Correction of Steady-State Mean Square Error of Unbiased Plain Gradient Algorithm for a Second-Order Adaptive IIR Notch Filter

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ABSTRACT

This paper presents the corrected steady-state mean square error (MSE) of the unbiased plain gradient (UPG) algorithm for a second-order adaptive IIR notch filter (ANF) for sinusoidal signal frequency detection. The corrected MSE in white Gaussian noise of the estimated filter coefficient is derived in closed form. Computer simulations are conducted to corroborate the theoretical analysis and to demonstrate its comparative performance with the previous analysis.

Keywords: Adaptive notch filter; MSE Analysis; Unbiased algorithm

1. INTRODUCTION

Frequency estimate based on adaptive method [1]-[9] plays a major role in digital signal processing application such as radar, sonar, telecommunication engineering, control engineering, biomedical engineering and so on. The frequency estimation based on ANF [1]-[8] is one of many methods that can be used to serve such applications. From the past several years to the present, the methods to improve the structure of ANF and the methods of adaptive algorithm have been intensively studied. It is found in the literature survey that the ANF with constrained poles and zeros [1] is the most popular type of filter which also will be considered in this work. In addition, many adopted gradient-based adaptive algorithms including the plain gradient (PG) [2], the modified PG (MPG) [3], the modified sign algorithm (MSA) [4], the unbiased PG (UPG) [5] algorithm and so on, are suitable for real time applications due to simplicity and economy. Moreover,

their performance/complexity trade-off is well understood. Recently, Loetwassana *et al.* [5] have proposed an efficient UPG algorithm and analyzed this algorithm using the theoretical framework studied in [2] and [3]. However, the accurate result for steady-state MSE has not been obtained due to the nonlinearity of the UPG and an inappropriate analysis method used.

In this paper, the steady-state MSE is re-analyze by modifying and applying the method studied in [6] to the case of an ANF using UPG algorithm. The corrected analytical expression for the steady-state MSE is derived and relationships between the MSE and the filter parameters are shown.

2. UPG AND PROPOSED MSE ANALYSIS

Assume that a noisy sinusoidal signal that we want to evaluate its frequency has the form of

$$x(k) = A\cos(\omega_0 k + \theta) + v(k), \tag{1}$$

where A is the signal amplitude, ω_0 is the signal frequency, and θ is the signal phase with uniformly distributed over $(0, 2\pi]$. $v(k)$ is a white Gaussian noise with zero mean and variance σ_v^2. The system function [1] of the ANF with constrained poles and zeros is given by

$$H(z) = \frac{N(z)}{D(z)} = \frac{1 + az^{-1} + z^{-2}}{1 + \rho a z^{-1} + \rho^2 z^{-2}}, \tag{2}$$

where $N(z)$ and $1/D(z)$ are, respectively, all zeros and all poles systems. The input $x(k)$ to outputs $e_1(k)$ and $e_2(k)$ of $N(z)$ and $H(z)$, respectively, can be described by the following difference equations:

$$e_1(k) = x(k) + ax(k-1) + x(k-2), \tag{3}$$

$$e_2(k) = e_1(k) - \rho a e_2(k-1) - \rho^2 e_2(k-2). \tag{4}$$

a is the filter coefficient and related with input frequency ω_0 by $a_0 = -2\cos\omega_0$. Note that, if a is adjusted till $a = a_0$, the evaluated frequency ω_0 is obtained. ρ is the pole radius and close but less than 1 to maintain the system stability. The UPG [5] that adjusts the coefficient a of (2) is given by

$$a(k+1) = a(k) - \mu e_2(k)g(k) + \mu C e_1(k)g(k), \tag{5}$$

where $a(k)$ is the estimate of a at time k, μ is the stepsize and real positive number, $g(k)$ [3] is the gradient signal and given by

$$g(k) = x(k-1), \tag{6}$$

and $C = 1 - \rho$. In order to re-analyze the MSE of (5), the steady-state expressions [3], [5] for the signal $e_1(k)$, $e_2(k)$ and $g(k)$ are employed and are of the following forms:

$$e_1(k) = A\delta_a(k)\cos(\omega_0 k + \theta - \omega_0) + v_1(k), \tag{7}$$

$$e_2(k) = AB\delta_a(k)\cos(\omega_0 k + \theta - \phi)$$
$$-\rho AB^2 \delta_a^2(k)\cos(\omega_0 k + \theta - 2\phi) + v_2(k), \tag{8}$$

$$g(k) = A\cos(\omega_0 k + \theta - \omega_0) + v_3(k), \tag{9}$$

where $v_1(k)$ and $v_2(k)$ are, respectively, the output noise components of $N(z)$ and $H(z)$ due to input noise

$v(k)$. $v_3(k)$ is the output noise component of gradient transfer function [3] $H_g(z) = z^{-1}$ due to noise $v(k)$. The variance [3], [5] of each noise component is, respectively, computed by

$$\sigma_1^2 = (2 + a_0^2)\sigma_v^2, \tag{10}$$

$$\sigma_2^2 = \frac{\sigma_v^2}{\rho^2} - \frac{1-\rho}{1+\rho} \frac{1+\rho^2}{\rho^2} \frac{1+\rho^2 - 2\rho^2 a_0^2}{1+\rho^2 - \rho^2 a_0^2}\sigma_v^2, \tag{11}$$

$$\sigma_3^2 = \sigma_v^2. \tag{12}$$

B and ϕ are, respectively, magnitude and phase of $H(z)$ and given by [3]

$$B = \frac{1}{(1-\rho)\sqrt{(1-\rho)^2 - \rho a_0^2}}, \tag{13}$$

$$\phi = \begin{cases} \tan^{-1}\left(\dfrac{1+\rho}{1-\rho}\dfrac{\sin\omega_0}{\cos\omega_0}\right), & \omega_0 \le \pi/2 \\[4mm] \pi + \tan^{-1}\left(\dfrac{1+\rho}{1-\rho}\dfrac{\sin\omega_0}{\cos\omega_0}\right), & \omega_0 > \pi/2. \end{cases} \tag{14}$$

In addition, the theoretical framework adopted in [6] is modified and applied to analyze the UPG algorithm instead of the adopted method in [2].

The analysis begins with subtracting both sides of (5) with a_0, squaring and taking expectation of the results. These operations yield

$$2E[\delta_a(k)\{e_2(k)g(k) - Ce_1(k)g(k)\}]_{k\to\infty}$$
$$= \mu E\left[\{e_2(k)g(k) - Cg(k)e_1(k)g(k)\}^2\right]_{k\to\infty}, \tag{15}$$

where $E[\bullet]$ is an expectation operator and $\delta_a(k) = a(k) - a_0$ is an estimation error. By using the steady-state expressions for $e_1(k)$, $e_2(k)$ and $g(k)$ given in (7) - (14), the component related to signal and noise in the right-hand side (RHS) of (16) is derived as follows:

$$\mu E\Big[\{e_2(k)g(k)-Ce_1(k)g(k)\}^2\Big]_{k\to\infty} \qquad (16)$$
$$\simeq \mu\psi_1 E[\delta_a^2(k)]+\mu\psi_2,$$

where

$$\psi_1 = \frac{1}{4}A^4B^4\left(1+\frac{1}{2}\cos(2\omega_0-2\phi)\right) \qquad (17)$$
$$-\frac{3}{4}A^4CB\cos(\omega_0-\phi)+\frac{3}{8}A^4C^2,$$

$$\psi_2 = \sigma_2^2\sigma_3^2+2R_{23}^2-2C(R_{12}\sigma_3^2+2R_{13}R_{23}) \qquad (18)$$
$$+C^2(\sigma_1^2\sigma_3^2+2R_{13}^2)$$

and [5]

$$R_{12} = E[v_1(k)v_2(k)] = (1+2C-C^2\{1-a_0^2\})\sigma_v^2, \qquad (19)$$
$$\simeq \mu\psi_1 E[\delta_a^2(k)]+\mu\psi_2,$$

$$R_{13} = E[v_1(k)v_3(k)] = a_0\sigma_v^2, \qquad (20)$$

$$R_{23} = E[v_2(k)v_3(k)] = CR_{13}. \qquad (21)$$

Furthermore, the component due to signal and noise in the left-hand side (LHS) of (16) can be evaluated as

$$2E[\delta_a(k)\{e_2(k)g(k)-Ce_1(k)g(k)\}]_{k\to\infty}$$
$$\simeq 2E[\delta_a(k)\{e_2(k)g(k)-Ce_1(k)g(k)\}]_{k\to\infty}$$
$$-2\mu E\Bigg[\sum_{i=1}^{2}(e_2(k-i)g(k-i)-Ce_1(k-i)g(k-i))$$
$$\times (e_2(k)g(k)-Ce_1(k)g(k))\Bigg]_{k\to\infty}$$
$$\approx (\psi_0-\mu\{\psi_3+\psi_5\})E[\delta_a^2(k)]-\mu\psi_4, \qquad (22)$$

where

$$\psi_0 = A^2(B\cos(\omega_0-\phi)-C), \qquad (23)$$

$$\psi_3 = \frac{1}{2}A^4B^2\left(\cos^2(\omega_0-\phi)+\frac{1}{2}\cos(2\omega_0)\right)$$
$$-\frac{1}{2}A^4CB\left(\cos(\phi-\omega_0)+\frac{1}{2}\cos(2\omega_0+\phi)\right)$$
$$-\frac{1}{2}A^4CB\left(\cos(\omega_0-\phi)+\frac{1}{2}\cos(3\omega_0-\phi)\right)$$
$$+\frac{A^4}{2}C^2\left(\frac{1}{2}+\cos^2\omega_0\right), \qquad (24)$$

$$\psi_4 = 2\ R_{23}^2-a_0CR_{23}\sigma_v^2+(C^2-C)\sigma_v^4 \qquad (25)$$
$$+(1-C)\{1-\rho^2+a_0^2(\rho^2-\rho)\}\sigma_v^4\ ,$$

$$\psi_5 = \frac{1}{2}A^4B^2\left(\cos^2(\phi-\omega_0)+\frac{1}{2}\cos(4\omega_0)\right)$$
$$-\frac{A^4}{2}CB\left(\cos(\phi-\omega_0)+\frac{1}{2}\cos(\phi-3\omega_0)\right)$$
$$-\frac{A^4}{2}CB\left(\cos(\phi-\omega_0)+\frac{1}{2}\cos(5\omega_0-\phi)\right)$$
$$+\frac{A^4}{2}C^2\left(1+\frac{1}{2}\cos(2\omega_0)\right). \qquad (26)$$

Arranging (16) and (22) and letting $k=\infty$ result in

$$E[\delta_a^2(\infty)] = \frac{\mu(\psi_2+\psi_4)}{\psi_0-\mu(\psi_1+\psi_3+\psi_5)}. \qquad (27)$$

From (27), the bound for stepsize parameter in the sense of MSE therefore is

$$0 < \mu < \frac{\psi_0}{\psi_1+\psi_3+\psi_5}\ . \qquad (28)$$

The corrected closed form expression for steady-state MSE given in (27) is obtained by using the following assumptions:

A1: All appeared sinusoidal components have zero mean and variance of 0.5.

A2: $\delta_a(k)$, all noise components and sinusoidal signal are uncorrelated with each other.

A3: $\delta_a^m(k)$ for $m>2$ are ignored for analytical simplicity.

A4: Stepsize μ used must be sufficient small.

A5: The Gaussian moment factoring theorem [2] is used to distribute the correlation $E[u_1(k)u_2(k)u_3(k)u_4(k)]$ where $u_i(k), i = 1,2,...,4$ all are Gaussian random variables of zero mean and the residue technique is applied to calculate the correlation $E[u_i(k)u_j(k)], j = 1,2,...,4$.

We now have ultimately derived the closed form expression for the steady-state MSE and the bound of stepsize parameter. Obviously, the MSE is a function of stepsize parameter. The smaller the stepsize value is, the lower the MSE will be. In the next section, the computer simulations are conducted to corroborate the theoretical result.

3. RESULTS AND DISCUSSION

Herein, the corrected MSE of the UPG algorithm is compared with the previous one [5]. To obtain the steady-state MSE of an algorithm from simulations, the iterations are allowed to be large enough so that the simulated MSE reaches its steady-state. The ensemble averages of fifty (50) runs are calculated to determine the simulated value of the steady-state MSE and are shown in the following figures. Figs. 1-4 demonstrate the comparisons between analytical MSEs and their simulated values versus frequency ω_0, pole contraction factor ρ, stepsize parameter μ, and noise variance σ_v^2, respectively. From these figures, the old analysis [5] has only the similar tendency of the MSE variation to the simulated results whereas the new analytical result show well agreement with the simulated values provided that the stepsize μ used is sufficient small $(\mu < 10^3$ see Fig. 3$)$. As a result, the corrected theoretical MSE of the UPG is therefore proved.

4. Conclusion

In this paper, the corrected steady-state MSE analysis framework has been developed for the UPG algorithm. It has been found that the proposed method result in high accuracy of the steady-state MSE and accordant with the simulated values provided that the stepsize value is sufficient small.
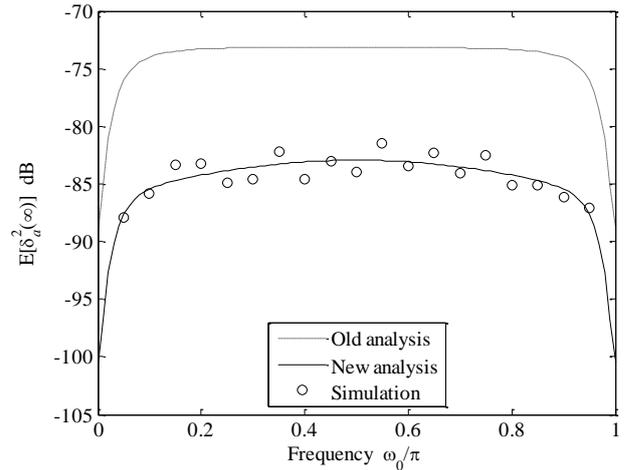

Fig. 1. MSE versus frequency
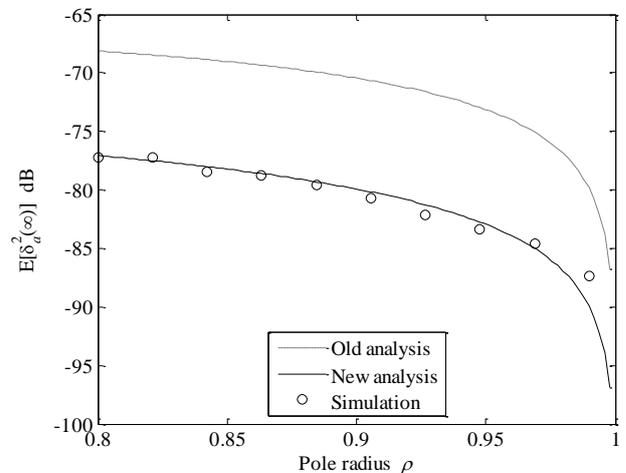$(A = \sqrt{2}, \ \theta = 0.1\pi, \ SNR = 10dB, \ \rho = 0.95, \ \mu = 10^{-5})$.

Fig. 2. MSE versus pole radius
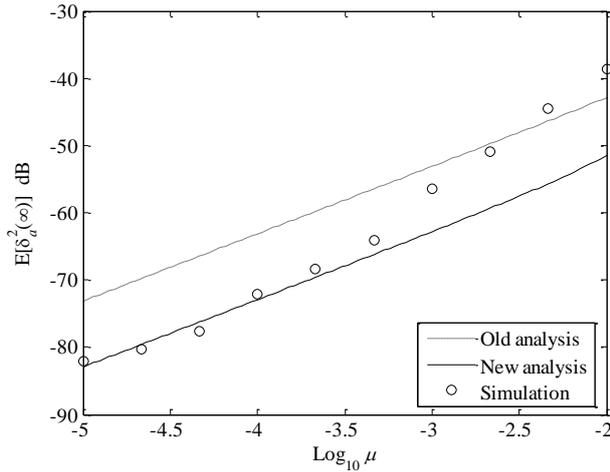$(A = \sqrt{2}, \ \theta = 0.1\pi, \ SNR = 10dB, \ \omega_0 = 0.5\pi, \ \mu = 10^{-5})$.

Fig. 3. MSE versus stepsize

$(A = \sqrt{2}, \ \theta = 0.1\pi, \ SNR = 10dB, \ \rho = 0.95, \ \omega_0 = 0.5\pi)$.
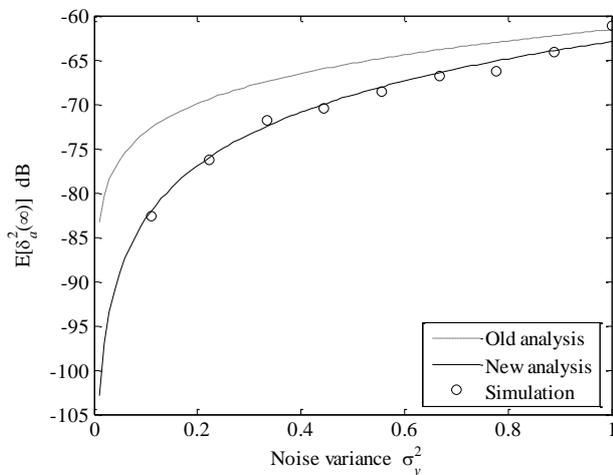

Fig. 4. MSE versus noise variance

$(A = \sqrt{2}, \ \theta = 0.1\pi, \ \omega_0 = 0.5\pi, \ \rho = 0.95, \ \mu = 10^{-5})$.

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