# Analysis of Variable Step-size Order Statistic LMS-based Algorithms with a Data-reuse Factor

**Suchada Sitjongsataporn**
Centre of Electronic Systems Design and Signal Processing (CESdSP)
Department of Electronic Engineering, Mahanakorn University of Technology
140 Cheumsamphan Road, Bangkok 10530 Thailand
E-mail: ssuchada@mut.ac.th

**Kodchakorn Na Nakornphanom†**
School of Science and Technology, Sukhothai Thammathirat Open University
9/9 Chaengwattana Road, Nonthaburi 11120 Thailand
E-mail: kodchakorn.nan@stou.ac.th

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**ABSTRACT**

***In this paper, we present variable step-size algorithms based on an order statistic normalized averaged least mean square (OS-NALMS) with a data-reuse factor. An optimal step-size parameter is investigated by means of mean square deviation criterion, we then modify in the recursion form. Two of new variable step-size approaches are introduced with regard to the optimal step-size algorithm. In order to reduce the complexity, a variable data-reuse factor approach is applied to the number of reused gradient based estimate sequences for averaged least mean square algorithm. Simulation results demonstrate that the performance of proposed variable step-size OS-NALMS algorithms can obtain in terms of fast convergence and robustness.***

***Keywords:*** *Adaptive filters, variable step-size algorithm, order statistic mechanism, least mean square (LMS) algorithm, data-reuse factor.*

## 1. INTRODUCTION

A class of adaptive algorithms based on least mean square (LMS) algorithm employing order statistic filtering of the sampled gradient estimates has been presented in [1] and [2]. These can perform in terms of simple and robust adaptive filter across a wide range of input environments. The order statistic LMS algorithms are similar to usual gradient-based LMS algorithm with robust order statistic filtering operations applied to the gradient estimate sequences. In [3], the adaptive step-size based on order statistic LMS-based time-domain equalisation has been presented with robust order statistic filtering and be quite flexible for discrete multitone systems.

The idea of data-reuse factor approach has been presented in [4] and [5] that utilised a variable number of past constraint sets at each coefficient update. In [6], the variable step-size based on normalised LMS algorithm has been proposed the adaptive filtering with regard to optimal performance by means of faster convergence rate and lower misadjustment error than existing schemes. The idea of an estimate averaging the prediction error in order to control the step-size parameter has been presented in [7].

In this paper, we describe the optimal variable step-size mechanism for order statistic normalised averaged LMS (OS-NALMS) algorithm by means of mean square deviation scheme in Section 2. A type of normalised LMS (NLMS) algorithm is applied in virtue of order statistic scheme which replaces linear smoothing of the gradient estimation by an averaged order statistic adaptive algorithm. An optimal step-size parameter is investigated based on the proposed OS-NALMS algorithm with regard to the optimal performance in Section 3. Two variable step-size algorithms are proposed by means of optimal performance in Section 4. The variable data-reuse factor approach adjusted adaptively by quantising the range of data-dependent step-size parameter is presented to reduce the complexity of number of taps of coefficient in Section 5. Simulation results are shown in Section 6 and Section 7 concludes the paper.

## 2. AN ORDER STATISTIC NORMALISED AVERAGED LEAST MEAN SQUARE (OS-NALMS) ALGORITHM

In this section, we present the NLMS algorithm which replace linear smoothing of gradient estimates by order statistic averaged LMS filter. The tap-weight vector $\widehat{\boldsymbol{w}}(k)$ minimises the solution following optimisation criterion as

$$\min \| \widehat{\boldsymbol{w}}(k) - \widehat{\boldsymbol{w}}(k-1) \| , \quad (1)$$

subject to the constraint

$$\widehat{\boldsymbol{w}}^H(k) \cdot \widetilde{\boldsymbol{x}}(k) = \widetilde{d}\,(k) , \quad (2)$$

where $\widetilde{d}\,(k)$ is the desired signal. The operator $(\cdot)^H$ denotes as the Hermitian operator. The vector $\widetilde{\boldsymbol{x}}(k)$is the input signal vector as $\widetilde{\boldsymbol{x}}(k) = [\tilde{x}(k), \cdots, \tilde{x}(k - L-1)\,]^T$ , where the operator $(\cdot)^T$ denotes as the transpose operator.

To solve the optimisation problem with the method of Lagrange multipliers, the update of complex-valued tap-weight estimated vector $\widehat{\boldsymbol{w}}(k)$ can be formulated by means of NLMS algorithm as [4], [9] as

$$\widehat{\boldsymbol{w}}(k) = \widehat{\boldsymbol{w}}(k-1) - \frac{\mu(k)}{\|\widetilde{\boldsymbol{x}}(k)^2\|}\, \widetilde{\boldsymbol{x}}(k)\tilde{e}^*(k) , \quad (3)$$

and

$$\tilde{e}\,(k) = \widetilde{d}\,(k) - \widehat{\boldsymbol{w}}^H(k-1) \cdot \widetilde{\boldsymbol{x}}(k) . \quad (4)$$

where $\tilde{e}\,(k)$ is *a priori* estimation error and $\mu(k)$ is the step-size parameter. The notation $\|\cdot\|$ is the Euclidean norm of a vector.

The advantage of a class of order statistic LMS algorithms has been presented in [1] which is similar to the usual gradient-based LMS algorithm with robust order statistic filtering operations applied to the gradient estimate sequence. In order to reduce the dimensions of the matrix of ordering transformation, a number of data reuses $\mathbb{D}$ in each update are applied.

We present an update of the complex-valued estimated vector $\widehat{\boldsymbol{w}}(k)$ with an OS-NALMS algorithm as [1]

$$\widehat{\boldsymbol{w}}(k) = \widehat{\boldsymbol{w}}(k-1) - \frac{\mu(k)}{\|\widetilde{\boldsymbol{x}}(k)^2\|}\, \widetilde{\boldsymbol{\mathcal{M}}}(k)\widetilde{\boldsymbol{a}}(k) , \quad (5)$$

and

$$\widetilde{\boldsymbol{\mathcal{M}}}(k) = \boldsymbol{\mathcal{T}}\{\widetilde{\boldsymbol{x}}\,(k) \cdot \tilde{e}^*(k),\ \widetilde{\boldsymbol{x}}\,(k-1) \cdot \tilde{e}^*(k-1), \cdots, \widetilde{\boldsymbol{x}}\,(k-\mathbb{D}+1) \cdot \tilde{e}^*(k-\mathbb{D}+1)\ \} , \quad (6)$$

$$\widetilde{\boldsymbol{a}} = [a(1), a(2), \dots, a(\mathbb{D})]\ , a\,(i) = \frac{1}{\mathbb{D}}\ ; \quad i = 1, 2, \dots, \mathbb{D} \quad (7)$$

where the estimation error $\tilde{e}(k)$ is given in (4). The operator $\boldsymbol{\mathcal{T}}\{\cdot\}$ denotes as the algebraic ordering transformation applied to the sequence as shown in (6).

The variable step-size $\mu\,(k)$ algorithm is derived firstly in order to achieve an optimal step-size parameter at each update in Section 3. The variable step-size algorithms based on the optimal step-size mechanism is proposed in Section 4. The vector $\widetilde{\boldsymbol{a}}(k)$ is an average of the gradient estimates of weighting of data reuses. The parameter $\mathbb{D}$ is the number of data reuses that is adjusted by quantising the range of data-dependent step-size parameter in Section 5.

## 3. AN OPTIMAL VARIABLE STEP-SIZE OS-NALMS ALGORITHM

In this section, we explain how to achieve an optimal step-size parameter based on the OS-NALMS algorithm at each update that indicates to define a new variable step-size algorithm as presented in [8]. The objective of an optimal step-size approach based on the LMS algorithm is to obtain the largest decrease of the mean square deviation (MSD) at each update.

We define the deviation vector $\boldsymbol{\vartheta}\,(k)$ that is between the optimal tap-weight vector $\boldsymbol{w}_{\text{opt}}$ and adaptive tap-weight estimated vector $\widehat{\boldsymbol{w}}(k)$ with the methods of OS-NALMS algorithm as

$$\boldsymbol{\vartheta}\,(k) = \boldsymbol{w}_{\text{opt}} - \widehat{\boldsymbol{w}}(k) , \quad (8)$$

where the tap-weight estimated vector $\widehat{\boldsymbol{w}}(k)$ is given in (5).

We consider the desired output $\{\widetilde{d}\,(k)\}$ that arise from this model as

$$\widetilde{d}\,(k) = \boldsymbol{w}_{\text{opt}}^H \widetilde{\boldsymbol{x}}(k) + \eta\,(k) , \quad (9)$$

where $\eta(k)$ defines for the measurement noise.

By following [6] and [10], the update recursion in (5) can be expressed in terms of weight deviation vector $\boldsymbol{\vartheta}\,(k)$ in (8) as

$$\boldsymbol{\vartheta}(k) = \boldsymbol{\vartheta}(k-1) - \frac{\mu(k)}{\|\widetilde{\boldsymbol{x}}(k)^2\|}\, \widetilde{\boldsymbol{\mathcal{M}}}(k)\widetilde{\boldsymbol{a}}(k) , \quad (10)$$

where $\widetilde{\boldsymbol{\mathcal{M}}}(k)$ and $\widetilde{\boldsymbol{a}}(k)$ are given in (6) and (7), respectively.

We then see that MSD can be satisfied by squaring both sides of (10) and taking expectation as

$$E\{\|\boldsymbol{\vartheta}(k)\|^2\} = E\{\|\boldsymbol{\vartheta}(k-1)\|^2\} - \Delta\mu(k) \ , \tag{11}$$

and

$$\Delta\mu(k) = 2\mu(k)\cdot\Re\left\{E\left\{\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2}\cdot\boldsymbol{\vartheta}(k-1)\right\}\right\} - \mu^2(k)E\left\{\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2}\cdot\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2}\right\}, \tag{12}$$

where $\Re\{\cdot\}$ and $E\{\cdot\}$ denote as the real and expectation operators, respectively. The operation $\|\cdot\|^2$ denotes the squared Euclidean norm operation.

The idea is that if $\Delta\mu(k)$ is maximized, then it guarantees that MSD will undergo with the largest decrease from each previous iteration $(k-1)$ to present iteration $(k)$.

By maximising $\Delta\mu(k)$ in (12) with respect to $\mu(k)$ and setting to zero, the optimal step-size parameter $\mu_{\text{opt}}(k)$ becomes

$$\mu_{\text{opt}}(k) = \frac{\Re\left\{E\left\{\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2}\cdot\boldsymbol{\vartheta}(k-1)\right\}\right\}}{E\left\{\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2}\cdot\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2}\right\}}, \tag{13}$$

where the matrix $\widetilde{\boldsymbol{\mathcal{M}}}(k)$ and the vector $\widetilde{\boldsymbol{\alpha}}(k)$ are given in (6) and (7), respectively.

For convenience of computation, let us define as

$$\boldsymbol{\zeta}(k) = \frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2} \ , \tag{14}$$

where $\boldsymbol{\zeta}(k)$ defines as the gradient vector.

Therefore, the optimal step-size parameter $\mu_{\text{opt}}(k)$ can be rewritten by substituting (14) in (13) as

$$\mu_{\text{opt}}(k) = \frac{\Re\{E\{\boldsymbol{\zeta}(k)\cdot\boldsymbol{\vartheta}(k-1)\}\}}{E\{\boldsymbol{\zeta}^H(k)\cdot\boldsymbol{\zeta}(k)\}} \, . \tag{15}$$

To facilitate the analysis, let us consider under a few assumptions as follows [6].

*Assumption 1*: We assume that the noise sequence $\eta(k)$ is identically and independently distributed (i.i.d.) and statistically independent of input signal $\widetilde{\boldsymbol{x}}(k)$.

*Assumption 2*: We assume that the previous deviation vector $\boldsymbol{\vartheta}(k-1)$ can be neglected on past noise.

By using the assumption 1 and assumption 2, the optimal step-size parameter $\mu_{\text{opt}}(k)$ can be performed approximately as

$$\mu_{\text{opt}}(k) \approx \frac{E\{\|\boldsymbol{\vartheta}(k-1)\|^2\}}{E\{\|\boldsymbol{\vartheta}(k-1)\|^2\} + \sigma_\eta^2 Tr\left\{E\left\{\left(\boldsymbol{\zeta}^H(k)\boldsymbol{\zeta}(k)\right)^{-1}\right\}\right\}} \ , \tag{16}$$

where

$$E\{\|\boldsymbol{\vartheta}(k-1)\|^2\} = \frac{E\{\boldsymbol{\vartheta}^H(k-1)\boldsymbol{\zeta}(k)\boldsymbol{\zeta}^H(k)\boldsymbol{\vartheta}(k-1)\}}{E\{\boldsymbol{\zeta}^H(k)\boldsymbol{\zeta}(k)\}} \ , \tag{17}$$

and $Tr\{\cdot\}$ denote as the trace of a matrix and $\sigma_\eta^2$ is the variance of the noise signal.

Hence, the optimal step-size $\mu_{\text{opt}}(k)$ in (16) can be written with method of the expression above as

$$\mu_{\text{opt}}(k) = \frac{E\{\|\boldsymbol{\vartheta}(k-1)\|^2\}}{E\{\|\boldsymbol{\vartheta}(k-1)\|^2\} + \mathbb{C}} \ , \tag{18}$$

where $\mathbb{C}$ is a small positive constant. It is noticed that $\mathbb{C}$ is related to (16) and (18) as

$$\mathbb{C} = \sigma_\eta^2 Tr\left\{E\left\{\left(\boldsymbol{\zeta}^H(k)\boldsymbol{\zeta}(k)\right)^{-1}\right\}\right\} \, . \tag{19}$$

where $\boldsymbol{\zeta}(k)$ is given in (14). The matrix $Tr\{\cdot\}$ denote as the trace of a matrix and $\sigma_\eta^2$ is the variance of the noise signal.

## 4. PROPOSED VARIABLE STEP-SIZE ALGORITHMS

This section describes the proposed variable step-size algorithms based on optimal step-size OS-NALMS algorithm as described in Section 3 as follows.

### 4.1 Proposed variable step-size OS-NALMS (VSOS-NALMS) algorithm

We propose a new variable step-size order statistic normalised averaged least mean square (VSOS-NALMS) algorithm by means of optimal step-size approach as analysed in Section 3 for the adaptive tap-weight estimated vector $\widehat{\boldsymbol{w}}(k)$ as

$$\widehat{\boldsymbol{w}}(k) = \widehat{\boldsymbol{w}}(k-1) - \mu(k)\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\widetilde{\boldsymbol{\alpha}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2} \ , \tag{20}$$

where $\widetilde{\boldsymbol{\mathcal{M}}}(k)$ and $\widetilde{\boldsymbol{a}}(k)$ are given in (6) and (7), respectively.

Let us define that

$$\boldsymbol{\vartheta}(k-1) \approx \frac{\widetilde{\boldsymbol{\mathcal{M}}}(k) \cdot \widetilde{\boldsymbol{a}}}{\|\widetilde{\boldsymbol{x}}(k)\|^2} \approx \boldsymbol{\zeta}(k-1)\,, \quad (21)$$

and

$$\left\|\widehat{\boldsymbol{\vartheta}}(k)\right\|^2 = \boldsymbol{\vartheta}^H(k-1)\frac{\boldsymbol{\zeta}(k)\boldsymbol{\zeta}^H(k)}{\boldsymbol{\zeta}^H(k)\boldsymbol{\zeta}(k)}\boldsymbol{\vartheta}(k-1)\,, \quad (22)$$

where $\boldsymbol{\zeta}(k)$ is given in (14).

Therefore, the proposed VSOS-NALMS algorithm can be expressed using time-averaging as

$$\mu(k) = \alpha \cdot \mu(k-1) + (1-\alpha)\frac{\left\|\widehat{\boldsymbol{\vartheta}}(k)\right\|^2}{\left\|\widehat{\boldsymbol{\vartheta}}(k)\right\|^2 + \mathbb{C}}\,, \quad (23)$$

where $\alpha$ is a smoothing factor, when $0.95 \leq \alpha < 1$. The constant $\mathbb{C}$ is a positive constant referred to (19).

### 4.2 Proposed Variable averaging Step-size OS-NALMS (VASOS-NALMS) algorithm

The objective is to ensure a large step-size parameter $\mu(k)$, when the algorithm is far from an optimal value with step-size parameter reducing as we approach the optimum [7].

We introduce a new variable averaging step-size order statistic normalised averaged least mean square (VASOS-NALMS) algorithm with the method of estimate of an averaging of $\left\|\widehat{\boldsymbol{\vartheta}}(k)\right\|^2$ as

$$\mu(k) = \gamma \cdot \mu(k-1) + \beta \cdot \hat{\kappa}(k)\,, \quad (24)$$

$$\hat{\kappa}(k) = \alpha \cdot \hat{\kappa}(k-1) + (1-\alpha)\frac{\left\|\widehat{\boldsymbol{\vartheta}}(k)\right\|^2}{\left\|\widehat{\boldsymbol{\vartheta}}(k)\right\|^2 + \mathbb{C}}\,, \quad (25)$$

where $0 \leq \gamma < 1$ and $\beta$ is an independent parameter for scaling the prediction of averaging of $\left\|\widehat{\boldsymbol{\vartheta}}(k)\right\|^2$. The exponentially-weighting parameter $\alpha$ should be close to 1. The parameter $\mathbb{C}$ is referred to (19).

Therefore, the proposed variable step-size OS-NALMS (VSOS-NALMS) algorithm and variable averaging step-size OS-NALMS (VSOS-NALMS) algorithm using adaptive data-reuse factor are summarized as introduced in Table 2 and 3, respectively.

## 5. PROPOSED DATA-REUSE FACTOR

In this section, we describe how to assign a proper data-reuse factor that utilised a variable number of taps at each coefficient update. We consider a rule for choosing the number of data-reuses $\mathbb{D}(k)$ at each update.

The idea of a suitable assignment rule is to increase the data-reuse factor when the solution is far from steady-state and to reduce the factor when close to stead-state [5].

The quantisation levels can be expressed into the maximum number of data-reuses allowed $\mathbb{D}_{\max}$ regions as

$$\mathcal{L}_d = \{l_{d-1} < \mu(k) \leq l_d\}\,,\quad d = 1, \ldots, \mathbb{D}_{\max} - 1 \quad (26)$$

that is defined by the decision level $l_d$. The variable data-reuse factor is given by

$$\mathbb{D}_{\max} = d\,,\text{if } \mu(k) \in \mathcal{L}_d\,. \quad (27)$$

The decision levels $l_d$ is used by the relation

$$l_d = e^{-\psi(\mathbb{D}_{\max}-d)/\mathbb{D}_{\max}}\,, \quad (28)$$

where $\psi$ is a positive constant and $l_0 = 0$. The number of data-reuse in the practical case should be changed for a maximum of five reuses as describes in [5].

Therefore, a variable data-reuse factor can be defined by

$$\mathbb{D}(k) = \max\left\{1, \left\lceil \mathbb{D}_{\max}\left(1 + \frac{1}{\psi}\ln \mu(k)\right)\right\rceil\right\}\,, \quad (29)$$

where $\lceil\cdot\rceil$ is the ceiling operator. The values of the decision variables $l_d$ of variable data-reuse factor approach provided in Table I are calculated with the above expression using $\psi = 4$ and $\mathbb{D}_{\max} = 5$.

***Table 1*** *Quantisation levels for* $\mathbb{D}_{max} = 5$ *and* $\psi = 4$.

| $\mathcal{L}_d$ | $\mathbb{D}(k)$ |
|---|---|
| $\mu(k) \leq 0.0408$ | 1 |
| $0.0408 < \mu(k) \leq 0.0907$ | 2 |
| $0.0907 < \mu(k) \leq 0.2019$ | 3 |
| $0.2019 < \mu(k) \leq 0.4493$ | 4 |
| $\mu(k) > 0.4493$ | 5 |

## 6. SIMULATION RESULTS

The performance for proposed VSOS-NALMS and VASOS-NALMS algorithms are presented by carrying out of the computer simulations. The experiment is implemented for discrete multitone (DMT) systems or high speed wired multicarrier transmission as asymmetric digital subscriber lines (ADSL) downstream simulation that comprises 512 coefficients of channel impulse response.

The carrier serving area (CSA) loop no. 2 was a representative of simulations with all 8 CSA loops detailed in [13]. The CSA#2 loop is of 26- and 24-gauge loop of length of 3000 and 700 ft, with 26 gauge bridged taps of length of 700 ft at 3700 ft and of 24- and 26-gauge loop of length of 350 and 3000 ft with 26-gauge bridged taps of length of 650 ft at 7050 ft. We implement the transmission channel based on parameters as follows: the sampling rate $f_s = 2.208$ MHz, the size of FFT $N = 512$, the synchronisation delay $\Delta = 28$, the length of cyclic prefix $\nu = 32$ and the input signal power of -40dBm/Hz [11]. The signal to noise ratio gap of 9.8dB, the coding gain of 4.2dB and the noise margin of 6dB are chosen for all active tones. The AWGN with a power of $-140$dBm/Hz and NEXT from 24 ADSL disturbers are included.

The initial parameters of proposed VSOS-NALMS and VASOS-OSNALMS algorithms are as follows: $\hat{\boldsymbol{w}}(0) = \mathbf{0}, \alpha = 0.975, \tilde{e}(0) = \sigma_\eta^2$. The other parameters of proposed VASOS-NALMS algorithm are as: $\alpha = 0.97,\ \gamma = 0.975$ and $\beta = 1.95 \times 10^{-3}$.

The proposed algorithms can computed with the soft-constrained initialization. The number of tap of proposed algorithm is set to 32. The chosen active tones at 200 was a representative of the high tone (frequency) causing the largest attenuation of active tones starting at tones 38 to 255 for ADSL downstream.

Fig. 1 shows the mean square deviation (MSD) of vector $\|\hat{\boldsymbol{w}}(k) - \hat{\boldsymbol{w}}(k-1)\|^2$ of proposed algorithms with the initial step-size parameter as: $\mu(0) = 0.005$ of proposed VSOS-NALMS and $\mu(0) = 0.001$ of proposed VASOS-NALMS algorithms. Fig. 2 illustrates the mean square errors (MSE) of proposed algorithms compared with the conventional OSNALMS [1] as described in (5)-(7).

Fig. 3 depicts the learning curves of step-size $\mu(k)$ of proposed algorithm with the different of initial step-size parameter as: $\mu(0) = 0.005$ and 0.100 of proposed VSOS-NALMS and $\mu(0) = 0.001$ and 0.100 of proposed VASOS-NALMS algorithms. It is shown to convege to its own equilibrium.

***Table 2*** *Summary of variable step-size OS-NALMS (VSOS-NALMS) algorithm with data-reuse factor*

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Starting with the soft-constrained initialization as: $\hat{\boldsymbol{w}}(0) = \mathbf{0}$

For $k = 1,2,\dots,K$

1) To arrange the estimated error $\tilde{e}(k)$ and order transformation matrix $\widetilde{\boldsymbol{\mathcal{M}}}(k)$ as:

$$\tilde{e}(k) = \tilde{d}(k) - \hat{\boldsymbol{w}}^H(k-1)\cdot\tilde{\boldsymbol{x}}(k) .$$

$$\widetilde{\boldsymbol{\mathcal{M}}}(k) = \boldsymbol{\mathcal{T}}\{\tilde{\boldsymbol{x}}(k)\cdot\tilde{e}^*(k),\ \tilde{\boldsymbol{x}}(k-1)\cdot\tilde{e}^*(k-1), \cdots, \tilde{\boldsymbol{x}}(k-\mathbb{D}(k)+1)\cdot\tilde{e}^*(k-\mathbb{D}(k)+1)\}\ ,$$

$$\tilde{\boldsymbol{a}} = [a(1), a(2), \dots, a(\mathbb{D}(k))]\ ,$$

$$a(i) = \frac{1}{\mathbb{D}(k)}\ ;\ i = 1, 2, \dots, \mathbb{D}(k)$$

2) To assign the deviation vector $\boldsymbol{\vartheta}(k)$ and $\|\boldsymbol{\vartheta}(k)\|^2$ as:

$$\|\hat{\boldsymbol{\vartheta}}(k)\|^2 = \boldsymbol{\vartheta}^H(k-1)\frac{\boldsymbol{\zeta}(k)\boldsymbol{\zeta}^H(k)}{\boldsymbol{\zeta}^H(k)\boldsymbol{\zeta}(k)}\boldsymbol{\vartheta}(k-1)\,,$$

where

$$\boldsymbol{\vartheta}(k) = \hat{\boldsymbol{w}}(k) - \hat{\boldsymbol{w}}(k-1)\ ,$$

$$\boldsymbol{\zeta}(k) = \frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\tilde{\boldsymbol{a}}}{\|\tilde{\boldsymbol{x}}(k)\|^2}\ .$$

3) To calculate $\hat{\boldsymbol{w}}(k)$ as:

$$\hat{\boldsymbol{w}}(k) = \hat{\boldsymbol{w}}(k-1) - \mu(k)\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k)\cdot\tilde{\boldsymbol{a}}}{\|\tilde{\boldsymbol{x}}(k)\|^2}\ .$$

4) To compute $\mu(k)$ as:

$$\mu(k) = \alpha\cdot\mu(k-1) + (1-\alpha)\frac{\|\hat{\boldsymbol{\vartheta}}(k)\|^2}{\|\hat{\boldsymbol{\vartheta}}(k)\|^2 + \mathbb{C}}\ ,$$

where $\mathbb{C}$ is a small positive constant.

5) To calculate the number of data-reuses $\mathbb{D}(k)$ as:

$$\mathbb{D}(k) = \begin{cases} 1\,, \text{if} & \mu(k) \le 0.0408 \\ 2\,, \text{if} & 0.0408 < \mu(k) \le 0.0907 \\ 3\,, \text{if} & 0.0907 < \mu(k) \le 0.2019 \\ 4\,, \text{if} & 0.2019 < \mu(k) \le 0.4493 \\ 5\,, & \text{otherwise.} \end{cases}$$

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***Table 3*** *Summary of variable averaging step-size OS-NALMS (VASOS-NALMS) algorithm with data-reuse factor*

Starting with the soft-constrained initialization as: $\hat{\boldsymbol{w}}(0) = \mathbf{0}$

For $k = 1,2,\dots,K$

1) To arrange the estimated error $\tilde{e}(k)$ and order transformation matrix $\widetilde{\boldsymbol{\mathcal{M}}}(k)$ as:

$$\tilde{e}(k) = \tilde{d}(k) - \hat{\boldsymbol{w}}^H(k-1) \cdot \tilde{\boldsymbol{x}}(k) .$$

$$\widetilde{\boldsymbol{\mathcal{M}}}(k) = \boldsymbol{\mathcal{T}}\{\tilde{\boldsymbol{x}}(k) \cdot \tilde{e}^*(k),\ \tilde{\boldsymbol{x}}(k-1) \cdot \tilde{e}^*(k-1), \cdots, \tilde{\boldsymbol{x}}(k - \mathbb{D}(k)+1) \cdot \tilde{e}^*(k-\mathbb{D}(k)+1)\} ,$$

$$\tilde{\boldsymbol{a}} = [a(1), a(2), \dots, a(\mathbb{D}(k))] ,$$

$$a(i) = \frac{1}{\mathbb{D}(k)} \ ; \ i = 1,2,\dots,\mathbb{D}(k)$$

2) To assign the deviation vector $\boldsymbol{\vartheta}(k)$ and $\|\boldsymbol{\vartheta}(k)\|^2$ as:

$$\|\hat{\boldsymbol{\vartheta}}(k)\|^2 = \boldsymbol{\vartheta}^H(k-1)\frac{\boldsymbol{\zeta}(k)\boldsymbol{\zeta}^H(k)}{\boldsymbol{\zeta}^H(k)\boldsymbol{\zeta}(k)}\boldsymbol{\vartheta}(k-1) ,$$

where

$$\boldsymbol{\vartheta}(k) = \hat{\boldsymbol{w}}(k) - \hat{\boldsymbol{w}}(k-1) ,$$

$$\boldsymbol{\zeta}(k) = \frac{\widetilde{\boldsymbol{\mathcal{M}}}(k) \cdot \tilde{\boldsymbol{a}}}{\|\tilde{\boldsymbol{x}}(k)\|^2} .$$

3) To calculate $\hat{\boldsymbol{w}}(k)$ as:

$$\hat{\boldsymbol{w}}(k) = \hat{\boldsymbol{w}}(k-1) - \mu(k)\frac{\widetilde{\boldsymbol{\mathcal{M}}}(k) \cdot \tilde{\boldsymbol{a}}}{\|\tilde{\boldsymbol{x}}(k)\|^2} .$$

4) To compute $\mu(k)$ as:

$$\mu(k) = \gamma \cdot \mu(k-1) + \beta \cdot \hat{\kappa}(k) ,$$

$$\hat{\kappa}(k) = \alpha \cdot \hat{\kappa}(k-1) + (1-\alpha)\frac{\|\hat{\boldsymbol{\vartheta}}(k)\|^2}{\|\hat{\boldsymbol{\vartheta}}(k)\|^2 + \mathbb{C}} ,$$

where $\mathbb{C}$ is a small positive constant.

5) To calculate the number of data-reuses $\mathbb{D}(k)$ as:

$$\mathbb{D}(k) = \begin{cases} 1, \text{if} & \mu(k) \le 0.0408 \\ 2, \text{if} & 0.0408 < \mu(k) \le 0.0907 \\ 3, \text{if} & 0.0907 < \mu(k) \le 0.2019 \\ 4, \text{if} & 0.2019 < \mu(k) \le 0.4493 \\ 5, & \text{otherwise.} \end{cases}$$

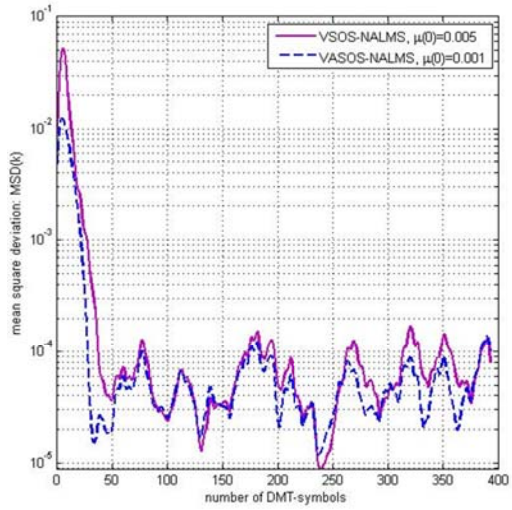


***Fig. 1*** *MSD of proposed algorithms; DMT-based systems.*

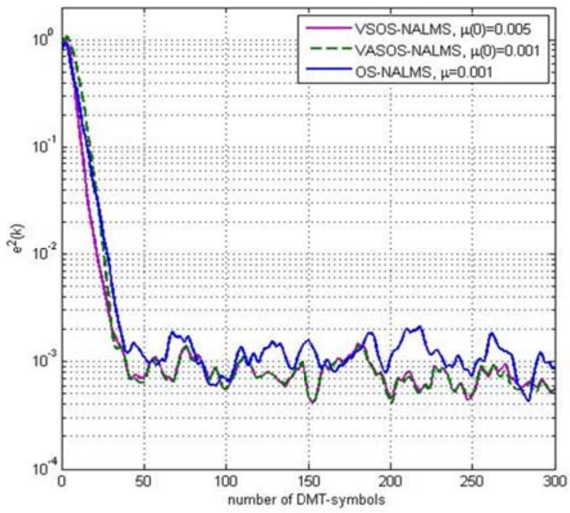


***Fig. 2*** *MSE of proposed algorithms; DMT-based systems.*

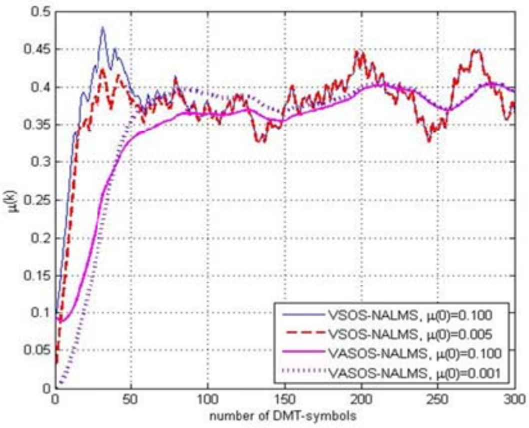


***Fig. 3** Learning curves of step-size $\mu(k)$ of proposed algorithms; DMT-based systems.*

## 7. CONCLUSIONS

In this paper, we have introduced the new variable step-size mechanism in order to achieve the optimal step-size parameter based on order statistic normalised least mean square algorithm. We have described how to investigate the optimal step-size algorithm in virtue of mean square deviation criterion. The idea of using time averaging can apply to the proposed variable step-size algorithms. Two variable step-size algorithms have been proposed with the method of OS-NALMS algorithm. The simulation results have demonstrated that the proposed algorithm can perform well and achieve a fast convergence.

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**Suchada Sitjongsataporn** received the B.Eng. (First-class honours) and D.Eng. degrees in Electrical Engineering from Mahanakorn University of Technology, Bangkok, Thailand in 2002 and 2009, respectively. She has worked as lecturer at department of Electronic Engineering, Mahanakorn University of Technology since 2002. Currently, she is an Assistant Professor in Mahanakorn University of Technology since 2012. She was a visiting professor in November 2012 at Utsunomiya University, Utsunomiya in Japan. Her research interests are in the area of adaptive signal processing for communications, image processing and Kansei Engineering. She is a member of IEEE and the Japan Society of Kansei Engineering (JSKE).

**Kodchakorn Na Nakornphanom** received the B.Ed. (Mathematics) and M.Sc. (Computational Science) degrees in department of Mathematics, Faculty of Science from Chulalongkorn University, Bangkok, Thailand in 1998 and 2001, respectively. Currently, she is working towards the Ph.D. degree in Mathematics, Chulalongkorn University. She is now a lecturer at school of Science and Technology, Sukhothai Thammathirat Open University, Nonthaburi, Thailand. Her research interests include neural network and classification.