Mathematical Formulation in Bending of Rectangular Thin Plates

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ABSTRACT

The main purpose of this paper is to review the fundamental feature of mathematical formulations involved the theory of thin elastic rectangular plates. All of the necessary equations are analytically given in terms of only one variable; namely the deflection of plate, which are the fourth-order partial differential equation governing the plate-bending behaviors, plate deformations (deflection, slope and curvature), and stress resultants of the plate. In order to obtain these equations, certain assumptions will be stated and referred to when related to any portion of the mathematical formulations and classical boundary support conditions. Therefore, a full description of the boundary conditions is also given. In addition, two classical well-known analytical solution approaches are presented in details for solving some basic problems of uniformly loaded rectangular plates.

Keywords : *Analytical Solution, Boundary Conditions, Partial Differential Equation, Rectangular Plate.*

1. INTRODUCTION

A plate is one of structural elements with planform dimensions that are large compared to its thickness, so the bending properties of a plate depend greatly on its thickness. In most cases, the thickness would not greater than one-tenth of the smallest in-plane dimension. Because of the smallness of thickness dimension, it is often not necessary to model them using three-dimensional elasticity equations. Only simple two-dimensional plate theories can be developed properly to study the deformation and stresses in plate structures.

Plate-like structures are one of the most commonly used structural components in the world throughout various fields of engineering design and application in which such problems have applications in the design of aerospace and aircraft for astronautics engineering, missile in military engineering, off-shore platforms in ocean engineering and ship structures as well as the construction of building floors or roof structures in civil engineering, etc. The plates, by nature, are used to carry lateral loads, in-plane loads, and sometimes both. Generally, the structural engineering components must resist not only static loads [1] but also dynamic loads [2]. It is, therefore, necessary to understand the behavior of these plate structures; namely, their deflections due to various static and dynamic loads, their tendency to resonate, and their susceptibility to buckling. Hence, knowing the static response analysis of the plates alone is not sufficient to ensure proper, even safe and performance. Their design needs to include the effects of periodic or random time varying forces causing vibration in order to determine the natural frequencies, mode shapes of vibration, and the dynamic responses [2], [3]. Additionally, among all the possible shapes in planform of the plates such as rectangular, circular, elliptical, triangular, or parallelogram plates, it may be noted that rectangular plate is of the greatest importance and interest in analysis. This can be seen in and confirmed by the most research works that found in the scientific and technical literature or in the Leissa's monograph [2].

2. OBJECTIVES AND SCOPES

Although the subject of static bending analysis of plate under arbitrary loads has a long history which goes back for more than two centuries since in 1766, it remains a fact that most theoretical solutions to this classical problem are considered to be at best approximate in nature. This is because of the difficulties which have been encountered in trying to obtain the solution which satisfy all of boundary conditions as well as the governing differential equation. Thus, the main objective of this paper is attended to review and focus on the derivation of basic mathematical formulations for the equation development in plate-bending behaviors. Significantly, the plate theory presented is only restricted and based on the Kirchhoff's plate assumptions [1] satisfying thin plate deformations. This means to other transverse shearing effects on the deformations of plate are not accounted for, and so that they are neglected in the equation formulations.

However, in the statements of equation formulation pertaining to the thin plates, rectangular plate is first emphasized on the problem consideration. Therefore, the outcomes of this paper can be listed and drawn in the following details in order to achieve: 1) the equations for the plate deformations; namely, the deflection, slopes, and curvatures of the plate; 2) the internal forces of the plate resisting to the applied external transverse loads, which are called the plate stress resultants; 3) the partial differential equation governing to the static plate-bending behaviors; 4) the equation conditions satisfied to the classical boundary conditions; namely, the conditions for the simply supported, clamped and free edges, and finally; 5) the solutions for problem example based on two well-known classical approaches, which are the Navier- and Levy-types analytical solution.

3. MATHEMATICAL FORMULATION

3.1 Basic Assumptions

The equation formulations of rectangular plate that considered in this paper as described in the aforementioned sections, the plate is assumed to be thin plates with small deflection assumption following the Kirchhoff's plate theory [1]. With this plate theory, all of the necessary equations can be derived analytically in terms of only one dependent variable called the lateral deflection of the plate. A fundamental feature of the mathematical formulations has to be subjected to certain assumptions which should be made clear in the analysis and derivation. Thus, the assumptions can be stated as follows: (1) the middle plane of the plate remains neutral during bending, that is, it does not undergo any extension (no stretching during bending on this middle plane); (2) any points of the plate lying initially on a normal to the middle plane of the plate remain on the normal to the middle surface of the plate after bending; (3) the normal stresses in the direction transverse to the plate can be disregarded; (4) the lateral load acting on the plate is normal to its surface; (5) all the deflections are small in comparison with the thickness of the plate; (6) plate thickness is small compared to its side dimensions so that effects of the transverse shear deformation are then negligible, and; (7) the plate material is homogeneous, isotropic and linearly elastic with obeying Hooke's law.

Using the first three assumptions (assumptions 1, 2, 3), all stress components that will be shown in the later stage can be expressed by deflection of the plate, which is a function of the two coordinates in the plane of the plate. It can be noted that assumption 2 is equivalent to the disregard of the effect of shear forces on the deflection of plates. This mentioned assumption is usually satisfactory, and served to assumption 6. Importantly, in addition to lateral loads (assumption 4), if there are external forces acting in the middle plane of the plate, assumption 1 does not hold any more and then, it is necessary to take into consideration the effect on bending of the plate of the stresses acting in the middle plane of the plate. This is out of the scope of the present paper.

Based on the assumptions stated above, the partial differential equation to be developed can be accepted in the theory of thin plates (sometimes called the classical plate theory: CPT). Also, other equations relating to the plate are all valid in corresponding to the transversely loaded rectangular thin plates.

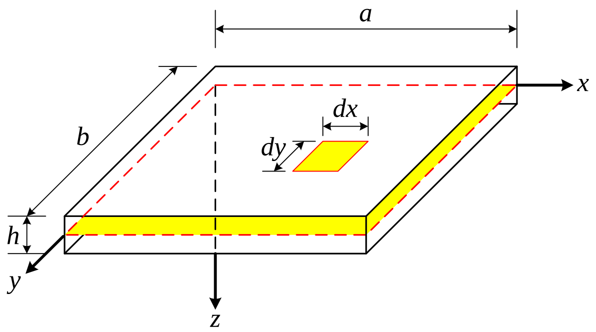

Fig. 1 *Coordinate system definition.*

The coordinate system to be used is illustrated in Fig. 1. The plate has the dimensions of length a and width b in the directions of x and y, respectively; and thickness h in z direction in which its thickness will start from $z=-h/2$ to $z=h/2$. The middle plane of the plate before bending occurs is taken as the xy plane and during bending; any points that were in the xy plane undergo small displacements normal to the xy plane (assumption 5) and form the middle surface of the plate. These displacements of the middle surface are called the deflections of a plate as demonstrated in Fig. 2.

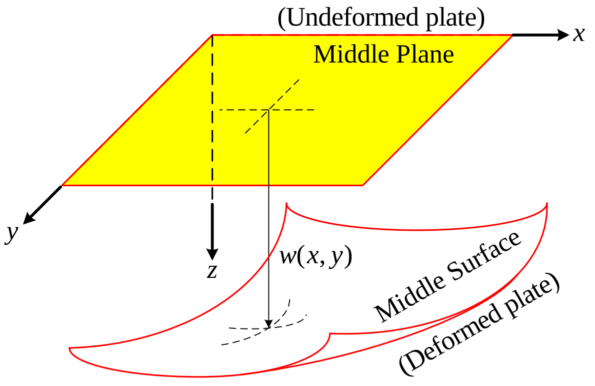

Fig. 2 *Definition of middle plane and middle surface.*

Particularly, it should be kept in mind that by assuming reasonably realistic variations, through the thickness, of displacements, strains and stresses, the thin plate theory is a simplification of the three-dimensional problem to one of two dimensions (those in the plane of the plate). The theory makes the analysis drastically simplified compared to the elasticity approach without unacceptable loss of accuracy. Therefore, the deflection of plate; $w(x,y,z)=w(x,y)$, is independent of z and is defined as the displacement of the middle plane of the plate after bending.

3.2 Plate Deformations

In this section, the slopes and curvatures of the plate will be defined physically and given mathematically in their formulae. From the assumption 1, it is possible to find the strains ($\varepsilon_x, \varepsilon_y$) after bending, of any plane of the plate a distance z from the middle surface.

The unit elongation of the surface for the xz plane as shown in Figure 3 can be determined by

$$\varepsilon_x=\frac{\alpha(r_x+z)-\alpha r_x}{\alpha r_x}=\frac{z}{r_x}, \quad (1)$$

and similarly; for the yz plane,

$$\varepsilon_y=\frac{z}{r_y}, \quad (2)$$

where r_x and r_y are the radii of curvature parallel to the x - and y - axes, respectively, and α is the angle sustained.

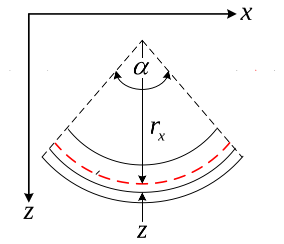

Fig. 3 *Deformed plate segment in xz plane.*

The reciprocal of the radius of curvature parallel to the x - axis or the xz plane that known as the bending curvature (κ_x) is described mathematically as,

$$\kappa_x=\frac{1}{r_x}=\frac{\left|\dfrac{\partial^2 w}{\partial x^2}\right|}{\left[1+\left(\dfrac{\partial w}{\partial x}\right)^2\right]^{3/2}}. \quad (3)$$

Refer to the assumption 5 for small deflection of the plate, the higher order terms in the denominator of Eq.(3) can be neglected that yields

$$\kappa_x=\frac{1}{r_x}=-\frac{\partial^2 w}{\partial x^2}, \quad (4)$$

and in the same manner, the bending curvature in a plane parallel to the yz plane is

$$\kappa_y=\frac{1}{r_y}=-\frac{\partial^2 w}{\partial y^2}, \quad (5)$$

in which the negative sign is introduced in Eqs.(4) and (5) because one consider a positive curvature to be convex downward and the second derivatives $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ are negative [2 p.34].

Considering the curvature in any direction n at a point of the middle surface on the xy plane, the quantity κ_{xy} is defined to be the twisting of the surface with respect to the x and y axes, or the twisting curvature. This can be expressed as

$$\kappa_{xy} = \frac{1}{r_{xy}} = \frac{\partial^2 w}{\partial x \partial y}, \tag{6}$$

where r_{xy} is called the radius of twisting curvature.

When the plate is deformed under the external load, the middle plane in xz plane of the plate rotates through an angle θ_x about the y-axis that called the slope of the plate. Thus, it can be expressed as

$$\theta_x = \frac{\partial w}{\partial x}, \tag{7}$$

and for the yz plane, the angle about the x-axis is

$$\theta_y = \frac{\partial w}{\partial y}. \tag{8}$$

3.3 Stress Resultants of the Plate

As given in the above relationships, it is now possible to express the normal stresses (σ_x, σ_y) acting on the plane in terms of the deflection (w). By making use of the generalized Hooke's law (assumption 7) for the plane-stress reduced constitutive law [4], the stresses can be determined from

$$\sigma_x = \frac{E}{1-\upsilon^2}\left(\varepsilon_x + \upsilon\varepsilon_y\right), \tag{9}$$

$$\sigma_y = \frac{E}{1-\upsilon^2}\left(\varepsilon_y + \upsilon\varepsilon_x\right), \tag{10}$$

where E and υ are both material properties of the plate; namely, Young's modulus and Poisson's ratio, respectively.

Applying Eqs.(1), (2), (4) and (5) in Eqs.(9) and (10), leads to

$$\sigma_x = \frac{Ez}{1-\upsilon^2}\left(\frac{1}{r_x} + \upsilon\frac{1}{r_y}\right) = -\frac{Ez}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right), \tag{11}$$

$$\sigma_y = \frac{Ez}{1-\upsilon^2}\left(\frac{1}{r_y} + \upsilon\frac{1}{r_x}\right) = -\frac{Ez}{1-\upsilon^2}\left(\frac{\partial^2 w}{\partial y^2} + \upsilon\frac{\partial^2 w}{\partial x^2}\right), \tag{12}$$

and also mentioning the assumption 3, which can be implied that

$$\sigma_z = 0. \tag{13}$$

The in-plane shearing stress (σ_{xy}) acting on the xy plane is given by the following

$$\sigma_{xy} = G\gamma_{xy}, \tag{14}$$

where

$$G = \frac{E}{2(1+\upsilon)}, \tag{15}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{16}$$

in which G is the shear modulus, γ_{xy} is the in-plane shearing strain and u, v denote the displacements in the x and y directions, respectively, that expressed in terms of w by

$$u = -z\frac{\partial w}{\partial x}, \tag{17}$$

and

$$v = -z\frac{\partial w}{\partial y}. \tag{18}$$

Finally, back substitution of Eqs.(17) and (18) into Eq.(16), together with using Eq.(15) and then, Eq.(14) becomes

$$\sigma_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y} = -\frac{Ez}{(1+\upsilon)}\frac{\partial^2 w}{\partial x \partial y}. \tag{19}$$

Since the plate is in the deformed state under the applied external load, the resulting action of the stresses is to cause bending and twisting of the plate. Thus, the appropriate stress resultants for the plate theory are bending moments (M_x, M_y) and twisting moment (M_{xy}) per unit length. These stress resultants can be obtained by integrating the stress times the distance from the mid-plane over the thickness, which are:

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz\,, \tag{20}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z dz\,, \tag{21}$$

$$M_{xy} = -\int_{-h/2}^{h/2} \sigma_{xy} z dz\,. \tag{22}$$

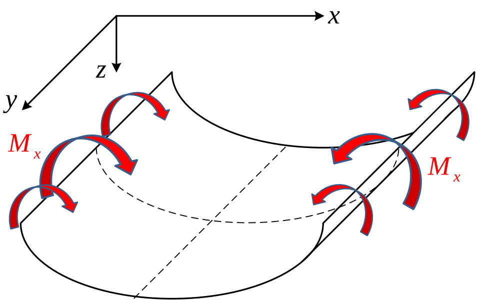

Fig. 4 *Positive direction of bending moment* M_x.

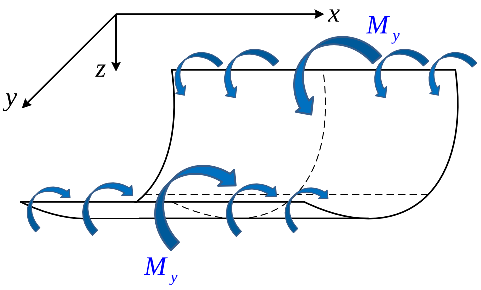

Fig. 5 *Positive direction of bending moment* M_y.

Substituting Eqs.(11), (12), and (19) into Eqs.(20), (21), and (22), respectively, the final results are found to be

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right), \tag{23}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right), \tag{24}$$

$$M_{xy} = -M_{yx} = D(1-\upsilon)\frac{\partial^2 w}{\partial x \partial y}, \tag{25}$$

where D is the flexural rigidity defined as

$$D = \frac{Eh^3}{12(1-\upsilon^2)}\,. \tag{26}$$

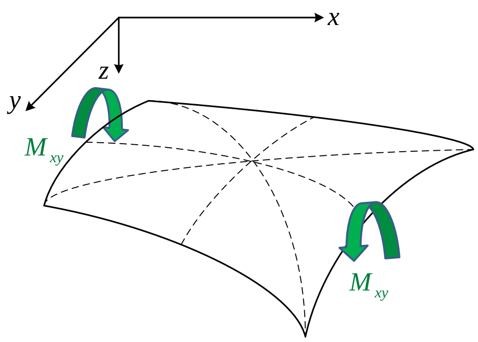

Fig. 6 *Positive direction of bending moment* M_{xy}.

The bending moments and twisting moment acting in the positive direction along the edges of a rectangular plate are demonstrated in Figs 4, 5, and 6. In addition to the bending and twisting moments, there are vertical shearing forces acting on the sides of the plate element to resist the external vertical load in z - direction. The magnitudes of these shearing forces per unit length parallel to the yz plane and xz plane denoted by Q_x and Q_y, respectively, so that

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz} dz\,, \tag{27}$$

$$Q_y = \int_{-h/2}^{h/2} \sigma_{yz} dz\,, \tag{28}$$

in which the stresses σ_{xz} and σ_{yz} are the transverse shearing stresses.

Consider an infinitesimal element $dxdy$ of the plate that presented in Figure 1, the middle plane of the element cut out of the plate is then illustrated in Fig. 7 with showing all the moments and shearing forces acting on the sides of the element and the load distributed over the upper surface of the plate. The intensity of the load is denoted by q, so that the load acting on the element is $qdxdy$.

Since the plate must be in the equilibrium states, three conditions for the equilibrium may be written by summing the forces in the z - direction and summing the moments about the x and y axes, setting them equal to zero as follows:

$$\frac{\partial Q_x}{\partial x}dxdy+\frac{\partial Q_y}{\partial y}dydx+qdxdy=0\,, \tag{29}$$

$$\frac{\partial M_{xy}}{\partial x}dxdy-\frac{\partial M_y}{\partial y}dydx+Q_y dxdy=0\,, \tag{30}$$

$$\frac{\partial M_{yx}}{\partial y}dxdy+\frac{\partial M_x}{\partial x}dydx-Q_x dxdy=0\,. \tag{31}$$

Dividing Eqs.(29)-(31) by $dxdy$, leads to

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0\,, \tag{32}$$

$$\frac{\partial M_{xy}}{\partial x}-\frac{\partial M_y}{\partial y}+Q_y=0\,, \tag{33}$$

$$\frac{\partial M_{yx}}{\partial y}+\frac{\partial M_x}{\partial x}-Q_x=0\,. \tag{34}$$

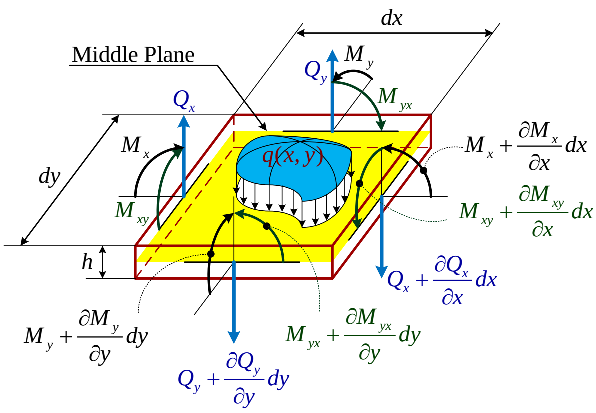

Fig. 7 *Positive sign conventions for stress resultants and applied load.*

Using Eqs.(23) - (25) in Eqs.(33) and (34) for the bending and twisting moments, the shearing forces can then be expressed in terms of the deflection function of the plate, which are

$$Q_x=-D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right), \tag{35}$$

$$Q_y=-D\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial y\partial x^2}\right). \tag{36}$$

It should be remarked that the shearing forces presented in Eqs.(35) and (36) mean to the shearing forces acting in an element of the plate. For the shearing forces distributed along the free edge or the support of the plate, they are called the supplemented or Kirchhoff shearing forces or the vertical edge reactions (V_x, V_y). These shearing forces or support reactions can be obtained by the consolidation of two quantities, which are the shearing force in element and the twisting moment, as shown in the followings below,

$$V_x=Q_x-\frac{\partial M_{xy}}{\partial y}=-D\left[\frac{\partial^3 w}{\partial x^3}+(2-\upsilon)\frac{\partial^3 w}{\partial x\partial y^2}\right], \tag{37}$$

$$V_y=Q_y+\frac{\partial M_{xy}}{\partial x}=-D\left[\frac{\partial^3 w}{\partial y^3}+(2-\upsilon)\frac{\partial^3 w}{\partial x^2\partial y}\right]. \tag{38}$$

However, a more details of discussion for two equations above have been given in [1, p.84]. Particularly, there is a concentrated reaction applied at the right-corner of the rectangular plate. This force is known to be the corner force (R).

Considering a corner of the plate at $x=a$ and $y=b$, the corner force acted upon the plate in the downward direction to prevent the plate corner from rising up during bending can be determined from the relation, together with using Eq.(25),

$$R(a,b)=2M_{xy}(a,b)=2D(1-\upsilon)\left.\frac{\partial^2 w}{\partial x\partial y}\right|_{x=a,\,y=b}. \tag{39}$$

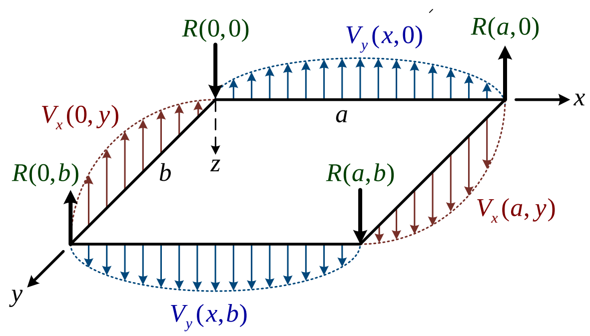

Fig. 8 *Positive directions of distributed vertical edge reactions and corner forces.*

The distribution of vertical edge reactions (V_x, V_y) and the corner forces (R) is also simultaneously demonstrated in Figure 8.

3.4 Plate Governing Equation

The equation that governed to the plate bending behaviors is the equilibrium equation of forces in the z-direction, which has been obtained in Eq.(32). However, it can further be reduced in terms of the deflection function. Substitution of Eq.(33) for Q_y and Eq.(34) for Q_x into Eq.(32), results in

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_{yx}}{\partial x \partial y}+\frac{\partial^2 M_y}{\partial y^2}-\frac{\partial^2 M_{xy}}{\partial x \partial y}=-q . \quad (40)$$

Applying Eq.(25) for the relation of $M_{yx}=-M_{xy}$ by virtue of $\sigma_{yx}=\sigma_{xy}$ that found in an elementary elasticity theory [4] and then, Eq.(40) becomes

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}-2\frac{\partial^2 M_{xy}}{\partial x \partial y}=-q . \quad (41)$$

By substituting Eqs.(23), (24), (25) for the moments M_x, M_y, and M_{xy} into Eq.(41), the final equation can be expressed in terms of the deflection function as

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D} . \quad (42)$$

This equation can also be written in the symbolic form

$$\nabla^4 w=\nabla^2\nabla^2 w=\frac{q}{D} , \quad (43)$$

and

$$\nabla^2 w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2} , \quad (44)$$

in which ∇^4 is called the biharmonic operator in the (x, y)-coordinates and ∇^2 is the harmonic (or Laplace) operator.

4. CLASSICAL BOUNDARY CONDITIONS

Whenever classical boundary conditions are mentioned, the attention is directed toward the three types of boundary conditions that have been studied thoroughly in the classical literature. They are simply supported, clamped, and free edge conditions as illustrated in Fig. 9.

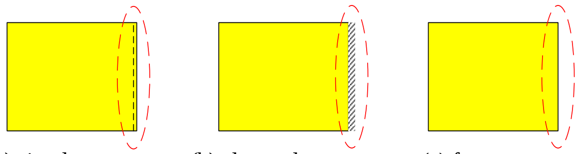

Fig. 9 *Boundary condition Symbols.*

4.1 Simply Supported Edge Condition

A simply supported edge as shown in Figure 9(a) is one which has zero deflection and zero bending moment about the edge direction. Consider a simply supported edge parallel to the y-axis at $x=a$, the boundary conditions can be given by

$$w(a, y)=0 , \quad (45)$$

and using Eq.(23) for the condition of zero bending moment about the edge parallel to the y-axis at $x=a$, thus

$$\left(\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}\right)\bigg|_{x=a}=0 . \quad (46)$$

Since the slope θ_y at $x=a$ is also zero, the curvature κ_y along this edge that defined in Eq.(5) is automatically vanished. Therefore, Eq.(46) reduces to

$$\frac{\partial^2 w}{\partial x^2}\bigg|_{x=a}=\kappa_x(a, y)=0 . \quad (47)$$

4.2 Clamped Edge Condition

For a clamped edge support as seen in Fig. 9(b), it allows no deflection and no slope in the direction perpendicular to the edge. Assuming that the edge parallel to the y-axis at $x=a$ is clamped, the first boundary condition is the same with Eq.(45) and the second boundary condition is

$$\frac{\partial w}{\partial x}\bigg|_{x=a}=\theta_x(a, y)=0 . \quad (48)$$

4.3 Free Edge Condition

A free edge support as shown in Figure 9(c) has no Kirchhoff shearing force along the edge direction and no bending moment about the edge direction. Consider a free edge parallel to the y -axis at $x = a$, the first boundary condition is for the zero Kirchhoff shearing force (V_x) that can be written as, with using Eq.(37),

$$\left[\frac{\partial^3 w}{\partial x^3} + (2-\upsilon)\frac{\partial^3 w}{\partial x \partial y^2}\right]\Bigg|_{x=a} = 0 . \qquad (49)$$

The second boundary condition is the zero bending moment about the edge direction which is similar to that defined in Eq.(46).

5. CLOSED-FORM SOLUTIONS

It is one of the objectives of this paper to present the analytical solution method for solving the bending of uniformly loaded rectangular plates. Two well-known analytical solutions are addressed here. The first is the Navier's solution and the second is the Levy's solution. These two methods will be used to solve the problem of a uniformly loaded simply supported rectangular plate as shown in Figure 10.

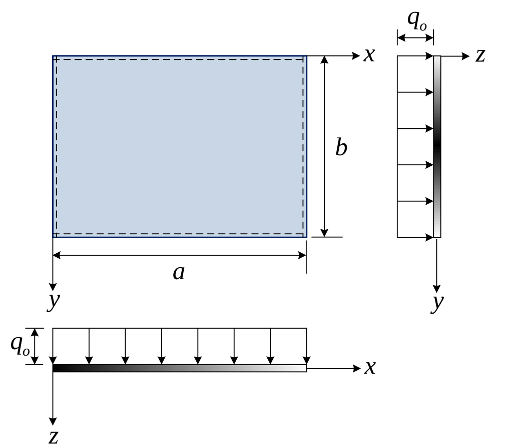

Fig. 10 *Simply supported rectangular plate with uniform loading.*

5.1 Navier-Type Solution

The first analytical solution to be developed for the lateral deflections of a simply supported rectangular plate under an arbitrary transverse loading, $q(x, y)$, is attributed to Navier, C.L.M.H. in 1820. He presented a paper to the French Academy of sciences on the solution of static bending of simply supported rectangular plates by using double trigonometric series. This solution is sometimes called the forced solution. Navier noted that the boundary conditions are automatically satisfied by choosing a deflection function in the form

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} . \qquad (50)$$

It is known from the theory of Fourier series [5], [6] that Eq.(50) can be made to represent any piecewise continuous function over the ranges of $0 \le x \le a$ and $0 \le y \le b$. Consequently, Eq.(50) will define the deflection $w(x, y)$ if one can determine the Fourier's coefficients W_{mn} . Therefore, the problem will complete by determining the coefficients W_{mn} in such a manner that Eq.(50) satisfies the partial differential equation of plates given in Eq.(42).

Substituting Eq.(50) into Eq.(42) yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = \frac{q}{D} . \qquad (51)$$

The coefficients W_{mn} can be determined by first expressing the known load distribution in the form of a double Fourier sine series, which is

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} , \qquad (52)$$

in which, using the orthogonality condition [5],

$$A_{mn} = \frac{4}{ab}\int_{y=0}^{b}\int_{x=0}^{a} q(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, dxdy , \qquad (53)$$

where A_{mn} is the Fourier's coefficients for the distributed load.

Substitution of Eq.(52) into Eq.(51) together with some manipulations, leads to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{W_{mn}\pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 - \frac{A_{mn}}{D}\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0 . \qquad (54)$$

It is also noted that Eq.(54) must be satisfied for all values of x and y . Consequently, the total coefficients must equal zero for all values of m and n and then, one obtains

$$W_{mn}\pi^4\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2-\frac{A_{mn}}{D}=0\,, \tag{55a}$$

and

$$W_{mn}=\frac{A_{mn}}{\pi^4 D\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\,. \tag{55b}$$

Substituting Eq.(53) for A_{mn} into Eq.(55b) yields the coefficients W_{mn} in the form below,

$$W_{mn}=\frac{4\int_{y=0}^{b}\int_{x=0}^{a} q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy}{ab\pi^4 D\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\,. \tag{55c}$$

Hence, the plate deflection function defined by Eq.(50) can be obtained if W_{mn} is determined from Eq.(55c) for a given load distribution $q(x,y)$. For the problem shown in Fig. 10, the case of a load uniformly distributed over the entire surface of the plate; $q(x,y)=q_o$, the coefficients W_{mn} in Eq.(55c) is equal to

$$W_{mn}=\frac{16q_o}{\pi^6 Dmn\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\,, \tag{56}$$

and the deflection function given in Eq.(50) becomes

$$w(x,y)=\frac{16q_o}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\,, \tag{57}$$

where $m=1,3,5,...$ and $n=1,3,5,...$

5.2 Levy-Type Solution

A more general technique which yields the lateral deflections of plates with boundary conditions other than simply supported was developed by Levy, M. in 1899 that known as the Levy's solution. The Levy solution method yields the lateral deflections of rectangular plates with two opposite edges simply supported, and with arbitrary boundary conditions imposed on the remaining two edges.

The two basic criteria which the Levy-type solution must satisfy are: (1) the boundary conditions, and (2) the partial differential equation presented in Eq.(42). By assuming the total solution to consist of a homogeneous part and a particular part as

$$w(x,y)=w_h(x,y)+w_p(x,y)\,, \tag{58}$$

in which w_h and w_p represent the homogeneous and particular solutions, respectively. This leads to the equations

$$\nabla^4 w_h=0\,, \tag{59}$$

and

$$\nabla^4 w_p=\frac{q}{D}\,. \tag{60}$$

Noted that Eq.(59) is independent of the loading and a single homogeneous solution can be developed for all rectangular plates that have two opposite edges simply supported. For the particular solution of Eq.(60), it must be determined for each individual loading condition, $q(x,y)$.

Consider the case of a plate with simply supported edges at $x=0$ and $x=a$, the homogeneous solution is assumed in the appropriate form of single Fourier series using the separation of variables method

$$w_h(x,y)=\sum_{m=1}^{\infty}Y_m(y)\sin\frac{m\pi x}{a}\,, \tag{61}$$

where $Y_m(y)$ is an arbitrary function of y.

The problem is solved by first determining the function $Y_m(y)$. Substituting Eq.(61) into Eq.(59) yields

$$\sum_{m=1}^{\infty}\left[\left(\frac{m\pi}{a}\right)^4 Y_m(y)-2\left(\frac{m\pi}{a}\right)^2\frac{d^2Y_m(y)}{dy^2}+\frac{d^4Y_m(y)}{dy^4}\right]\sin\frac{m\pi x}{a}=0\,. \tag{62}$$

It is seen that Eq.(62) must be satisfied for all values of x when the bracket term is equal to zero. Thus,

$$\frac{d^4Y_m(y)}{dy^4}-2\left(\frac{m\pi}{a}\right)^2\frac{d^2Y_m(y)}{dy^2}+\left(\frac{m\pi}{a}\right)^4Y_m(y)=0\,, \quad (63)$$

and its general solution can be found as [7],

$$Y_m(y)=A_m\cosh\frac{m\pi y}{a}+B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}$$
$$+C_m\sinh\frac{m\pi y}{a}+D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\,, \quad (64)$$

where A_m, B_m, C_m, and D_m are the unknown constants of integration and can be determined by adjusting the boundary conditions in each specific problem at $y=0$ and $y=b$.

Therefore, the homogeneous solution presented in Eq.(61) takes in the following form,

$$w_h(x,y)=\sum_{m=1}^{\infty}\left(A_m\cosh\frac{m\pi y}{a}+B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right.$$
$$\left.+C_m\sinh\frac{m\pi y}{a}+D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\,. \quad (65)$$

The four constants will later be determined by requiring that the total general solution of Eq.(58) satisfies the four boundary conditions on the two arbitrary edges which have yet to be satisfied. Therefore, the particular solution presented in Eq.(60) must be obtained. However, there are many ways to determine particular solution. A frequently used technique is described here in the following details.

Similar to those of the Navier's solution, the loading function can be taken in the form of a single Fourier series,

$$q(x,y)=\sum_{m=1}^{\infty}q_m(y)\sin\frac{m\pi x}{a}\,, \quad (66)$$

in which

$$q_m(y)=\frac{2}{a}\int_0^a q(x,y)\sin\frac{m\pi x}{a}dx\,. \quad (67)$$

Next, assume w_p in the form

$$w_p(x,y)=\sum_{m=1}^{\infty}w_m(y)\sin\frac{m\pi x}{a}\,, \quad (68)$$

where $w_m(y)$ is an arbitrary function of y.

Substitution of Eqs.(66) and (68) into Eq.(60), leads to

$$\frac{d^4w_m(y)}{dy^4}-2\left(\frac{m\pi}{a}\right)^2\frac{d^2w_m(y)}{dy^2}+\left(\frac{m\pi}{a}\right)^4w_m(y)$$
$$=\frac{q_m(y)}{D}\,. \quad (69)$$

The solution of w_p is completed by finding $w_m(y)$ of Eq.(69). Therefore, the final solution can be found by requiring that $w=w_h+w_p$ satisfy the four boundary conditions along the edges at $y=0$ and $y=b$. These conditions yield four equations from which the constants A_m, B_m, C_m, and D_m are determined.

Consider the loaded plate as shown in Figure 10, the loading function is $q(x,y)=q_o$ and then, the function $q_m(y)$ in Eq.(67) can be obtained as

$$q_m(y)=\frac{4q_o}{m\pi}\,, \quad (70)$$

where $m=1,3,5,...$ and further substituting $q_m(y)$ into Eq.(69) yields

$$w_m(y)=\frac{4q_oa^4}{m^5\pi^5D}\,. \quad (71)$$

Therefore, the particular solution for this problem is

$$w_p(x,y)=\sum_{m=1,3,5,...}^{\infty}\frac{4q_oa^4}{m^5\pi^5D}\sin\frac{m\pi x}{a}\,, \quad (72)$$

and the total solution becomes

$$w(x,y)=\sum_{m=1,3,5,...}^{\infty}\left(\frac{4q_oa^4}{m^5\pi^5D}+A_m\cosh\frac{m\pi y}{a}\right.,$$
$$+B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}+C_m\sinh\frac{m\pi y}{a}$$
$$\left.+D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\,. \quad (73)$$

Similarly, applying the boundary conditions for the simply supported edges at $y=0$ and $y=b$, which are:

$$w(x,0)=0, \tag{74a}$$

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0} = \kappa_y(x,0)=0, \tag{74b}$$

$$w(x,b)=0, \tag{74c}$$

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=b} = \kappa_y(x,b)=0, \tag{74d}$$

and then, four simultaneous equations above yield the constants in $w(x,y)$.

6. SUMMARY

In this paper, the mathematical formulations that involved in the bending theory of rectangular thin plates are reviewed. All necessary equations are derived analytically; namely, the slopes, curvatures, stress resultants, and the 4th-order partial differential equation governing the plate behaviors. Three kinds of classical boundary conditions are given and described physically. Finally, two classical analytical methods, which are the Navier's double Fourier series solution and the Levy's single Fourier series solution, are presented for solving the problem of uniformly loaded, simply supported rectangular plate.

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