Mathematical Solutions for Bending of Uniformly Loaded Rectangular Plates with Mixed Edge Conditions

Adisak Muenkling [1] and **Yos Sompornjaroensuk** [2]

[1] Graduate Student, Department of Civil Engineering, Mahanakorn University of Technology, Bangkok 10530, Thailand, Email: steel2521@hotmail.com
[2] Lecturer, Department of Civil Engineering, Mahanakorn University of Technology, Bangkok 10530, Thailand, Email: yos@mut.ac.th

ABSTRACT

The present paper deals with the method of finite Hankel integral transforms for solving two specific cases of uniformly loaded rectangular plate simply supported on two opposite edges, and mixed between partially simply supported and free on the third edge where the remaining fourth edge may be specified by: (a) clamped support or (b) free support. The mathematical solution for the problems analyzed can be written in terms of single Fourier series satisfying the fourth-order partial differential equation governing to the plate behaviors. Therefore, the mixed boundary conditions on the third edge are formulated through dual-series equations, which can be reduced to determining the solution of inhomogeneous Fredholm integral equation of the second kind for the unknown auxiliary function. The most important consideration is that the inverse-square-root moment singularities are taken into account in the analysis at the points of transition from a simple support to a free edge and treated analytically. The solutions of integral equation are evaluated numerically for two different cases of the plate. The obtained results are demonstrated graphically and also given numerically in tabular form for assessing other analytical or numerical methods.

Keywords : *Dual-series equations, Fredholm integral equation, Hankel integral transform, Mixed boundary conditions, Partial differential equation, Rectangular plate, Singularities.*

1. INTRODUCTION

Plates have widely been used as part of fundamental structural components throughout various engineering designs and applications. Many solutions concerning the bending [1], vibration [2], and buckling [3] of plates with regular and/or common boundary conditions can be carried out by applying the analytical and numerical methods. Therefore, either exact closed-form solutions or the well-known standard numerical techniques for solving these plate problems were generally found in several scientific or technical literatures, but not for the case of irregular or mixed boundary conditions due to the existence of stress singularities [4].

The plates which are partially cracked [5], [6], clamped [7]- [10], or simply supported [11]- [13] all belong to a class of plates with mixed boundary conditions. In order to investigate such problem of plates, the method of finite integral transforms have been utilized with considerable success to solve many boundary value problems (static bendings) and eigenvalue problems (free vibrations and bucklings) of plates having mixed edge conditions that led to dual-series equations.

Finite Hankel integral transform method is one of the efficient analytical methods that can be used to convert the dual-series equations into the proper form of integral equations where the stress singularities can also be taken into account in the analysis. With utilizing this mentioned method, a related problem was treated by Kiattikomol et al. [14] for solving the bending problem of uniformly loaded rectangular plates that are simply supported on two opposite edges, but may only be partially constrained along the other two edges. Two specific problems were considered in their works. In the first case, the plate has symmetrically placed partial simple supports at the other two edges, while the rest of these edges are free. The second case is involved to the plate that is simply supported on three edges, but partially constrained by a simple support on the fourth edge.

Recently, another related problem that also used the finite Hankel integral transform method was presented by Damang et al. [15], who gave the analytical formulation

and then derived the closed-form expressions for the bending of rectangular plate having a partial edge support under a uniformly distributed strip load. Numerical results concerning the deflections, slopes, bending moments, and corner forces of the plate were carried out only for the case of a square plate with varying the widths of uniformly distributed strip load [16]. It is interesting to note that for the problems analyzed in references [5], [6] and [10]-[16], the singularities are in the order of an inverse-square-root type in moments at the points of discontinuity of the boundary conditions.
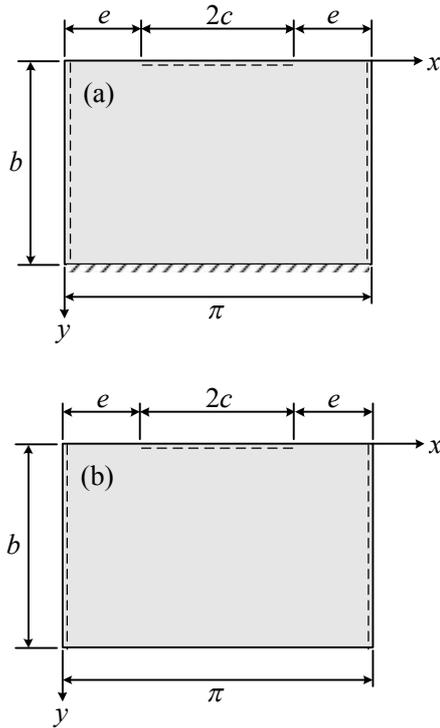

Fig. 1 *Rectangular plates with a partially supported edge:*
(a) clamped edge at $y = b$, and
(b) free edge at $y = b$.

2. PROBLEM STATEMENT AND BASIC EQUATIONS

In the present study, the main objective is to deal with the method of finite Hankel integral transforms for analytically solving two specific cases of rectangular plate having mixed boundary conditions arising from the partial simple support at $y = 0$ as shown in Fig. 1. The plates have the actual dimensions of length $\bar{a}$ and width $\bar{b}$ in the directions of $\bar{x}$ and $\bar{y}$, respectively, and are of

uniform thickness (h) .

For all two cases of the plate, the applied transverse load in the z - direction perpendicular to the x-y plane of the plate is a uniformly distributed load (q_o) . In order to facilitate the formulation of problems, it is expedient to scale the lengths involved to the actual plane dimensions of plate by the factor $\pi / \bar{a}$. Thus, the new coordinates and dimensions with respect to Fig. 1 can be introduced as follows:

$$(x, y) = (\pi / \bar{a})(\bar{x}, \bar{y}) , \tag{1}$$

and

$$(b, c, e) = (\pi / \bar{a})(\bar{b}, \bar{c}, \bar{e}) , \tag{2}$$

while c is the scaled half length of partial simple support, and e is the scaled length of free edges at $y = 0$.

Moreover, the fourth-order partial differential equation governing the deflection function (w) of plate [1] in the new coordinates (x, y) is

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q \bar{a}^4}{D \pi^4} , \tag{3}$$

in which the applied external load (q) is equal to q_o for this paper, and D is the bending stiffness of the plate defined by

$$D = \frac{E h^3}{12(1 - \nu^2)} , \tag{4}$$

where E and ν are the Young's modulus and Poisson's ratio of the plate, respectively.

Refer to Fig. 1, the rectangular plate is simply supported on two opposite edges at $x = 0$ and $x = \pi$ and partially simply supported on the third edge at $y = 0$, while the fourth edge may be either clamped or free at $y = b$ in corresponding to each case of the plates studied. Consequently, the deflection function, which is the solution of Eq.(3) can be taken in the same form as in the Levy-Nadai solution [1] of the plate having at least two opposite simply supported edges. Thus, the total deflection is the sum of the particular (w_p) and complementary (w_c) solutions of Eq.(3), which is

$$w = w_p + w_c , \tag{5}$$

with

$$w_p = \frac{q_o \bar{a}^4}{D} \sum_{m=1,3,5,\dots}^{\infty} Q_m \sin(mx) , \tag{6}$$

$$w_c = \frac{q_o \bar{a}^4}{D} \sum_{m=1,3,5,\dots}^{\infty} Y_m(y) \sin(mx) , \tag{7}$$

And

$$Q_m = \frac{4}{\pi^5 m^5} , \tag{8}$$

$$Y_m(y) = A_m \cosh(my) + B_m my \sinh(my) + C_m \sinh(my) + D_m my \cosh(my) , \tag{9}$$

where A_m, B_m, C_m, and D_m are the unknown constants to be determined from the prescribed boundary conditions at $y = 0$ and $y = b$.

Because of the symmetry in deflection function about $x = \pi/2$, the boundary conditions need only be written in the region bounded by $0 \le x \le \pi/2$ and $0 \le y \le b$. Therefore, the boundary conditions along the mixed edges at $y = 0$ for two cases of the plate are given by

$$M_y = 0 \quad ; \qquad 0 \le x \le \frac{\pi}{2} , \tag{10}$$

$$w = 0 \quad ; \qquad e < x \le \frac{\pi}{2} , \tag{11}$$

$$V_y = 0 \quad ; \qquad 0 \le x < e , \tag{12}$$

where the bending moment in the y-direction (M_y) and the supplemented shearing force normal to the y-axis (V_y) can be expressed in terms of deflection function as

$$M_y = -D\left(\frac{\pi}{\bar{a}}\right)^2 \left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) , \tag{13}$$

$$V_y = -D\left(\frac{\pi}{\bar{a}}\right)^3 \left[\frac{\partial^3 w}{\partial y^3} + (2-v)\frac{\partial^3 w}{\partial x^2 \partial y}\right] . \tag{14}$$

Since the deflection (w) and slope ($\partial w/\partial x$) are forced to vanish along the partial simple support at $y = 0$, the condition presented in Eq.(11) can be replaced with the condition of zero curvature in the x-direction below,

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad ; \qquad e < x \le \frac{\pi}{2} . \tag{15}$$

It is notable that the mixed boundary conditions become Eqs.(12) and (15) which will be used further to formulate the problems in the later stage.

3. PLATE HAVING CLAMPED EDGE AT $y = b$

The first case studied, as shown in Fig. 1(a), is that of a rectangular plate having clamped edge at $y = b$. Then, the boundary conditions for this edge are:

$$w = 0 \quad ; \qquad 0 \le x \le \frac{\pi}{2} , \tag{16}$$

$$\frac{\partial w}{\partial y} = 0 \quad ; \qquad 0 \le x \le \frac{\pi}{2} . \tag{17}$$

Application of the boundary conditions that presented in Eqs.(10), (16), (17) and together with using Eq.(13) for M_y reduces the problem to the determination of a single constant B_m, and the other three constants in Eq.(9) are then found in the following relations:

$$A_m = v\eta' Q_m - 2\eta' B_m , \tag{18}$$

$$C_m = c_m Q_m + \bar{c}_m B_m , \tag{19}$$

$$D_m = d_m Q_m - \bar{d}_m B_m , \tag{20}$$

in which

$$c_m = -\frac{\beta \sinh\beta + \cosh\beta + v\eta'\cosh^2\beta}{\sinh\beta\cosh\beta - \beta} , \tag{21}$$

$$\bar{c}_m = \frac{\beta^2 + 2\eta'\cosh^2\beta}{\sinh\beta\cosh\beta - \beta} , \tag{22}$$

$$d_m = \frac{v\eta' + \cosh\beta}{\sinh\beta\cosh\beta - \beta} , \tag{23}$$

$$\overline{d}_m = \frac{2\eta' + \sinh^2 \beta}{\sinh \beta \cosh \beta - \beta}, \qquad (24)$$

and

$$\beta = mb, \qquad (25)$$

$$\eta' = \frac{1}{(1-\nu)}. \qquad (26)$$

After applying the remaining boundary conditions as indicated in Eqs.(12) and (15) that are mixed with respect to the shear and curvature, they are led to the following dual-series equations, with the help of Eq.(14) for V_y,

$$\sum_{m=1,3,5,\ldots}^{\infty} m^3 P_m (1+F_m) \sin(mx) = \sum_{m=1,3,5,\ldots}^{\infty} G_m \sin(mx)$$
$$; \quad 0 \le x < e, \qquad (27)$$

$$\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m \sin(mx) = 0; \qquad e < x \le \frac{\pi}{2}, \qquad (28)$$

where P_m, F_m, and G_m are the unknown function, weight function, and known function, respectively. They can be expressed as follows:

$$P_m = Q_m - 2B_m, \qquad (29)$$

$$F_m = f_m - 1, \qquad (30)$$

$$G_m = g_m Q_m, \qquad (31)$$

and

$$f_m = \frac{4\eta' + (1-\nu)\beta^2 + (3+\nu)\sinh^2 \beta}{(3+\nu)(\sinh \beta \cosh \beta - \beta)}, \qquad (32)$$

$$g_m = m^3 \left[\frac{4+(1-\nu)\beta^2 + (3-\nu)\sinh^2 \beta}{(3+\nu)(\sinh \beta \cosh \beta - \beta)} \right.$$
$$\left. - \frac{2(1-\nu)\beta \sinh \beta + 4\cosh \beta}{(3+\nu)(\sinh \beta \cosh \beta - \beta)} \right]. \qquad (33)$$

At the present stage, it can be noted that the problem is now reduced to determine the unknown function P_m in a pair of dual-series equations shown in Eqs.(27) and (28). In solving these two equations simultaneously, the method is made by choosing the

unknown function P_m in the proper form of a finite Hankel integral transform [11], [12], [14]:

$$m^2 P_m = \overline{E} J_1(me) + \int_0^e t\varphi(t) J_1(mt) dt ; \ m = 1,3,5,\ldots, \quad (34)$$

where $\overline{E}$ is the constant to be determined from the condition of zero deflection ($w = 0$) given in Eq.(11) at only one point of x between e and $\pi/2$; i.e., $w(\pi/2,0)$, $\varphi(t)$ is the unknown auxiliary function, t is a dummy variable, and $J_1(u)$ is the Bessel function of the first kind and first order [17] with argument u.

Thus integrating Eq.(28) twice with respect to x and setting $x = \pi/2$ yields the expression,

$$\sum_{m=1,3,5,\ldots}^{\infty} P_m \sin\left(\frac{m\pi}{2}\right) = \sum_{m=1,3,5,\ldots}^{\infty} (-1)^{(m-1)/2} P_m = 0, \qquad (35)$$

and then, substituting P_m from Eq.(34) into the above equation together with changing the order of summation and integration results in

$$\overline{E} = -\frac{\int_0^e t\varphi(t) \sum_{m=1,3,5,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^2} J_1(mt) dt}{\sum_{m=1,3,5,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^2} J_1(me)}. \qquad (36)$$

The constant $\overline{E}$ can be determined by utilizing the identity [6], [14]:

$$\sum_{m=1,3,5,\ldots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^2} J_1(mt) = \frac{\pi t}{8} ; \ t < \frac{\pi}{2}, \qquad (37)$$

into Eq.(36), therefore, one obtains the constant $\overline{E}$ as

$$\overline{E} = -\frac{1}{e} \int_0^e t^2 \varphi(t) dt. \qquad (38)$$

After that, with using $\overline{E}$ as shown in Eq.(38), the integral representation form of P_m presented by Eq.(34) becomes

$$m^2 P_m = \int_0^e t\varphi(t) \left[J_1(mt) - \frac{t}{e} J_1(me) \right] dt. \qquad (39)$$

It is important to note here that the choice of P_m

given by Eq.(34), which led to Eq.(39), automatically satisfies the second dual-series in Eq.(28). In order to verify this requirement, it is easily made by substituting P_m from Eq.(39) into Eq.(28) and also changing the order of summation and integration, therefore, leads to

$$\int_0^e t\varphi(t) \sum_{m=1,3,5,\ldots}^{\infty} \left[J_1(mt) - \frac{t}{e} J_1(me) \right] \sin(mx) dt = 0$$

$$; \ e < x \le \frac{\pi}{2} . \tag{40}$$

To demonstrate the satisfaction of Eq.(40), it is useful to consider the identity [6]

$$\sum_{m=1,3,5,\ldots}^{\infty} J_1(mt) \sin(mx) = \frac{xH(t-x)}{2t(t^2-x^2)^{1/2}} ; \ x+t < \pi . \tag{41}$$

Thus, the left-hand side of Eq.(40) is seen to vanish because x is always larger than t and e, which leads to the Heaviside's function $H(t-x) = 0$.

Similarly, the condition of zero slope $(\partial w/\partial x = 0)$ on the partial simple support as mentioned before Eq.(15) can also be verified by the same procedure as treated in Eq.(40). Therefore, integrating Eq.(28) once with respect to x and using Eq.(39) for P_m, together with the assistance of identity [6] defined by

$$\sum_{m=1,3,5,\ldots}^{\infty} \frac{J_1(mt)}{m} \cos(mx) = \frac{H(t-x)}{2t(t^2-x^2)^{-1/2}} ; \ x+t < \pi , \tag{42}$$

hence, the condition of zero slope can be represented in the following integral form,

$$\frac{1}{2} \int_0^e \varphi(t) \left[\frac{H(t-x)}{(t^2-x^2)^{-1/2}} - \left(\frac{t}{e}\right)^2 \frac{H(e-x)}{(e^2-x^2)^{-1/2}} \right] dt = 0$$

$$; \ e < x \le \frac{\pi}{2} . \tag{43}$$

Obviously, the terms involved with the Heaviside's functions in the bracket of Eq.(43) are all vanished because both t and e are always less than x. The next step is to show the singularity order at the points of transition from a simple support to a free edge at $y = 0$.

As it has been pointed out by Williams [4] that the singularities are of order $O(\varepsilon^{-1/2})$ in the moments or of order $O(\varepsilon^{-3/2})$ in the shearing forces, where ε is defined to be an infinitesimal length measured from the singular points.

To verify the nature of singularity that existed in the problem analyzed, one may consider Eq.(27) which has been obtained from Eq.(12) for the condition of zero supplemented, or Kirchhoff, shearing force $(V_y = 0)$ along the line outside of partial simple support at $y = 0$ and $0 \le x < e$. Thus, the distribution of shearing force exerted by the partial simple support can be written in the form as follows:

$$V_y(x,0) \sim -\frac{d}{dx} \sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m \cos(mx) + \sum_{m=1,3,5,\ldots}^{\infty} (m^3 F_m P_m$$

$$-G_m) \sin(mx) ; \ e < x \le \frac{\pi}{2} . \tag{44}$$

Substitution of Eq.(34) for P_m into the first series term on the right-hand side of Eq.(44) yields

$$V_y(x,0) \sim -\bar{E} \frac{d}{dx} \sum_{m=1,3,5,\ldots}^{\infty} J_1(me) \cos(mx)$$

$$-\int_0^e t\varphi(t) \frac{d}{dx} \sum_{m=1,3,5,\ldots}^{\infty} J_1(mt) \cos(mx) dt$$

$$+ \sum_{m=1,3,5,\ldots}^{\infty} (m^3 F_m P_m - G_m) \sin(mx) ; \ e < x \le \frac{\pi}{2} . \tag{45}$$

Further, the first series term of Eq.(45) can also be expressed as

$$\frac{d}{dx} \sum_{m=1,3,5,\ldots}^{\infty} J_1(me) \cos(mx) = \frac{e}{2(x^2-e^2)^{3/2}}$$

$$+ \int_0^{\infty} \frac{sI_1(es) \sinh(xs)}{\exp(\pi s)+1} ds , \tag{46}$$

that obtained by the aid of identity [6],

$$\sum_{m=1,3,5,\ldots}^{\infty} J_1(mt) \cos(mx) = \frac{1}{2t} - \frac{xH(x-t)}{2t(x^2-t^2)^{1/2}}$$

$$+ \int_0^{\infty} \frac{I_1(ts) \cosh(xs)}{\exp(\pi s)+1} ds ; \ x+t < \pi , \tag{47}$$

where $I_1(u)$ is the modified Bessel function of the first kind and first order with argument u [17].

Replacing $x = e + \varepsilon$ in the first right-hand side term of Eq.(46), after performing the binomial expansion theorem, results in

$$\frac{d}{dx}\sum_{m=1,3,5,\ldots}^{\infty} J_1(me)\cos(mx) = \frac{e}{2(2e\varepsilon)^{3/2}} + O(\varepsilon^{-1/2})$$

$$+\int_0^{\infty}\frac{sI_1(es)\sinh(xs)}{\exp(\pi s)+1}ds \ . \tag{48}$$

By consideration of Eq.(48), it can immediately be seen that there is the singularity of order $O(\varepsilon^{-3/2})$ contributed in Eq.(45) for the shearing force, which is in agreement with the conclusions by Williams [4] when P_m is introduced in the integral form of Eq.(34).

The last task is to reduce the remaining dual-series in Eq.(27) to a tractable form of integral equation, which can be solved numerically by standard methods [18]. Therefore, the process is first made by integration of Eq.(27) once with respect to x and substitution of P_m from Eq.(39). The result is

$$\int_0^e t\varphi(t)\sum_{m=1,3,5,\ldots}^{\infty}(1+F_m)\left[J_1(mt)-\frac{t}{e}J_1(me)\right]\cos(mx)dt$$

$$=\sum_{m=1,3,5,\ldots}^{\infty}\frac{G_m}{m}\cos(mx) \ ; \ 0 \le x < e \ . \tag{49}$$

Using the identity given by Eq.(47) into Eq.(49) leads to the well-known integral equation of Abel-type as

$$\int_0^x \frac{x\varphi(t)}{\sqrt{x^2-t^2}}dt = h(x) \ ; \ 0 \le x < e \ , \tag{50}$$

Where

$$h(x) = e\int_0^1\varphi(er)\{1-r^2$$

$$+2er\int_0^{\infty}\frac{[I_1(ser)-rI_1(se)]\cosh(xs)}{\exp(\pi s)+1}ds\}dr$$

$$+2e^2\int_0^1 r\varphi(er)\sum_{m=1,3,5,\ldots}^{\infty}F_m[J_1(mer)-rJ_1(me)]\cos(mx)\,dr$$

$$-2\sum_{m=1,3,5,\ldots}^{\infty}\frac{G_m}{m}\cos(mx) \ . \tag{51}$$

Noted that the change of variable $t=er$ and $0 \le r \le 1$ has been introduced in Eq.(51) in which r is a dummy variable.

The solution of Eq.(50) is generally found to be of the form

$$\varphi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{h(x)}{\sqrt{t^2-x^2}}dx \ ; \ 0 \le t < e \ . \tag{52}$$

Inserting Eq.(51) for the function $h(x)$ into the right-hand side of Eq.(52), the terms of differentiating under integral sign can be evaluated analytically. These terms are obtained by using the identities that found in Gradshteyn and Ryzhik [19], which are

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{1}{\sqrt{t^2-x^2}}dx = 0 \ , \tag{53}$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{\cosh(sx)}{\sqrt{t^2-x^2}}dx = sI_1(st) \ , \tag{54}$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{\cos(mx)}{\sqrt{t^2-x^2}}dx = -mJ_1(mt) \ . \tag{55}$$

With the help of Eqs.(53) to (55) and changing the variable $t=e\rho$ and $0 \le \rho \le 1$ with ρ being a dummy variable, the final result of Eq.(52) can be cast in the form of inhomogeneous Fredholm integral equation of the second kind:

$$\Phi(\rho)+\int_0^1 K(\rho,r)\Phi(r)\,dr = f(\rho) \ ; \ 0 \le \rho \le 1, \tag{56}$$

in which

$$\Phi(\rho) = \varphi(e\rho) \ ; \ \Phi(r) = \varphi(er) \ , \tag{57}$$

$$K(\rho,r) = 2e^2 r\sum_{m=1,3,5,\ldots}^{\infty}mF_m[J_1(mer)-rJ_1(me)]J_1(me\rho)$$

$$-2e^2 r\int_0^{\infty}\frac{s[I_1(ser)-rI_1(se)]I_1(se\rho)}{\exp(\pi s)+1}ds \ , \tag{58}$$

$$f(\rho) = 2\sum_{m=1,3,5,\ldots}^{\infty}G_m J_1(me\rho) \ , \tag{59}$$

where the functions F_m and G_m are previously defined in Eqs.(30) and (31), respectively, and the Fredholm integral equation that presented in Eq.(56) can be solved to obtain the unknown auxiliary function $\Phi(\rho)$ by using the standard numerical techniques [18]. This numerical procedure will be explained in more details in the later stage.

4. PLATE HAVING FREE EDGE AT $y = b$

For this second case, the geometry of plate is shown in Fig.1(b). Only the boundary condition at $y = 0$ that given in Eq.(10) is still governed, but the boundary conditions satisfying the edge at $y = b$ as presented in Eqs.(16) and (17) are changed. Thus, the boundary conditions are:

$$M_y = 0 \quad ; \qquad 0 \le x \le \frac{\pi}{2}, \qquad (60)$$

$$V_y = 0 \quad ; \qquad 0 \le x \le \frac{\pi}{2}. \qquad (61)$$

Substituting Eq.(5) for the deflection function (w) into Eqs.(60) and (61) and together with using Eqs.(13) and (14) for M_y and V_y, respectively, leads to the relations of unknown constants given in Eqs.(19) and (20).

However, the coefficients c_m, $\bar{c}_m$, d_m, and $\bar{d}_m$ that defined by Eqs.(21) to (24), respectively, have to be redefined in accordance with the present case as follows:

$$c_m = -v \left[\frac{\eta''(1 - \cosh\beta) + \beta\sinh\beta}{(1-v)\beta + (3+v)\sinh\beta\cosh\beta} \right. \\ \left. + \frac{\eta'(3+v)\sinh^2\beta}{(1-v)\beta + (3+v)\sinh\beta\cosh\beta} \right], \qquad (62)$$

$$\bar{c}_m = \frac{2(3+v)\eta'\sinh^2\beta - (1-v)\beta^2}{(1-v)\beta + (3+v)\sinh\beta\cosh\beta}, \qquad (63)$$

$$d_m = \frac{v(\cosh\beta - 1)}{(1-v)\beta + (3+v)\sinh\beta\cosh\beta}, \qquad (64)$$

$$\bar{d}_m = \frac{(3+v)\sinh^2\beta}{(1-v)\beta + (3+v)\sinh\beta\cosh\beta}, \qquad (65)$$

and

$$\eta'' = \frac{1+v}{1-v}. \qquad (66)$$

In the same manner with the first case of the plate having clamped edge at $y = b$, the dual-series equations resulting from the mixed boundary conditions at $y = 0$ and the functions P_m, F_m, and G_m can also be expressed as in the same form of Eqs.(27) to (31). Exceptionally,

the coefficients f_m and g_m that found in Eqs.(32) and (33) are needed to be changed into the new following relations:

$$f_m = \frac{(3+v)^2 \sinh^2\beta - (1-v)^2\beta^2}{(3+v)\left[(1-v)\beta + (3+v)\sinh\beta\cosh\beta\right]}, \qquad (67)$$

$$g_m = m^3 \left\{ \frac{(9-v^2)\sinh^2\beta - (1-v)^2\beta^2}{(3+v)[(1-v)\beta + (3+v)\sinh\beta\cosh\beta]} \right. \\ \left. - \frac{2v(1-v)\beta\sinh\beta}{(3+v)[(1-v)\beta + (3+v)\sinh\beta\cosh\beta]} \right\}. \qquad (68)$$

Representing the unknown function P_m in the identical form of Eqs.(34) or (39) and then, the dual-series equations can further be reduced to the integral equation presented by Eq.(56) in terms of the unknown auxiliary function $\Phi(\rho)$.

5. NUMERICAL PROCEDURE

As clearly derived in sections 3 and 4 for two different cases of the plate, the problem considered can be reduced to determine the solution of Eq.(56) for the unknown auxiliary function $\Phi(\rho)$. Nevertheless, its solution can only be carried out by means of numerical treatment with using the standard methods [18]. This is due to the complexity of the kernel as seen in Eq.(58). Therefore, Simpson's rule is chosen for this purpose in order to solve Eq.(56) for the discretized values of $\Phi(\rho)$ within the desired degree of accuracy.

In the framework of numerical treatment, Eq.(56) will be approximated by considering a sum over discrete values in each r and ρ with N numbers of discrete points. Applying the Simpson's rule for numerical integration yields a system of linear simultaneous equation as follows:

$$([I] + [K])\{\Phi\} = \{F\}, \qquad (69)$$

and

$$[I] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{N \times N}, \qquad (70)$$

$$[K] = \begin{bmatrix} W_1 K(r_1, r_1) & W_2 K(r_1, r_2) & \cdots & W_N K(r_1, r_N) \\ W_1 K(r_2, r_1) & W_2 K(r_2, r_2) & \cdots & W_N K(r_2, r_N) \\ \vdots & \vdots & \ddots & \vdots \\ W_1 K(r_N, r_1) & W_2 K(r_N, r_2) & \cdots & W_N K(r_N, r_N) \end{bmatrix}_{N \times N} \quad ,(71)$$

$$\{\Phi\} = [\Phi(r_1) \quad \Phi(r_2) \quad \cdots \quad \Phi(r_N)]^T , \tag{72}$$

$$\{F\} = [f(r_1) \quad f(r_2) \quad \cdots \quad f(r_N)]^T , \tag{73}$$

where $[I]$ is the identity matrix and $[K]$, $\{F\}$ and $\{\Phi\}$ are the matrices of discrete value for the kernel multiplied with weight function W_i and $i = 1, 2, 3, ..., N$ based on Simpson's rule, the right-hand side function of integral equation defined in Eq.(59), and the unknown auxiliary function, respectively.

The infinite series of the kernel and function $f(\rho)$

that presented in Eqs.(58) and (59), respectively, are evaluated to a relative error criterion of 0.0000001, i.e., the series evaluation is terminated when the ratio of the absolute value of the last term calculated to the absolute value of the sum of all previous terms became less than 0.0000001. Additionally, it can be observed that the integrand of the improper infinite integral in the kernel is a monotonically increasing function up to some maximum values. After the maximum is reached, the integrand is then decayed exponentially. In order to evaluate this infinite integral, two methods of numerical quadrature are used in comparison of the results, which are the 32-point Gauss-Legendre and 15-point Gauss-Laguerre quadrature formula [17]. Since a system of equations presented in Eq.(69) is formed, the matrix $\{\Phi\}$ can be solved by using the direct method; namely, the Gaussian elimination with partial pivoting.
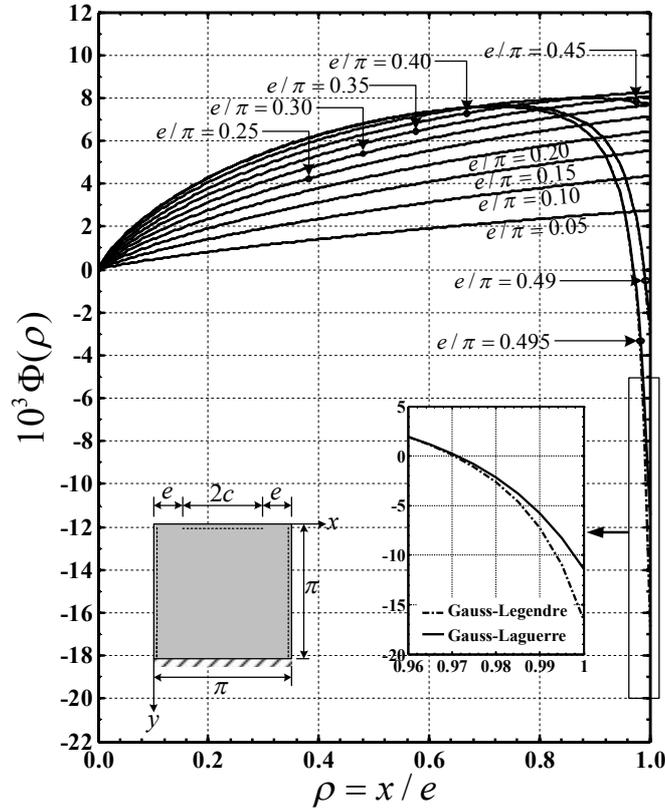

Fig. 2 *Auxiliary function* $\Phi(\rho)$ *for square plate having clamped edge at* $y = \pi$.

6. RESULTS

All numerical results carried out and demonstrated in this paper are obtained from computer programming on MATLAB program language [20] and computing with

the highest accuracy attainable using double precision.

Results for the integral equation solution in Eq.(56) in terms of the unknown auxiliary function $\Phi(\rho)$ are presented for two different cases of scaled square plate of side π, and Poisson's ratio is only taken as 0.3. Also

the length e of free edge at $y = 0$ is varied from 0.05π to 0.495π. Therefore, their numerical results are graphically presented in Figs. 2 and 3 for the cases of plate having clamped edge at $y = \pi$ and free edge at $y = \pi$, respectively. In addition, their numerical values are also given in the tabular form as shown in Tables 1 and 2.

As can be seen in Figs 2 and 3, all curves of $\Phi(\rho)$ in each e/π-ratio are computed from two different types of Gaussian quadrature as explained previously in section 5. It is, however, noted that the curve $\Phi(\rho)$ shows the different values only for the case of $e/\pi = 0.495$, otherwise are the same. This observation

can clearly be seen in Tables 1 and 2.

To evaluate the value of $\Phi(\rho)$ in Eq.(56), the integral equation for both cases of the plate is approximated by the system of linear simultaneous N-equations. The highest number of equations used is found to be 81 equations for the 32-point Gauss-Legendre quadrature and 61 equations for the 15-point Gauss-Laguerre quadrature in order to the determination of improper infinite integral in the kernel. These numbers of equation are based on the requirement of degree of accuracy of $\Phi(\rho)$-values in which their differences between the present value and the previous value of $\Phi(\rho)$ have to be less than 0.001.
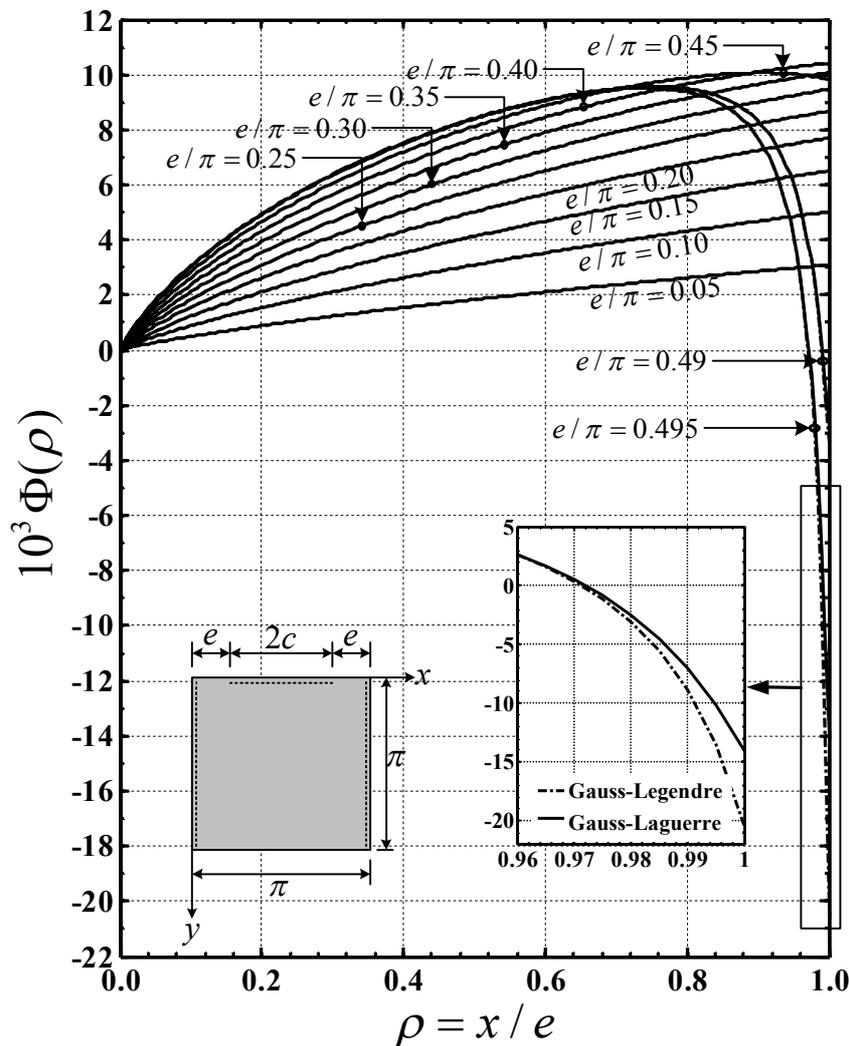

Fig. 3 *Auxiliary functions* $\Phi(\rho)$ *for square plate having free edge at* $y = \pi$.

Table 1 *Values of Auxiliary Function* $\Phi(\rho)$ *for Scaled Square Plate with Clamped Edge at* $y = \pi$.

ρ	$10^3 \Phi(\rho)$ (Gauss-Legendre)		
	$e = 0.10\pi$	$e = 0.30\pi$	$e = 0.495\pi$
0.0	0.0000	0.0000	0.0000
0.2	1.4127	3.1304	4.2888
0.4	2.3600	4.8711	6.2783
0.6	3.1319	6.0986	7.3566
0.8	3.7906	7.0039	7.4195
1.0	4.3655	7.6789	-16.6236
ρ	$10^3 \Phi(\rho)$ (Gauss-Laguerre)		
	$e = 0.10\pi$	$e = 0.30\pi$	$e = 0.495\pi$
0.0	0.0000	0.0000	0.0000
0.2	1.4127	3.1304	4.2887
0.4	2.3600	4.8711	6.2782
0.6	3.1319	6.0986	7.3563
0.8	3.7906	7.0039	7.4182
1.0	4.3655	7.6789	-11.4458

Table 2 *Values of Auxiliary Function* $\Phi(\rho)$ *for Scaled Square Plate with Free Edge at* $y = \pi$.

ρ	$10^3 \Phi(\rho)$ (Gauss-Legendre)		
	$e = 0.10\pi$	$e = 0.30\pi$	$e = 0.495\pi$
0.0	0.0000	0.0000	0.0000
0.2	1.5464	3.5301	4.9411
0.4	2.6269	5.6596	7.5282
0.6	3.5314	7.2549	9.0866
0.8	4.3213	8.4968	9.3713
1.0	5.0260	9.4686	-20.6056
ρ	$10^3 \Phi(\rho)$ (Gauss-Laguerre)		
	$e = 0.10\pi$	$e = 0.30\pi$	$e = 0.495\pi$
0.0	0.0000	0.0000	0.0000
0.2	1.5464	3.5301	4.9411
0.4	2.6269	5.6596	7.5280
0.6	3.5314	7.2549	9.0862
0.8	4.3213	8.4968	9.3697
1.0	5.0260	9.4686	-14.1072

7. CONCLUSIONS

The bending problem of rectangular plates with a partially simply supported edge and subjected to a uniformly distributed load is considered in the present paper. Mathematically, this type of problem can be classified into the mixed boundary value problems in structural mechanics. Importantly, the moment singularities in the order of an inverse-square-root type at the ends of partial simple support are also taken into account in the analysis in which the solution can be found by using the method of finite Hankel integral transforms. Therefore, the dual-series equation resulting from the mixed boundary conditions can be reduced to that of solving an inhomogeneous Fredholm integral equation of the second kind, and the solution of this integral equation in terms of unknown auxiliary function can be computed numerically by using the standard techniques. The obtained results are both given numerically and graphically for assessing other analytical and numerical methods.

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Adisak Muenkling received his B.Eng. in civil engineering from Sripatum University in 2002. Currently he is the graduate student in the Master Program of Civil Engineering, Department of Civil Engineering, Mahanakorn University of Technology.

Yos Sompornjaroensuk received his B.Eng., M.Eng, Ph.D. in Civil Engineering from King Mongkut's University of Technology Thonburi in 1997, 1999, and 2007, respectively. Currently he is a lecturer at the Department of Civil Engineering, Mahanakorn University of Technology. His areas of interest include Structural Engineering and Mechanics, Fracture Mechanics, Contact Mechanics, Composite Structures and Materials, and Mathematical Modeling.