Stress Intensity Factors of a Uniformly Line Loaded Rectangular Plate with Mixed Edge Conditions

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ABSTRACT

The main purpose of the paper is to investigate the stress intensity factors at the end points of discontinuous support in rectangular thin elastic plate. The plate is simply supported on two opposite edges, clamped on the third edge and partially simply supported on the fourth edge, and also loaded by a uniformly distributed line load. Since the plate has the partial simple support along one edge leading to a pair of dual-series equations that resulted from the mixed boundary conditions, this causes the existence of moment singularities in the order of an inverse-square-root type. In order to analytically determine the stress intensity factors, the finite Hankel integral transform techniques are applied for solving the dual-series equations which can reduce to finding the solution of Fredholm integral equation. Numerical results concerning the solution of integral equation, stress intensity factors, and additionally, the change in strain energy due to the presence of a partial simple support are given for the case of a square plate in the form of graphs and tables for easy reference.

Keywords : *Dual-series equations, Elastic plate, Fredholm integral equation, Hankel integral transform, Mixed boundary conditions, Moment singularities, Strain energy, Stress intensity factors.*

1. INTRODUCTION AND PROBLEM SIGNIFICANCE

It is well-known that most problems containing the initial cracks in the body can familiarly be classified into the field of fracture mechanics in which the applied mechanics framework is required for the discussion of the behaviour of cracked bodies under load [1]. Because these mentioned problems with cracked body will usually lead to yielding or some other kinds of failure in practice due to the stress singularities arising in the vicinity at the end of the cracks [2]- [5], then it is necessary to determine the parameters or factors in order to characterize the crack growth processes [6] and to practically identify the strengths of materials under a critical load. Thus, one of the major parameters which have widely been used in fracture analysis to describe and distinguish between different cracks depending on different modes of failure is the stress intensity factor. This factor can easily be determined from the stress fields or stress components ahead of a crack-tip for denoting the strengths of singularities [7].

Based upon the theory of elasticity [8], the definition of stress singularities at some points means that the infinite stresses are able to predict at those points in the domain of problem. Thus, any situation in the problem considered having an infinite stresses will lead to the violation of the basic assumptions behind mathematical solutions and also the reflection of limited physical applications in the linear elasticity theory.

However, there are many other engineering applications whose problems are involved with stress singularities that may be encountered by geometric, loading conditions, and/or material discontinuities [9], [10]. In the present paper, the problem analyzed is of the bending problem of thin elastic plates [11] where stress singularities are due to the boundary geometry having discontinuous support, sometimes, called the mixed support conditions. For this type of support condition, Williams [12] first analytically investigated the singularity orders at the transition points of two different boundary supports by using the method of eigenfunction expansions. It was found that the moment distributions are singular in the order of an inverse-square-root type [13]- [15].

As generally described in the significance of stress singularities and because of the analogy between the two different classes of problems; namely, the crack problem in fracture mechanics and the plate problem with mixed boundary conditions in continuum mechanics, it can be recommended that the points having stress singularities are often the points of initiation of cracks due to the fact that no real materials are capable of sustaining an infinite stress. Eventhough the stress singularities are not of the real world, nevertheless, their presences in stress analysis are still to be a real fact in mathematical theory of elasticity.

Therefore, the necessity of considering the present problem as stipulated in the title motivates the authors to investigate the stress intensity factors within the frame of linear elastic fracture mechanics (LEFM) following the theory of elasticity, which is the principle aim of this paper to present a new aspect results in the field of structural engineering mechanics.

2. GOVERNING DIFFERENTIAL EQUATION AND STRESS RESULTANTS OF THE PLATE

Assuming that the rectangular plate has the actual length $\bar{a}$ and actual width $\bar{b}$ in the directions of $\bar{x}$ and $\bar{y}$, respectively. However, in order to investigate the specific problem presented in this paper, it is convenient to scale the lengths involved with the plate so that $\bar{a}$ becomes π.

Therefore, the scaled coordinates and dimensions of plate as demonstrated in Fig. 1 can be determined from the relation,

$$(x, y, b, c, e) = (\pi / \bar{a})(\bar{x}, \bar{y}, \bar{b}, \bar{c}, \bar{e}), \tag{1}$$

which c is the scaled half length of partial simple support and e is the scaled length of free edges at $y = 0$.

Consequently, the differential equation in coordinates (x, y) governing the deflection (w) of the plate is found to be [11]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q\bar{a}^4}{D\pi^4}, \tag{2}$$

with

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{3}$$

where q is the external load applied in the z-direction, D is the flexural rigidity of the plate, h is the plate thickness, and the material properties E and ν are the Young's modulus and Poisson's ratio, respectively.

Moreover, the internal quantities in an element of plate that called the stress resultants can be expressed in terms of the deflection function as follows:

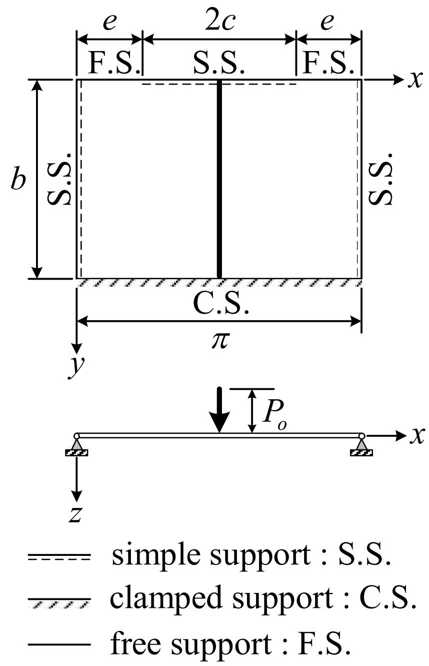

Fig. 1 *Rectangular plates with a partially simply supported edge loaded by uniformly distributed line load.*

$$M_x = -D\left(\frac{\pi}{a}\right)^2 \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \tag{4}$$

$$M_y = -D\left(\frac{\pi}{a}\right)^2 \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right), \tag{5}$$

$$M_{xy} = D(1-\nu)\left(\frac{\pi}{a}\right)^2 \frac{\partial^2 w}{\partial x \partial y}, \tag{6}$$

$$Q_x = V_x + \frac{\partial M_{xy}}{\partial y}, \tag{7}$$

$$Q_y = V_y - \frac{\partial M_{xy}}{\partial x}, \tag{8}$$

and

$$V_x = -D\left(\frac{\pi}{a}\right)^3 \left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right], \tag{9}$$

$$V_y = -D\left(\frac{\pi}{a}\right)^3 \left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right], \tag{10}$$

in which the notations of M_x, M_y, M_{xy}, Q_x, Q_y, V_x, and V_y are, respectively, defined in the followings below:

M_x :- bending moment in the x -direction,
M_y :- bending moment in the y -direction,
Q_x :- transverse shearing force normal to x -axis,
Q_y :- transverse shearing force normal to y -axis,
V_x :- supplemented shearing force normal to x -axis,
V_y :- supplemented shearing force normal to y -axis.

Because the plate has two opposite simply supported edges placed at $x = 0$ and $x = \pi$ and is loaded by a uniformly distributed line load (P_o), thus, the deflection function satisfying the solution of Eq.(2) can be assumed to be of the form of single Fourier sine series according to the Levy-Nadai type solution [11] as

$$w(x,y) = \frac{P_o \bar{a}^3}{D} \sum_{m=1,3,5,\ldots}^{\infty} \left[Q_m + Y_m(y)\right]\sin(mx), \tag{11}$$

where Q_m is the Fourier's coefficient resulting from the expansion of applied load (P_o), and $Y_m(y)$ is the function of y only which contains the unknown constants (A_m, B_m, C_m, D_m) to be adjusted from the support conditions at $y = 0$ and $y = b$.

Therefore, the coefficient of Q_m and function of $Y_m(y)$ can be expressed by

$$Q_m = \frac{2(-1)^{(m-1)/2}}{\pi^4 m^4}, \tag{12}$$

and

$$Y_m(y) = A_m \cosh(my) + B_m my \sinh(my) + C_m \sinh(my) + D_m my \cosh(my). \tag{13}$$

Since the geometry of the plate treated here is the same with the first case of the plates analyzed by Adisak and Sompornjaroensuk [16], hence, the expressions for the boundary conditions as shown in that previous work are also the same and they will not be repeated in the present work. However, the fundamental difference between the problem considered herein and that analyzed in Ref. [16] is found in loading condition, which can explicitly be indicated in Eq.(12) for Q_m.

In order to solve the governing differential equation of the plate as shown in Eq.(2) for finding the solution in terms of the deflection function (w), the unknown constants in Eq.(13) have to be determined for this purpose. This can be made by applying the common boundary conditions at $y = 0$ for the zero moment $(M_y = 0)$ and at $y = b$ for the zero deflection and zero slope $(w = \partial w / \partial y = 0)$ with $0 \le x \le \pi / 2$ due to the symmetry in deflection function about $x = \pi / 2$. In addition, the boundary conditions along the partially simply supported edge at $y = 0$ are physically mixed between the simple support and free edge. Thus, the conditions used for this edge are chosen to be mixed with respect to the zero shear ($V_y = 0$ with $0 \le x < e$) and zero curvature ($\partial^2 w / \partial x^2 = 0$ with $e < x \le \pi / 2$).

As described above for the determination of unknown constants, the results can be taken in the following forms, after some mathematical manipulations,

$$A_m = \nu \eta' Q_m - 2\eta' B_m, \tag{14}$$

$$C_m = c_m Q_m + \bar{c}_m B_m, \tag{15}$$

$$D_m = d_m Q_m - \bar{d}_m B_m, \tag{16}$$

with

$$c_m = -\frac{\beta \sinh\beta + \cosh\beta + \nu\eta' \cosh^2\beta}{\sinh\beta\cosh\beta - \beta}, \tag{17}$$

$$\bar{c}_m = \frac{\beta^2 + 2\eta' \cosh^2\beta}{\sinh\beta\cosh\beta - \beta}, \tag{18}$$

$$d_m = \frac{\nu\eta' + \cosh\beta}{\sinh\beta\cosh\beta - \beta}, \tag{19}$$

$$\bar{d}_m = \frac{2\eta' + \sinh^2\beta}{\sinh\beta\cosh\beta - \beta}, \tag{20}$$

and

$$\beta = mb, \tag{21}$$

$$\eta' = \frac{1}{(1-\nu)}. \tag{22}$$

It is notable that three unknown constants are now expressed in terms of B_m which can be determined from the dual-series equations given below,

$$\sum_{m=1,3,5,\ldots}^{\infty} m^3 P_m (1+F_m)\sin(mx) = \sum_{m=1,3,5,\ldots}^{\infty} G_m \sin(mx) \quad ; \quad 0 \le x < e, \tag{23}$$

$$\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m \sin(mx) = 0 \; ; \qquad e < x \le \frac{\pi}{2}, \tag{24}$$

where

$$P_m = Q_m - 2B_m, \tag{25}$$

$$F_m = f_m - 1, \tag{26}$$

$$G_m = g_m Q_m, \tag{27}$$

with

$$f_m = \frac{4\eta' + (1-\nu)\beta^2 + (3+\nu)\sinh^2\beta}{(3+\nu)(\sinh\beta\cosh\beta - \beta)}, \tag{28}$$

and

$$g_m = m^3 \left[\frac{4 + (1-\nu)\beta^2 + (3-\nu)\sinh^2\beta}{(3+\nu)(\sinh\beta\cosh\beta - \beta)} - \frac{2(1-\nu)\beta\sinh\beta + 4\cosh\beta}{(3+\nu)(\sinh\beta\cosh\beta - \beta)} \right]. \tag{29}$$

Noting, however, that Eqs.(14) to (29) are taken in the same form as previously presented in that of Adisak and Sompornjaroensuk [16], except only for the term of Q_m given in Eq.(12) due to particular loading being treated in the present paper.

In a pair of dual-series equations as seen in Eqs.(23) and (24), the unknown function is P_m that only related to the unknown constant B_m. It has been shown that the dual-series equations can analytically be solved for the unknown function P_m by taking the advantage of finite Hankel integral transform techniques leading to the integral equation governing the problem analyzed [16].

Therefore, with introducing the function P_m in the proper form, which is

$$P_m = \left(\frac{e}{m}\right)^2 \int_0^1 \rho\,\Phi(\rho)\left[J_1(me\rho) - \rho J_1(me)\right] d\rho, \tag{30}$$

where ρ is a dummy variable, $\Phi(\rho)$ is a new unknown auxiliary function, and $J_1(u)$ is the Bessel function of the first kind and first order with argument u [17].

Substituting Eq.(30) into Eq.(23) and together with the assistance of identities found in Stahl and Keer [4] and Gradshteyn and Ryzhik [18], it can be reduced to an inhomogeneous Fredholm integral equation of the second kind:

$$\Phi(\rho) + \int_0^1 K(\rho, r)\,\Phi(r)\,dr = f(\rho) \; ; \; 0 \le \rho \le 1, \tag{31}$$

in which

$$K(\rho, r) = 2e^2 r \sum_{m=1,3,5,\ldots}^{\infty} m F_m [J_1(mer) - rJ_1(me)] J_1(me\rho) - 2e^2 r \int_0^{\infty} \frac{s[I_1(ser) - rI_1(se)] I_1(se\rho)}{\exp(\pi s) + 1} ds, \tag{32}$$

and

$$f(\rho) = 2 \sum_{m=1,3,5,\ldots}^{\infty} G_m J_1(me\rho), \tag{33}$$

while r is also a dummy variable, $I_1(u)$ and $\exp(u)$ are the modified Bessel function of the first kind and first order [17] and exponential function with argument u, respectively.

Further details concerning the method for derivation of Eq.(31) and its numerical evaluation for obtaining the unknown auxiliary function $\Phi(\rho)$ are provided in Adisak and Sompornjaroensuk [16].

3. THE STRESS FIELDS AND STRESS INTENSITY FACTORS

Because the problem considered has the moment singularities in the order of an inverse-square-root type at the points where the simple support changes to a free edge on the same side of the plate [12], it is then interested in determining the stress intensity factors due to the existence of singularities at those points.

In the present paper, the emphasis is placed on the determination of stress intensity factors near the ends of partial simple support; namely, the bending stress intensity factors (k_x, k_y), in-plane shearing stress intensity factor (k_{xy}), and also the transverse shearing stress intensity factors (k_{xz}, k_{yz}). These stress intensity factors will be first analytically derived in closed-form expression and then, numerically carried out for a variety of lengths of partial simple support.

To determine the stress intensity factors, the method can be made by considering the stress components in the vicinity of the end of partial simple support $(x \to e, y = 0)$ and then, isolating the singular dominant terms from these stress components [14], [19]. Thus, stress intensity factors are defined by the quantity involving the singular term of stresses at that singular point [20].

Within the framework of thin plate theory [11], the relationships between the stress components and stress resultants of the plate can be seen in the form as follows:

$$\sigma_x = \frac{12M_x\delta}{h^3}, \tag{34}$$

$$\sigma_y = \frac{12M_y\delta}{h^3}, \tag{35}$$

$$\sigma_{xy} = -\frac{12M_{xy}\delta}{h^3}, \tag{36}$$

$$\sigma_{xz} = \frac{3Q_x}{2h}\left[1-\left(\frac{2\delta}{h}\right)^2\right], \tag{37}$$

$$\sigma_{yz} = \frac{3Q_y}{2h}\left[1-\left(\frac{2\delta}{h}\right)^2\right], \tag{38}$$

where σ_{ij} is defined as the stress components acting on the right side of cubic element in the rectangular coordinates (x, y, z) [8], and δ is the coordinate measured from and perpendicular to the middle plane of the plate.

Since the singularities are of $O(\varepsilon^{-1/2})$ in the moments or of $O(\varepsilon^{-3/2})$ in the transverse shearing forces [12] where ε is an infinitesimal distance measured from the singular point (ends of partial simple support), it may be expressed those stress components presented in Eqs.(34) to (38) in terms of stress intensity factors as shown below [1], [3], [7],

$$\sigma_x = \frac{k_x}{\varepsilon^{1/2}}, \tag{39}$$

$$\sigma_y = \frac{k_y}{\varepsilon^{1/2}}, \tag{40}$$

$$\sigma_{xy} = \frac{k_{xy}}{\varepsilon^{1/2}}, \tag{41}$$

$$\sigma_{xz} = \frac{k_{xz}}{\varepsilon^{3/2}}, \tag{42}$$

$$\sigma_{yz} = \frac{k_{yz}}{\varepsilon^{3/2}}. \tag{43}$$

Again, in views of Eqs.(34) to (38) the stress intensity factors are immediately found in terms of stress resultants when they are compared with Eqs.(39) to (43). Thus, the relationships between the stress intensity factors and stress resultants of the plate are:

$$k_x = \frac{12M_x\delta\varepsilon^{1/2}}{h^3}, \tag{44}$$

$$k_y = \frac{12M_y\delta\varepsilon^{1/2}}{h^3}, \tag{45}$$

$$k_{xy} = -\frac{12M_{xy}\delta\varepsilon^{1/2}}{h^3}, \quad (46)$$

$$k_{xz} = \frac{3Q_x}{2h}\left[1-\left(\frac{2\delta}{h}\right)^2\right]\varepsilon^{3/2}, \quad (47)$$

$$k_{yz} = \frac{3Q_y}{2h}\left[1-\left(\frac{2\delta}{h}\right)^2\right]\varepsilon^{3/2}. \quad (48)$$

Noted that the negative sign convention presented in Eq.(46) is resulted from the expression of σ_{xy} in Eq.(36), which can be found in Ref.[11, pp.167]. Moreover, the expressions of stress intensity factor as given in Eqs.(44) to (48) are valid since stress resultants are considered to be the quantities in the vicinity of singular point.

At the present stage, it remains to determine the stress resultants in Eqs.(44) through (48) near the end of partial simple support $(x \to e, y = 0)$. Before proceeding further to the determinations of these stress resultants, it is useful to consider two identities found in Stahl and Keer [4] that will be required for the subsequent analyses. Thus, they are listed here for convenience and will be reference as needed:

$$\sum_{m=1,3,5,...}^{\infty} J_1(mt)\sin(mx) = \frac{xH(t-x)}{2t(t^2-x^2)^{1/2}}\ ;\ x+t<\pi\ , \quad (49)$$

$$\sum_{m=1,3,5,...}^{\infty} J_1(mt)\cos(mx) = \frac{1}{2t} - \frac{xH(x-t)}{2t(x^2-t^2)^{1/2}} + \int_0^{\infty}\frac{I_1(ts)\cosh(xs)}{\exp(\pi s)+1}ds\ ;\ x+t<\pi\ , \quad (50)$$

where $H(t-x)$ is the Heaviside unit step function [21] that can be defined by

$$H(t-x) = \begin{Bmatrix} 1 & \text{for} & t \ge x \\ 0 & \text{for} & t < x \end{Bmatrix}. \quad (51)$$

Therefore, by substituting Eq.(11) for w into Eqs.(4) to (8) with setting $y = 0$ while the unknown constants as presented in Eq.(13) can be found in Eqs.(14)-(16), (25), and (30) and then, together with using the identities of Eqs.(49) and (50), one can obtain the singular dominant terms of stress resultants as follows:

$$M_x(x,0)\big|_{0\le x<e} = -\frac{(1+\nu)\pi^2 P_o\bar{a}e\xi}{2(1-\xi^2)^{1/2}}\int_0^1 \rho^2\Phi(\rho)d\rho\ , \quad (52)$$

$$M_{xy}(x,0)\big|_{e<x\le\frac{\pi}{2}} = -\frac{(1+\nu)\pi^2 P_o\bar{a}e\xi}{4(\xi^2-1)^{1/2}}\int_0^1 \rho^2\Phi(\rho)d\rho\ , \quad (53)$$

$$Q_x(x,0)\big|_{0\le x<e} = -\frac{\pi^3 P_o}{2(1-\xi^2)^{3/2}}\int_0^1 \rho^2\Phi(\rho)d\rho\ , \quad (54)$$

$$Q_y(x,0)\big|_{e<x\le\frac{\pi}{2}} = -\frac{\pi^3 P_o}{2(\xi^2-1)^{3/2}}\int_0^1 \rho^2\Phi(\rho)d\rho\ , \quad (55)$$

in which $\xi = x/e$ and $0 \le \xi, \rho \le 1$.

It is notable that at $y = 0$, the bending moment M_y is automatically vanished due to the characteristic of support conditions being simple and free supports.

Further substitution of Eqs.(52), (53), (54), (55) into Eqs.(44), (46), (47), and (48), respectively, and also with introducing $x = e-\varepsilon$ in $M_x(x,0)$, $Q_x(x,0)$ and $x = e+\varepsilon$ in $M_{xy}(x,0)$, $Q_y(x,0)$, then, the stress intensity factors can be obtained in the following forms:

$$k_x = -\frac{6(1+\nu)\pi^2 P_o\bar{a}e^{3/2}\delta}{\sqrt{2}h^3}\int_0^1 \rho^2\Phi(\rho)d\rho\ , \quad (56)$$

$$k_{xy} = \frac{3(1+\nu)\pi^2 P_o\bar{a}e^{3/2}\delta}{\sqrt{2}h^3}\int_0^1 \rho^2\Phi(\rho)d\rho\ , \quad (57)$$

$$k_{xz} = -\frac{3\pi^3 P_o e^{3/2}}{8\sqrt{2}h}\left[1-\left(\frac{2\delta}{h}\right)^2\right]\int_0^1 \rho^2\Phi(\rho)d\rho\ , \quad (58)$$

$$k_{yz} = -\frac{3\pi^3 P_o e^{3/2}}{8\sqrt{2}h}\left[1-\left(\frac{2\delta}{h}\right)^2\right]\int_0^1 \rho^2\Phi(\rho)d\rho\ . \quad (59)$$

It is notable that the bending moment M_y along the edge at $y = 0$ is vanished due to the characteristic of support conditions being simple and free supports. As seen in Eqs.(58) and (59), the expression for k_{yz} is equal to k_{xz}, hence, there are only three different stress intensity factors to be determined as mentioned before in this problem considered.

From the literatures, especially in the field of fracture mechanics, it is emphasized on finding their maximum values. However, the bending and in-plane shearing stress intensity factors (k_x, k_{xy}) are found to be the maximum values at the top surface of the plate, while the transverse shearing stress intensity factor (k_{xz}) is at the middle plane of the plate [11].

Substituting $\delta = -h/2$ into Eqs.(56) and (57) and $\delta = 0$ into Eq.(58) yields the maximum values for:

$$k_x = \frac{3(1+\nu)\pi^2 P_o \bar{a} e^{3/2}}{\sqrt{2}h^2} \int_0^1 \rho^2 \Phi(\rho) d\rho \,, \tag{60}$$

$$k_{xy} = -\frac{3(1+\nu)\pi^2 P_o \bar{a} e^{3/2}}{2\sqrt{2}h^2} \int_0^1 \rho^2 \Phi(\rho) d\rho \,, \tag{61}$$

$$k_{xz} = -\frac{3\pi^3 P_o e^{3/2}}{8\sqrt{2}h} \int_0^1 \rho^2 \Phi(\rho) d\rho \,. \tag{62}$$

Another quantity that interested in analysis of fracture mechanics is the strain energy (U). The strain energy of the plate can be computed from the external work done since the plate is in equilibrium [3], [22], which is

$$U = \frac{\bar{a}^2}{2\pi^2} \int_0^b \int_0^\pi wq\,dx\,dy \,. \tag{63}$$

Substitution of Eq.(11) for w and the line load P_o into the above equation leads to

$$U = \frac{P_o^2 \bar{a}^4}{2\pi^2 D} \int_0^b \int_0^\pi \sum_{m=1,3,5,...}^{\infty} \big[Q_m + A_m \cosh(my) + B_m my \sinh(my) + C_m \sinh(my) + D_m my \cosh(my)\big] \sin(mx)\,dx\,dy \,. \tag{64}$$

Upon the integration of Eq.(64) and using Eqs.(14) to (16), (25), and (30), the result becomes

$$U = \frac{P_o^2 \bar{a}^4}{2\pi^2 D} \sum_{m=1,3,5,...}^{\infty} [\{2\beta - (2c_m + \bar{c}_m) + (2d_m - \bar{d}_m) - [3 - \beta(2d_m - \bar{d}_m)]\sinh\beta + [\beta + (2c_m + \bar{c}_m) - (2d_m - \bar{d}_m)]\cosh\beta\} \frac{Q_m}{m} + \frac{e^2}{m^4}\{\bar{c}_m + \bar{d}_m + [(3-\nu)\eta' + \beta\bar{d}_m]\sinh\beta - (\beta + \bar{c}_m + \bar{d}_m)\cosh\beta\} \times \int_0^1 \rho\Phi(\rho)[J_1(me\rho) - \rho J_1(me)]\,d\rho] \,. \tag{65}$$

To find the strain energy change due to the presence of partial simple support, it can be determined by the terms which involve with the function $\Phi(\rho)$ presented in Eq.(65). Therefore, the change in strain energy is

$$\Delta U = \frac{P_o^2 \bar{a}^4}{2\pi^2 D} \sum_{m=1,3,5,...}^{\infty} \frac{e^2}{m^4} \{\bar{c}_m + \bar{d}_m + [(3-\nu)\eta' + \beta\bar{d}_m]\sinh\beta - (\beta + \bar{c}_m + \bar{d}_m)\cosh\beta\} \times \int_0^1 \rho\Phi(\rho)\big[J_1(me\rho) - \rho J_1(me)\big]\,d\rho \,. \tag{66}$$

It can be remarked that for the limiting case of $e = 0$, the strain energy given in Eq.(65) is the strain energy for the plate having full length of simple support at $y = 0$ which is the first bracket term in the series. Thus, Eq.(65) can be represented by the sum of two parts. The first is $\bar{U}$ for the plate supported by simple support along the edge at $y = 0$ and the second is ΔU for the additional deflection due to the partial simple support at the same edge.

4. NUMERICAL RESULTS

After performing the integral equation for obtaining the function $\Phi(\rho)$ in Eq.(31), the stress intensity factors that found in Eqs.(60), (61), (62) and the change in strain energy in Eq.(66) can be computed numerically.

In order to evaluate Eq.(31), the method is first made by transforming the integral equation into a system of N-linear algebraic equations using the Simpson's rule of integration. With using the 15-point Gauss-Laguerre quadrature formula [17], the improper infinite integral in the kernel of Eq.(32) can numerically be obtained. In addition, the infinite series as seen in Eqs.(32), (33), and (66) are all calculated to a relative error of 0.0000001.

The results are carried out for a scaled square plate of side π with the Poisson's ratio taken as 0.3. The free edge length (e) at $y = 0$, when parts of simple support are removed, is varied between 0.05π and 0.495π.

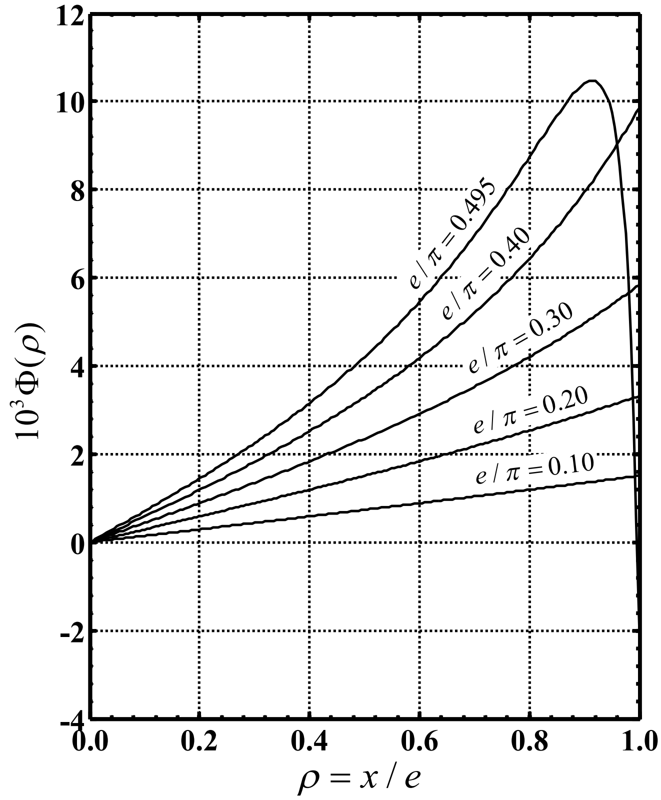

Fig. 2 *Auxiliary function for square plate*

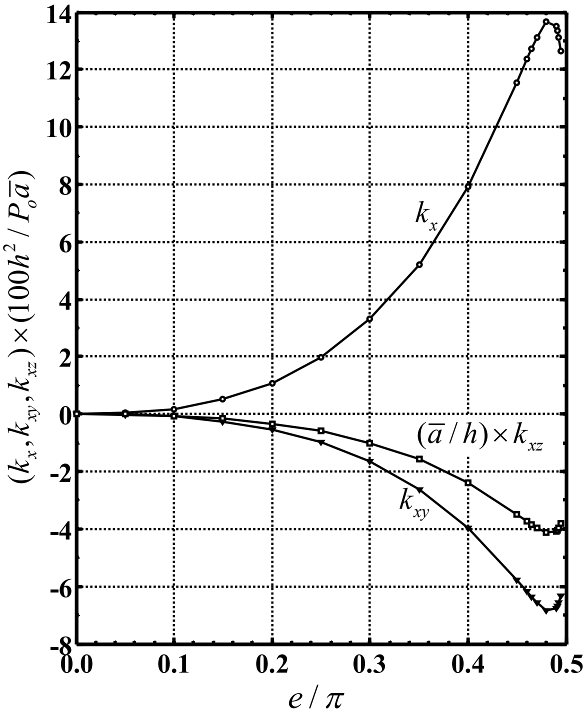

Fig. 3 *Stress intensity factors for square plate*

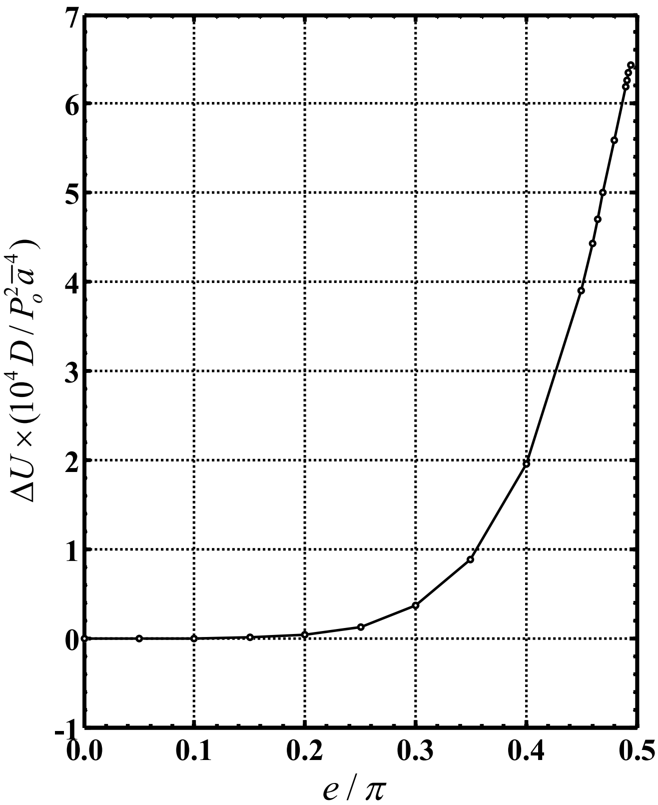

Fig. 4 *Change in strain energy for square plate*

As explained above, the solution of Eq.(31) in terms of function $\Phi(\rho)$ can be obtained as shown in Fig.2. After the determination of this function, the calculation of stress intensity factors (k_x, k_{xy}, k_{xz}) and the change in strain energy (ΔU) are evaluated and graphically presented in Figs.3 and 4, respectively. All numerical results are carried out based on using MATLAB program [23] and given in the tabular form as seen in Tables 1 to 3. Note that stress intensity factors have the highest magnitude at $e/\pi = 0.48$.

Table 1 *Numerical Values for Auxiliary Function.*

ρ	$10^3\Phi(\rho)$		
	$e = 0.10\pi$	$e = 0.30\pi$	$e = 0.495\pi$
0.0	0.0000	0.0000	0.0000
0.1	0.1462	0.4392	0.7066
0.2	0.2926	0.8861	1.4452
0.3	0.4396	1.3483	2.2489
0.4	0.5875	1.8339	3.1547
0.5	0.7364	2.3518	4.2059
0.6	0.8868	2.9121	5.4531
0.7	1.0389	3.5258	6.9546
0.8	1.1930	4.2057	8.7394
0.9	1.3495	4.9696	10.4003
1.0	1.5084	5.8368	-2.3783

Table 2 *Numerical Values for Stress Intensity Factors.*

e/π	$\text{SIFs}\times(100h^2/P_o\overline{a})^*$		
	k_x	k_{xy}	$(\overline{a}/h)\times k_{xz}$
0.00	0.0000	0.0000	0.0000
0.05	0.0311	-0.0156	-0.0094
0.10	0.1788	-0.0894	-0.0540
0.15	0.5061	-0.2530	-0.1529
0.20	1.0782	-0.5391	-0.3257
0.25	1.9761	-0.9880	-0.5969
0.30	3.3069	-1.6535	-0.9989
0.35	5.2177	-2.6088	-1.5761
0.40	7.9082	-3.9541	-2.3889
0.45	11.5502	-5.7751	-3.4890
0.46	12.3511	-6.1755	-3.7310
0.47	13.1018	-6.5509	-3.9577
0.48	**13.6563**	**-6.8282**	**-4.1253**
0.49	13.5187	-6.7594	-4.0837
0.495	12.6538	-6.3269	-3.8224
* SIFs = stress intensity factors; k_x, k_{xy}, and k_{xz}			

Table 3 *Numerical Values for Change in Strain Energy.*

e/π	$\dfrac{10^4\Delta U}{P_o^2\overline{a}^4/D}$	e/π	$\dfrac{10^4\Delta U}{P_o^2\overline{a}^4/D}$
0.00	0.0000	0.40	1.9491
0.05	0.0000	0.45	3.9021
0.10	0.0008	0.46	4.4284
0.15	0.0070	0.47	4.9978
0.20	0.0351	0.48	5.5966
0.25	0.1277	0.49	6.1846
0.30	0.3646	0.495	6.4417
0.35	0.8875		

5. CONCLUDING REMARKS

An exact analytical method based on the finite Hankel integral transform techniques is presented for the analysis of uniformly line loaded rectangular plate with mixed edge conditions. The method is valuable in view of the fact that the moment singularities in the order of an inverse-square-root type have been taken into account at the ends of partial simple support. Therefore, in accordance with the linear elastic fracture mechanics, the stress intensity factors due to the existence of moment singularities and the change in strain energy due to the presence of partial simple support can analytically be determined in closed-form expressions. In the present paper, some numerical values are also prepared in the form of tables for easy reference, which can be used as benchmarks for other alternative methods.

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