk and Kiattikoml [15] based on the finite Hankel integral transforms, respectively.

2 MATHEMATICAL FORMULATION

To simplify the analysis and formulation, the first case of the plates is shown in Fig. 1.

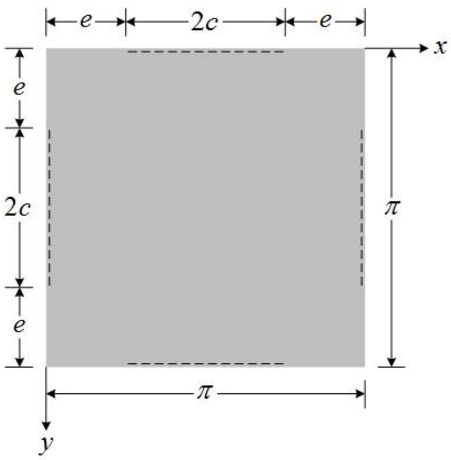

***Fig. 1** Square plate with partially simply supported along the middle edges.*

The lengths involved are scaled by the factor $\pi/\bar{a}$ where the actual length of the square plate is $\bar{a}$, and e, c are the length of free edge and the half-length of partial simple support, respectively.

Because of two-fold symmetry of the geometry and the lateral load, the deflection function (w) of the plate having uniform thickness (h) loaded by a uniformly distributed load (q) that satisfied with this case of the plate can then be taken in the form as [16]

$$w(x,y) = \frac{q\bar{a}^4}{2D} \sum_{m=1,3,5,...}^{\infty} \left[W_m(x,y) + W_m(y,x) \right] + W_c \, , \qquad (1)$$

and

$$D = \frac{Eh^3}{12(1-\nu^2)} \, , \qquad (2)$$

$$W_m(x,y) = \left[\frac{4}{m^5 \pi^5} + Y_m(x) \right] \sin(my) \, , \qquad (3)$$

$$Y_m(u) = A_m \cosh(mu) + B_m mu \sinh(mu) + C_m \sinh(mu) + D_m mu \cosh(mu) \, , \qquad (4)$$

with

$$A_m = \frac{4\nu\eta'}{m^5\pi^5} + 2D_m \eta' \coth\beta \, , \qquad (5)$$

$$B_m = -D_m \coth\beta \, , \qquad (6)$$

$$C_m = -\frac{4\nu\eta' \tanh\beta}{m^5\pi^5} - D_m \left[2\eta' + \beta(\tanh\beta - \coth\beta) \right] , \qquad (7)$$

in which

$$\eta' = \frac{1}{1-\nu} \, , \qquad (8a)$$

$$\beta = \frac{m\pi}{2} \, , \qquad (8b)$$

where D is defined to be the plate's bending rigidity [1], E and ν are thus the plate's material properties, namely, the Young's modulus and Poisson's ratio, respectively, W_c is the deflection of the corner that required in order to insure the region of the plate corners to be free to movement, and A_m, B_m, C_m, and D_m are the unknown constants.

The formulation of problem can be treated only in the region bounded by the upper left quadrant of the plate due to the condition of symmetry. Application of the mixed boundary conditions leads to the dual-series equations as follows:

$$\sum_{m=1,3,5,...}^{\infty} mP_m \cos(mx) = 0 \; ; \; e < x \le \frac{\pi}{2} \, , \qquad (9)$$

$$\sum_{m=1,3,5,...}^{\infty} \Big\{ m^3 P_m (1+F_m^{(1)}) \sin(mx) + m^3 P_m \Big[F_m^{(2)} \sinh(mx) - 2\eta \cosh(mx) + F_m^{(3)} mx \cosh(mx) - \eta mx \sinh(mx) \Big] \Big\}$$
$$= \sum_{m=1,3,5,...}^{\infty} \Big[F_m^{(4)} \sin(mx) + F_m^{(5)} + F_m^{(6)} \sinh(mx) - F_m^{(5)} \cosh(mx) + F_m^{(7)} mx \cosh(mx) - F_m^{(8)} mx \sinh(mx) \Big] ; \; 0 \le x < e \, , \qquad (10)$$

where

$$P_m = \frac{2}{m^5 \pi^5} + D_m \coth\beta \, , \qquad (11)$$

$$1+F_m^{(1)} = \frac{(3+\nu)\sinh\beta\cosh\beta - (1-\nu)\beta}{(3+\nu)\cosh^2\beta} \, , \qquad (12)$$

$$F_m^{(2)} = \eta\left(2\tanh\beta + \beta \mathrm{sech}^2\beta\right), \qquad (13)$$

$$F_m^{(3)} = \eta \tanh\beta \, , \qquad (14)$$

$$F_m^{(4)} = \frac{2\left[(3-\nu)\tanh\beta - (1-\nu)\beta\mathrm{sech}^2\beta\right]}{(3+\nu)m^2\pi^5} \, , \qquad (15)$$

$$F_m^{(5)} = \frac{4}{(3+\nu)m^2\pi^5} \, , \qquad (16)$$

$$F_m^{(6)} = \frac{2\left[2\tanh\beta + (1-\nu)\beta\mathrm{sech}^2\beta\right]}{(3+\nu)m^2\pi^5} \, , \qquad (17)$$

$$F_m^{(7)} = \frac{2\eta\tanh\beta}{m^2\pi^5} \, , \qquad (18)$$

$$F_m^{(8)} = \frac{2\eta}{m^2\pi^5} \, , \qquad (19)$$

and

$$\eta = \frac{1-\nu}{3+\nu} \, . \qquad (20)$$

Further details concerning the derivation of Eqs.(1) to (20) have provided previously in Kongtong et al. [16].

3 SINGULARITY IN THE SHEAR FIELD

It can generally be noted from the aforementioned works [8], [10]-[13], [15] that at the ends of partial simple supports, the singularity should be expected in the order of an inverse-square-root type in the shearing forces. With this purpose, the required form of an unknown function P_m is taken as [16], based on the finite Hankel integral transforms,

$$m^2 P_m = \int_0^e t\varphi(t) J_1(mt)dt \; ; \; m = 1,3,5,... \, , \qquad (21)$$

where t is a dummy variable, $\varphi(t)$ is the unknown function introduced in the method of Hankel integral transforms, and $J_1(u)$ is the Bessel function of the first kind and first order of argument u [17], [18].

Substitution of Eq.(21) into the second dual-series equations of Eq.(10) and using the identity presented in Kontong et al. [16: Eq.(50)], yields

$$V_y(e+\varepsilon, 0) = \left(\frac{e}{2}\right)\varphi(e)(2e\varepsilon)^{-1/2} + O(\varepsilon^{1/2}) \; ; \; \varepsilon \to 0 \, . \qquad (22)$$

It is clearly seen that the first term on the right side of Eq.(22) indicates an inverse-square-root singularity in the shear (V_y) existed at the end of simple support ($x = e + \varepsilon$, $y = 0$ and $\varepsilon \to 0$).

Subsequently, the deflection of the plate corner (W_c) as introduced in Eq.(1) can be determined analytically and expressed as,

$$W_c = -\frac{q\bar{a}^4\eta'\pi}{8D}\int_0^e t^2\varphi(t)dt \, . \qquad (23)$$

It is interesting to note here that the corners of the plate have a tendency to rise up during bending because of the minus sign presented in Eq.(23).

4 SINGULARITY IN THE MOMENT FIELD

Consider the second case of the plates as shown in Fig. 2, the plate is partially simply supported on each edge similar to the first case analyzed and has an additional point support at all corners.

An inverse-square-root shear singularity in the vicinity of the transition points from a simple support to a free edge as presented in the previous case is not allowed in the current case and it should be changed to be an inverse-square-root moment singularity [5], which is proper at those transition points [2]-[4], [6], [7], [19].

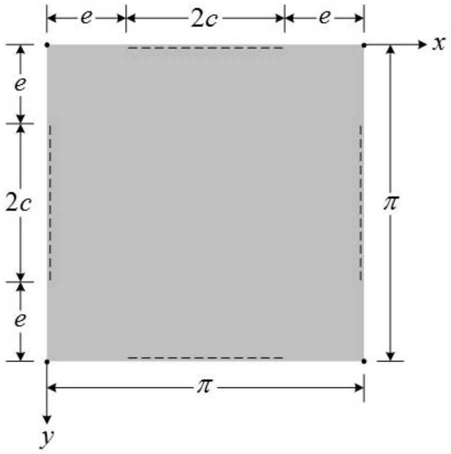

***Fig. 2** Square plate with partially simply supported along the middle edges and point column supported at all corners.*

To confirm the statement above, the function P_m as given in Eq.(21) must be replaced by

$$m^2 P_m = \bar{E} J_1(me) + \int_0^e t\varphi(t) J_1(mt)dt \; ; m = 1,3,5,\ldots , \qquad (24)$$

where

$$\bar{E} = -\frac{1}{e}\int_0^e t^2 \varphi(t)dt \, . \qquad (25)$$

Substituting Eq.(24) in Eq.(10) and using the identity that previously shown in Eq.(40) of [16], Eq.(22) then becomes

$$V_y(e+\varepsilon,0) = -\left(\frac{\bar{E}}{2}\right)\Big[e(2e\varepsilon)^{-3/2} + O(\varepsilon^{-1/2}) \\ -(2e\varepsilon)^{-1/2}/e - O(\varepsilon^{1/2})\Big]. \qquad (26)$$

This clearly reveals that the singularity is of $O(\varepsilon^{-3/2})$ in the shear field, or of $O(\varepsilon^{-1/2})$ in the moment field, since P_m is assumed in the proper form of Eq.(24). The next is to determine the quantity of the corner deflection (W_c), which can be obtained in the same manner with the previous case and then, leads to the result as

$$W_c = 0 \, . \qquad (27)$$

This can be implied immediately and concluded that the deflection at the plate corners is unable to lift up freely due to the existence of a point column support at the corners of the plate, which is automatically satisfied with the problem under consideration exactly.

5 DISCUSSION AND CONCLUSION

As is presented and explained in the detailed analysis shown in Section 3, it has generally disclosed that for the specific case of a square plate supported by simple supports with certainly specified support lengths (see Fig. 1), the nature of singularity is shown to be the order of an inverse-square-root type in the shear. Nevertheless, within the authors' opinion, the problem is not seemed to be a natural receding contact between a plate and the unilateral supports as analytically treated in Dempsey et al. [13] and Sompornjaroensuk and Kiattikomol [15] because of no presence of zero corner force constraint condition. This is one of important points to discuss in the following details.

Consider a uniformly loaded square plate supported by the unilateral supports as demonstrated in Fig. 3, the plate configuration seems to be the same with the plate shown in Fig. 1.

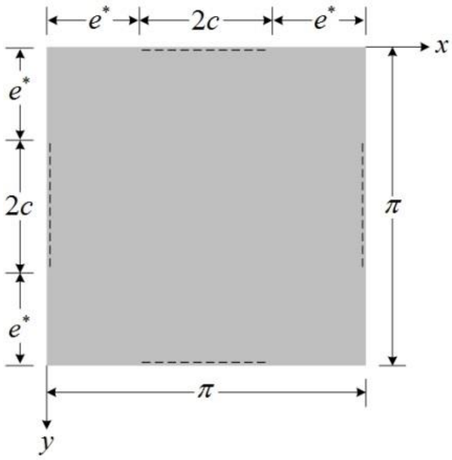

***Fig. 3** Square plate with unilaterally supported edges.*

The difference is that the length of the supports is a certain length for the previous case, but not for the present in which e^* is the loss of contact between a square plate and the unilateral supports, sometimes called the noncontact length. However, a significant additional constraint condition must be provided, for the zero corner forces in this problem [15], which is

$$e^2 \int_0^e T(er) r\varphi(er)dr = B \, , \qquad (28)$$

where

$$T(er) = \sum_{m=1,3,5,...}^{\infty} \left[(1+\nu)\eta' \tanh\beta - \beta \text{sech}^2 \beta \right] J_1(mer), \quad (29)$$

and

$$B = 2 \sum_{m=1,3,5,...}^{\infty} \left[\frac{\tanh\beta - \beta\text{sech}^2\beta}{m^3 \pi^{5.}} \right]. \quad (30)$$

It is notable that the presence of corner forces cannot be allowed in the plate having unilaterally supported edges [12], [13], [15], because it is forced by the condition that given in Eq.(28).

Thus, the problem considered herein is one of the receding contact problems and the loss of contact is not depended on the level of loading but strongly depended on the Poisson's ratio of the plate. Refer to the results that presented in Sompornjaroensuk and Kiattikomol [15], the loss of contact was continually decreased upon the increasing of the Poisson's ratio [9].

In views of Section 4, it has clearly shown that the singularity must be the order of an inverse-square-root type in the moment field and the deflection of the corners is zero. This means that the term W_c presented in Eq.(1) is automatically vanished for this case of the plate.

However, since the length of free edges in both cases of the plates illustrated in Figs. 1 and 2 equals zero (e = 0), the nonzero term of W_c for the first case of the plate (Fig. 1) is vanished automatically. This can immediately be observed in Eq.(23) with setting e = 0 for the upper limit of integration, yields

$$W_c = -\frac{q\bar{a}^4 \eta' \pi}{8D} \int_0^{e=0} t^2 \varphi(t) dt = 0. \quad (31)$$

In the case of e = 0, both plates have become an identical problem of simply supported square plate as demonstrated in Fig. 4. There is no singularity existed in either the shear or the moment. The deflection solution can thus be easily determined for the limiting case.

By considering Eqs.(21) and (24) for the plates shown in Figs. 1 and 2, respectively, with setting e = 0, they are

$$m^2 P_m = \int_0^{e=0} t\varphi(t) J_1(mt) dt = 0, \quad (32)$$

for the plate in Fig. 1, and

$$m^2 P_m = -\frac{J_1(me)}{e} \int_0^{e=0} t^2 \varphi(t) dt + \int_0^{e=0} t\varphi(t) J_1(mt) dt = 0, \quad (33)$$

for the plate in Fig. 2.

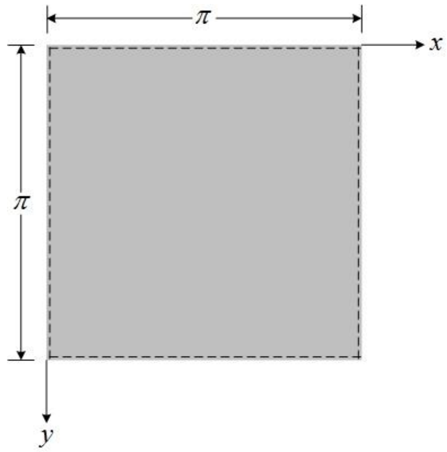

Fig. 4 *Square plate with simply supported edges.*

It is notable that the first term in the right-hand side of Eq.(33) is resulted from the substitution of Eq.(25) for $\bar{E}$.

Further consideration of Eq.(11) leads to

$$D_m = -\frac{2\tanh\beta}{m^5 \pi^5}, \quad (34)$$

and also the unknown constants presented in Eqs.(5) to (7) are rewritten as follows:

$$A_m = -\frac{4}{m^5 \pi^5}, \quad (35)$$

$$B_m = \frac{2}{m^5 \pi^5}, \quad (36)$$

and

$$C_m = \frac{2}{m^5 \pi^5} \left(2\tanh\beta - \beta \text{sech}^2 \beta \right). \quad (37)$$

Application of the constants given in Eqs.(34) to (37) into Eq.(4), the plate deflection function of Eq.(1) can be obtained in closed-form of the Levy-type solution [1] in correspondence with the simply supported square plate as shown in Fig. 4.

When the free edge lengths are equal to $\pi/2$ (e = $\pi/2$), solution of both plates cannot be determined within the method of finite Hankel integral transforms [6], [7], [19], [20]. Moreover, this case of e = $\pi/2$ is corres-ponding to the plates having point column support placed at all middle edges. The problems are thus singular due to concentrated reaction forces, which are not intentionally interested here for the plates under consideration.

Nevertheless, if the length e approaches $\pi/2$ such as $e = 0.495\pi$ or $2c = \pi/100$, then the solution for both plates (Figs. 1 and 2) can still be obtained as the limiting cases that illustrated in Figs. 5 and 6.

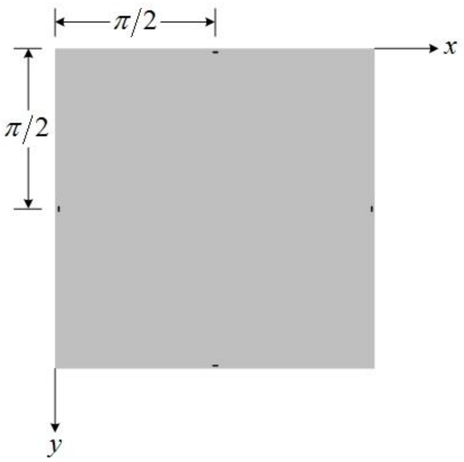

Fig. 5 *Square plate with narrow strip column supported at the middle edges.*

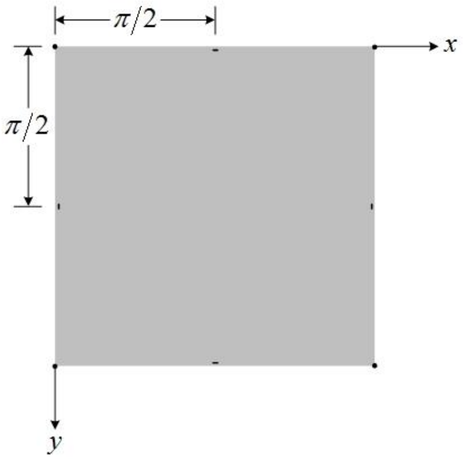

Fig. 6 *Square plate with narrow strip column supported at the middle edges and point column supported at all corners.*

It is remarkable that since the length of partial simple supports in both cases of the plate is set to be $\pi/100$, these supports can be considered to be the narrow strip columns placed at the middle edges.

In the conclusion, this paper theoretically presents the bending problem of uniformly loaded square plates having different types of singularity, which can be allowed to exist in the mathematical senses. The method of analysis is thus based on the finite Hankel integral transform techniques.

Some observations and discussions for each case of the plates have given and clearly explained in detailed content of the paper. In the authors' opinion, these are, however, complete within the scope of investigation.

REFERENCES

[1] S.P. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells, 2nd ed., McGraw-Hill, Singapore, 1959.

[2] B. Stahl and L.M. Keer, "Vibration and buckling of a rectangular plate with an internal support," Quarterly Journal of Mechanics and Applied Mathematics, vol. 25, pp. 467 - 478, 1972.

[3] B. Stahl and L.M. Keer, "Vibration and stability of cracked rectangular plates," International Journal of Solids and Structures, vol. 8, pp. 69 - 91, 1972.

[4] L.M. Keer and B. Stahl, "Eigenvalue problems of rectangular plates with mixed edge conditions," Journal of Applied Mechanics, ASME, vol. 39, pp. 513 - 520, 1972.

[5] M.L. Williams, "Surface stress singularities resulting from various boundary conditions in angular corners of plates under bending," in Proc. the First U.S. National Congress of Applied Mechanics, ASME, vol. 1, pp. 325 - 329, 1952.

[6] Y. Sompornjaroensuk and K. Kiattikomol, "Exact analytical solutions for bending of rectangular plates with a partial internal line support," Journal of Engineering Mathematics, vol.62, pp. 261 - 276, 2008.

[7] Y. Sompornjaroensuk and K. Kiattikomol, "Inverse square root moment singularities in rectangular plates containing an internal line support," International Journal of Computational Methods, vol. 6, no. 1, pp. 1 - 21, 2009.

[8] J. Dundurs, K. Kiattikomol, and L.M. Keer, "Contact between plates and sagged supports," Journal of the Engineering Mechanics Division, ASCE, vol. 100, pp. 445 - 456, 1974.

[9] J. Dundurs and M. Stippes, "Role of elastic constants in certain contact problems," Journal of Applied Mechanics, ASME, vol. 37, pp. 965 - 970, 1970.

[10] Y. Sompornjaroensuk and K. Kiattikomol, "Singularity of advancing contact problems of plate bending," in Proc. The Tenth East Asia-Pacific Conference on Structural Engineering & Construction (EASEC-10), Bangkok, vol. 2, pp. 193 - 198, August 3 - 5, 2006.

[11] Y. Sompornjaroensuk, K. Kiattikomol, and D. Dy "Extent of advancing contact between rectangular plates and sagged supports," in Proc. 14th National Convention on Civil Engineering (NCCE14), Nakhon Ratchasima, vol. 6, pp. 2183 - 2188, May 13-15, 2009.

[12] L.M. Keer and A.F. Mak, "Loss of contact in the vicinity of a right-angle corner for a simply supported, laterally loaded plate," Journal of Applied Mechanics, ASME, vol. 48, pp. 597 - 600, 1981.

[13] J.P. Dempsey, L.M. Keer, N.B. Patel, and M.L. Glasser, "Contact between plates and unilateral supports," Journal of Applied Mechanics, ASME, vol. 51, pp. 324 - 328, 1984.

[14] N.J. Salamon, T.P. Pawlak, and F.F. Mahmoud, "Plates in unilateral contact with simple supports: pressure loading," Journal of Applied Mechanics, ASME, vol. 53, pp. 141 - 145, 1986.

[15] Y. Sompornjaroensuk and K. Kiattikomol, "The solution of certain loss of contact between a plate and unilateral supports," Theoretical and Applied Mechanics, vol. 34, no. 4, pp. 289 - 308,

2007.
[16] P. Kongtong, Y. Sompornjaroensuk, and D. Sukawat, “Singularities in bending behavior of plates with free corner supports,” Applied Mathematical Sciences, vol. 7, no. 25, pp. 1213 - 1222, 2013.
[17] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 55, Dover Publications, New York, 1964.
[18] J.J. Tuma, Engineering Mathematics Handbook, 2nd enlarged and revised ed., McGraw-Hill, New York, 1979.
[19] Y. Sompornjaroensuk and D. Dy, “Corners lifted up square plate with specifying partial simple support lengths,” in Proc. 15th National Convention on Civil Engineering (NCCE15), Ubonratchatani, May 12 - 14, 2010, paper STR020.
[20] K. Kiattikomol, S. Porn-anupapkul, and N. Thonchangya, “Application of dual series equations to a square plate with varying corner supported lengths,” in Proc. The Recent Advances in Structural Engineering, Bangkok, March 14 - 15, 1985, pp. 115-130.

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