The Procedure for Generating Random Numbers with Crack Distribution

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Abstract

The paper investigates an algorithm that can generate random numbers that follow the three-parameter of the Crack lifetime distribution. The process combines analytical and composite methods.

Keywords: Random numbers generation, Crack Distribution, Inverse Gaussian Distribution, Length Biased Inverse Gaussian Distribution

Introduction

Survival Analysis is a branch of statistics for handling the analysis of time duration until one or more events happen, such as a death of biological organisms, and a failure in mechanical systems. Survival Analysis consists of techniques for positive valued random variables that model time to death, time to onset (or relapse) of a disease, length of stay in a hospital, duration of a strike, money paid by health insurance, viral load measurements, time of crack development in a plastic concrete, fatigue life of aluminum, fatigue life of spring, and fatigue limit load. Sometimes this topic is called Reliability Theory or Reliability Analysis in Engineering, Event History Analysis in Sociology, and Duration Analysis or Duration Modeling in economics. Statistical models that were developed for any of these topics are generically called Time-to-Event models. In Reliability Theory, failure is called an event, and the goal is to project or forecast the rate of events for a given population, or the probability of an event, or the frequency of an event for an individual. In order to achieve the objectives, it is necessary to define lifetime or failure time.

In the Reliability Theory, a lifetime is the period of time during which a property, or an object, or a process, or a phenomenon exists or functions. A lifetime distribution gives a useful information which motivates users to protect damages of the industrial or financial occurred after the lifetime is terminated. It is not safe, if users do not know the lifetime of their machines or systems because it will be meant a danger of their lives or health.

The lifetime distributions that are common in Data Analysis and Reliability Theory include Log-Normal, Extreme Value, Birnbaum-Saunders, Weibull, Inverse Gaussian, Length Biased Inverse Gaussian and Crack distributions among many others. Distributions mentioned above were studied in many research papers and monographs. In this paper, we emphasize the Crack distribution which contains of Inverse Gaussian, Length Biased Inverse Gaussian and Birnbaum-Saunders distributions as particular cases.

Now we provide a brief literature survey on Inverse Gaussian distribution shortly denoted as IG distribution. It has been coming to the attention of the authors and researchers with its usefulness in Reliability Theory for more than a century already. The IG distribution is a right skewed distribution also known as the first passage time distribution of Brownian motion with positive drift, which was discovered by Schrodinger (1915). Later, Tweedie (1957) proposed the name Inverse Gaussian for this distribution since its cumulant generating function is the inverse of the cumulant generating function of a normal random variable. It has many interesting statistical and probabilistic properties that are similar to the normal distribution. Chhikara and Folks (1989) mentioned that the normal distribution describes the distance traveled by a particle at fixed time the standard Brownian motion, while the Inverse Gaussian distribution describes the distribution of the time a Brownian motion with positive drift takes to reach a fixed positive level. They also showed the connection of IG distribution and χ^2 and F distributions and applied these facts to the Sampling Theory. Chaubey *et al*. (2014) proved that the likelihood ratio test for one sided hypotheses concerning the coefficient of variation in the Inverse Gaussian family is the uniformly most powerful invariant test under scale transformation. They also investigated some approximations to the cumulative distribution function of the test statistic. The Inverse Gaussian distribution is an interesting alternative to the normal distribution for modeling non-negative data with positive skewness.

Next, we provide a brief literature survey on Length Biased Inverse Gaussian distribution shortly denoted as LB distribution. This is the length biased version of the Inverse Gaussian distribution, which was studied by Ahsanullah and Kirmani (1984), and Khattree (1989). It may be proved that the Length Biased Inverse Gaussian distribution is the reciprocal of the Inverse Gaussian distribution and hence sometimes it is called Complementary Reciprocal of Inverse Gaussian distribution. The notion of a Length Biased distribution has been received considerable attention due to its various applications. Sen (1987) studied the properties of the arithmetic, geometric and harmonic mean for length biased distributions in a nonparametric fashion. He also presented the coefficient of variation and the characterization of length biased distributions. Gupta and Akman (1998) apply some results from Sen (1987) in order to develop confidence intervals and tests regarding the mean and the coefficient of variation of the Inverse Gaussian distribution based on the length biased data.

Birnbaum and Saunders (1969a) proposed a lifetime time distribution for fatigue failure caused by reliability a crack development under cyclic loading. The model is established under the assumption that the failure is due to development and growth of a dominant crack. They also considered some closure properties of this family and compared with other families such as the lognormal distribution. This distribution is called the two-parameter Birnbaum-Saunders distribution (herein after BS distribution). Birnbaum and Saunders (1969b) presented theoretical and practical review of the fitting this distribution to several extensive sets of fatigue data. Desmond (1986) proposed a more general derivation based on a biological model and strengthened the physical justification for the use of this distribution. His derivation follows from considerations of renewal theory for the number of cycles needed to force a fatigue crack extension to exceed a critical value. Ahmad (1988) proposed the estimation of the scale parameter by the jackknife method to eliminate first-order bias. This estimate has the same limiting behavior as that of Birnbaum and Saunders (1969b). Lemonte *et al*. (2007) developed nearly unbiased estimators for the Birnbaum-Saunders distribution. They derived modified maximum likelihood estimators that are bias-free to second order and considered bootstrap-based bias correction. Additionally, they derived a Bartlett correction that improves the finite-sample performance of the likelihood ratio test in finite samples.

Kamon *et al*. (2008) proposed the new parametrization of the Birnbaum-Saunders distribution. Essentially, this re-parametrization fits the physics of studying phenomena since the proposed parameters characterize the thickness and the nominal treatment loading on the metallic plate where a crack is developing. The usual shape and scale parameters of the distribution do not allow this physical interpretation. They also presented the relationship between the usual parameters and the proposed parameters. Kundu *et al*. (2010) presented bivariate absolutely continuous Birnbaum-Saunders distribution and discussed different properties and parameter estimation of this distribution. Some recent publications on the Birnbaum-Saunders distribution we refer to Ng *et al*. (2006), From and Li (2006), Ng *et al*. (2007) and Cordeiro and Lemonte (2011).

The Crack distribution is a positively skewed model, which is widely applicable to model failure times of fatiguing materials. Up to our knowledge, this distribution was introduced in Jørgensen et al. (1991) as JSW distribution and it was discussed from the reliability point of view. It is also known as the Inverse Gaussian Mixture distribution and discussed by Gupta and Akman (1995a). Gupta and Akman (1995b) studied the Bayesian estimation of this distribution. Volodin and Dzhungurova (2000) introduced a five-parameter family of so-called General Crack distributions, which contains, in particular, the Inverse Gaussian Mixture distribution, normal distribution, the Inverse Gaussian distribution, and the Birnbaum-Saunders distribution, as well as others which are used in applications of the Reliability Theory. Balakrishnan *et al.* (2009) considered Inverse Gaussian Mixture distribution and produced a lifetime analysis by developing the EM-algorithm for maximum likelihood estimation of parameters and illustrating the obtained results with real data showing the robustness of the estimation procedure.

Bowonrattanaset (2011a) and Bowonrattanaset and Budsaba (2011b) re-introduced the Inverse Gaussian mixture distribution based on re-parametrization model presented in Ahmed *et al.* (2008) and called it Crack distribution. In the following we use this term and it will be denoted by $CR(\lambda,\theta,p)$. Bowonrattanaset (2011a) and Bowonrattanaset and Budsaba (2011b) also established some basic probability properties of the Crack distribution and derived distribution, moment generating, and characteristic functions in the closed form. Duangsaphon (2014) studied Crack distribution in the view of regression-quantile estimation, Bayesian estimation and confidence interval estimation. Additionally, Saengthong and Bodhisuwan (2014) proposed a new two-parameter Crack distribution which is obtained by adding a new weight parameter to the Crack distribution.

In this article, we provide a new procedure to generate random number that follow three parameter Crack distribution. To generate Crack random number by composition method, first we generate random number from already known two parameter distributions: Inverse Gaussian distribution, and Length Biased Inverse Gaussian distribution. Finally, we derive Crack random number generation procedure.

Inverse Gaussian Distribution

According to Chhikara and Folks (1989), the classical parametrization of the inverse Gaussian distribution is a two parameter family of continuous probability distributions with support on (0, ∞). Suppose a random variable X has the inverse Gaussian distribution, and the corresponding probability density function (pdf.) is

$$f_{IG}(x;\mu,\beta)=\sqrt{\frac{\beta}{2\pi}}x^{-\frac{3}{2}}\exp\left(-\frac{\beta(x-\mu)^2}{2\mu^2x}\right)\quad ;x>0\,.$$

where parameter $\mu>0$ is the mean of the distribution and $\beta>0$ is a scale parameter.

Shuster (1968) mentioned a method to obtain the exact probabilities for Inverse Gaussian distribution by using Standard Normal tables and Log tables.

The new parametrization of the inverse Gaussian probability density function, denote as $f_{IG}(\lambda,\theta)$, is

$$f_{IG}(x;\lambda,\theta)=\lambda\sqrt{\frac{\theta}{2\pi}}x^{-\frac{3}{2}}\exp\left(-\frac{(x-\lambda\theta)^2}{2\theta x}\right)\quad ;x>0\,.$$

The new parameters are $\lambda > 0$ and $\theta > 0$ corresponding to the thickness of the machine element and nominal treatment pressure on the machine element, respectively. The relations between classical parameters α, β and new parameter λ, θ can be expressed as

$$\lambda = \frac{\beta}{\mu} \text{ and } \theta = \frac{\mu^2}{\beta};$$

$$\mu = \lambda\theta \text{ and } \beta = \lambda^2\theta.$$

Length Biased Inverse Gaussian Distribution

According to Khattree (1989) , the length biased density of its original density function is defined as follows. Let X be a non-negative random variable having an absolutely continuous pdf. $f(\cdot)$ and a finite first moment $E[X]$. We say that a non-negative random variable Y with pdf. $h(\cdot)$ has the length biased random variable associated with X, if its density function is given by the formula

$$h(x) = \frac{xf(x)}{E[X]}, x > 0.$$

We are interested in the Length Biased Inverse Gaussian distribution. Thus, we will find the density of the Length Biased inverse Gaussian distribution in form of parameters λ, θ.

We know that the first moment of the inverse Gaussian distribution is $E(X) = \mu = \lambda\theta$. Hence the density of the Length Biased Inverse Gaussian distribution is given by the following formula

$$f_{LB}(x;\lambda,\theta) = \frac{1}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right] \quad ; x > 0.$$

Here, $\lambda > 0$ and $\theta > 0$ are the shape and scale parameters, respectively. Hereafter, this distribution will be denoted by $LB(\lambda, \theta)$.

Birnbaum-Saunders Distribution

The Birnbaum-Saunders distribution arises as fatigue life model. This distribution helps us to explain how material failure occurs to the development and growth of a dominant crack.

First we provide the density function of the Birnbaum-Saunders distribution in the classical parametrization. Let a random variable X has the Birnbaum-Saunders distribution, its density function can be written as

$$f_{BS}(x;\alpha,\beta) = \frac{x+\beta}{2\alpha(2\pi\beta)^{\frac{1}{2}} x^{\frac{3}{2}}} \exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right] \quad ; x > 0$$

here, $\alpha > 0$ and $\beta > 0$ are the shape and the scale parameter. (see Birnbaum and Saunders (1969a, 1969b); Ng, Kundu and Balakrishnan (2003); Patel and Read (1996); Rausand and

Høyland (2004).

Ahmed *et al* (2008) introduced the new parametrization of the Birnbaum-Saunders distribution and discussed various estimation strategies for this new parametrization. Their proposed parameters are important by fitting the physical phenomena of fatigue cracks. The parameters $\lambda > 0$ and $\theta > 0$ correspond to the thickness of the machine element and the nominal treatment pressure on the machine element, respectively. The relation between classical parameters α, β and proposed parameters λ, θ in physical interpretation can be expressed as

$$\lambda = \frac{1}{\alpha^2} \text{ and } \theta = \alpha^2\beta;$$

$$\alpha = \frac{1}{\sqrt{\lambda}} \text{ and } \beta = \alpha\theta.$$

The new parametrization of the Birnbaum-Saunders density function, denoted as $f_{BS}(\lambda, \theta)$, is

$$f_{BS}(x;\lambda,\theta) = \frac{1}{2\theta\sqrt{2\pi}}\left[\lambda\left(\frac{\theta}{x}\right)^{\frac{3}{2}} + \left(\frac{\theta}{x}\right)^{\frac{1}{2}}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right] \quad ; x > 0.$$

Note that

$$f_{BS}(x;\lambda,\theta) = \frac{1}{2}\left[f_{IG}(x;\lambda,\theta) + f_{LB}(x;\lambda,\theta)\right].$$

Crack Distribution

The three-parameter crack distribution had been proposed by Bowonrattanaset and Budsaba (2011). This distribution is formed by adding the weighted parameter and including the two parameters inverse Gaussian distribution and the two parameters length biased inverse Gaussian distribution as follows:

$$f_{CR}(x;\lambda,\theta,p) = pf_{IG}(x;\lambda,\theta) + (1-p)f_{LB}(x;\lambda,\theta)$$

where $\lambda > 0, \theta > 0$ and $0 \le p \le 1$.

The density function of three-parameter Crack distribution is given by

$$f_{CR}(x;\lambda,\theta,p) = \frac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(\frac{\theta}{x}\right)^{\frac{3}{2}} + (1-p)\left(\frac{\theta}{x}\right)^{\frac{1}{2}}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right] \quad ; x > 0$$

here, $\lambda > 0, \theta > 0$ and $0 \le p \le 1$. Hereafter, this distribution will be denoted by $CR(\lambda, \theta, p)$.

The cumulative distribution function of $X : CR(\lambda, \theta, p)$ is

$$F_{CR}(\lambda,\theta,p) = \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) - 1 - 2pe^{2\lambda}\left[1 - \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)\right] \quad ; x > 0$$

where $\Phi(x)$ is the distribution function of the standard normal distribution.

The relevance of the density functions of four distributions i.e. the Crack distribution, the Inverse Gaussian distribution, the Length Biased Inverse Gaussian distribution and the Birnbaum-Saunders distribution is as follows. Suppose X_1 and X_2 be independent random variables such as $X_1 : IG(\lambda,\theta)$ and $X_2 : LB(\lambda,\theta)$. For the Crack distribution, we consider the new random variable X such that,

$$X = \begin{cases} X_1 & \text{with probability } p \\ X_2 & \text{with probability } 1-p \end{cases}$$

where $0 \le p \le 1$. Obviously, X is a mixture of X_1 and X_2. Namely

$$f_{CR}(x;\lambda,\theta,p) = \begin{cases} f_{IG}(x;\lambda,\theta) & ; p = 1 \\ f_{BS}(x;\lambda,\theta) & ; p = \dfrac{1}{2} \\ f_{LB}(x;\lambda,\theta) & ; p = 0 \end{cases}$$

where, $\lambda > 0, \theta > 0$ and $0 \le p \le 1$.

Random numbers generation methods

There are many algorithms to generate a random numbers with some specific distribution, for example, the Inversion method, Convolution method, Composite method, the Acceptance-Rejection method and etc.

The Acceptance-Rejection method, applicable to continuous, discrete, and mixed distributions, is widely used in generating random variables from a specified probability distribution on a computer. The method can be used alone, but more typically it is used together with other methods, especially the Mixture method, in creating exact and efficient algorithms. It is a common ingredient in many of the proposed methods for generating random variables from various distributions.

Composite method is applied for generation of random number for a distribution whose density function $f(x)$ can be written as a weighted sum of n density functions $f_i(x)$, where $1 \le i \le n$, that is

$$f(x) = \sum_{i=1}^{n} p_i f_i(x), p_i > 0 \;\; and, \sum_{i=1}^{n} p_i = 1.$$

In this case, as a random number for such distribution we can take random number for a distribution whose density function $f_i(x)$ with probability p_i. This method is called composition algorithm, see for example Ross (2013).

It is easy to see that the Crack distribution is the weighted linear combination of the Inverse Gaussian and Length Biased Inverse Gaussian distributions

$$f_{CR}(x;\lambda,\theta,p) = pf_{IG}(x;\lambda,\theta) + (1-p) f_{LB}(x;\lambda,\theta).$$

This representation gives us an idea to use the composition method to generate random number which follows the three parameter Crack distribution.

Outline of the procedure for generating Crack random number

The following steps are the procedure to generate Crack random number :

1. Find the connection between Inverse Gaussian, Length Biased Inverse Gaussian and $\chi^2(1)$ (Chi-square) distribution.
2. Find the connection between Inverse Gaussian, and $\chi^2(1)$ distribution.
3. Find the Inverse Gaussian random number generation procedure based on step 2.
4. Find the Length Biased Inverse Gaussian random number generation procedure based on steps 1 and 3.
5. Derive the Crack random number generation procedure based on steps 3 to 4.

Connection between Inverse Gaussian, Length Biased Inverse Gaussian and $\chi^2(1)$ distribution

Theorem 1 *If random variable Y has $IG(\lambda,\theta)$ distribution, random variable Z has $\chi^2(1)$ distribution (Chi - square distribution with 1 degree of freedom) and random variables Y and Z are independent, then random variable $X = Y + \theta Z$ has $LB(\lambda,\theta)$ distribution.*

Proof: The Moment Generating Function of $\chi^2(n)$ distribution is $(1-2t)^{-n/2}$, hence the Moment Generating Function of random variable θZ is

$$\varphi_{\theta Z}(t) = (1-2\theta t)^{-1/2}.$$

Since random variables Y and Z are independent, the Moment Generating Function of $X = Y + \theta Z$ is

$$\varphi_X(t) = \varphi_{Y+\theta Z}(t) = \varphi_Y(t)\cdot\varphi_{\theta Z}(t) = \varphi_{IG}(t;\lambda,\theta)(1-2\theta t)^{-1/2}$$

$$= \exp\{\lambda[1-(1-2\theta t)^{1/2}]\}(1-2\theta t)^{-1/2} = \varphi_{LB}(t;\lambda,\theta). \qquad \square$$

The next theorem is the connection between IG and $\chi^2(1)$ distribution which is related to Theorem 4.6 Chhikara and Folks (1989) and Shuster (1968).

Connection between Inverse Gaussian and $\chi^2(1)$ distribution

Theorem 2 *If Y has $IG(\lambda,\theta)$ distribution, then $Z = \dfrac{(Y-\lambda\theta)^2}{\theta Y}$ has $\chi^2(1)$ distribution with one degree of freedom.*

Proof: The distribution function of a random variable $Z = \dfrac{(Y-\lambda\theta)^2}{\theta Y}$ is

$$F_Z(t) = P(Z \le t)$$

$$
\begin{aligned}
&= P\left(\frac{(Y-\lambda\theta)^2}{\theta Y} \le t\right) \\
&= P\left\{Y^2 - \theta(2\lambda+t)Y + \lambda^2\theta^2 \le 0\right\} \\
&= P\left(u_1(t) \le Y \le u_2(t)\right),
\end{aligned}
$$

where

$$u_1(t) = \lambda\theta + \frac{\theta}{2}\left[t - \sqrt{t^2+4\lambda t}\right] \quad and \quad u_2(t) = \lambda\theta + \frac{\theta}{2}\left[t + \sqrt{t^2+4\lambda t}\right]$$

are the solutions of the quadratic equation $u^2 - \theta(2\lambda+t)u + \lambda^2\theta^2 = 0$. Hence, $F_Z(t) = F_{IG}(u_2;\lambda,\theta) - F_{IG}(u_1;\lambda,\theta)$ and the density function $f_Z(t) = f_{IG}(u_2;\lambda,\theta)\cdot u_2'(t) - f_{IG}(u_1;\lambda,\theta)\cdot u_1'(t)$. Since u_1 and u_2 are solutions of the quadratic equation $u^2 - \theta(2\lambda+t)u + \lambda^2\theta^2 = 0$, we have $u_1 + u_2 = \theta(2\lambda+t)$ and $u_1 \cdot u_2 = \lambda^2\theta^2$.

Then,

$$\left(u_2^{1/2} - u_1^{1/2}\right)^2 = u_2 + u_1 - 2(u_1u_2)^{1/2} = \theta t.$$

Also note that $\frac{u_2-u_1}{\theta} = \sqrt{t^2+4\lambda t}$, $2\lambda + t = \frac{u_2+u_1}{\theta}$ and therefore

$$
\begin{aligned}
u_1'(t) &= \frac{\theta}{2}\left(1 - \frac{2\lambda+t}{\sqrt{t^2+4\lambda t}}\right) = \frac{\theta}{2}\left(1 - \frac{u_2+u_1}{u_2-u_1}\right) = \frac{\theta u_1}{u_2-u_1}, \\
u_2'(t) &= \frac{\theta}{2}\left(1 + \frac{2\lambda+t}{\sqrt{t^2+4\lambda t}}\right) = \frac{\theta}{2}\left(1 + \frac{u_2+u_1}{u_2-u_1}\right) = \frac{\theta u_2}{u_2-u_1}.
\end{aligned}
$$

Moreover, $\frac{(u_1-\lambda\theta)^2}{u_1\theta} = \frac{(u_2-\lambda\theta)^2}{u_2\theta} = t$.

Hence,

$$\exp\left\{-\frac{(u_1-\lambda\theta)^2}{2u_1\theta}\right\} = \exp\left\{-\frac{(u_2-\lambda\theta)^2}{2u_2\theta}\right\} = \exp\left\{-\frac{t}{2}\right\}$$

and

$$
\begin{aligned}
f_Z(t) &= f_{IG}(u_2;\lambda,\theta)\cdot u_2'(t) - f_{IG}(u_1;\lambda,\theta)\cdot u_1'(t) \\
&= \frac{\lambda\theta^{1/2}}{\sqrt{2\pi}} u_2^{-3/2} \exp\left\{-\frac{(u_2-\lambda\theta)^2}{2u_2\theta}\right\}\cdot\frac{\theta u_2}{u_2-u_1} \\
&\quad + \frac{\lambda\theta^{1/2}}{\sqrt{2\pi}} u_1^{-3/2} \exp\left\{-\frac{(u_1-\lambda\theta)^2}{2u_1\theta}\right\}\cdot\frac{\theta u_1}{u_2-u_1} \\
&= \frac{\lambda\theta^{3/2}\exp\{-t/2\}}{\sqrt{2\pi}}\cdot\frac{u_2^{-1/2}+u_1^{-1/2}}{u_2-u_1}
\end{aligned}
$$

$$= \frac{\lambda\theta^{3/2}\exp\{-t/2\}}{\sqrt{2\pi}} \cdot \frac{u_2^{1/2}+u_1^{1/2}}{(u_2u_1)^{1/2}\left(u_2^{1/2}+u_1^{1/2}\right)\left(u_2^{1/2}-u_1^{1/2}\right)}$$

$$= \frac{\lambda\theta^{3/2}\exp\{-t/2\}}{\sqrt{2\pi}\lambda\theta} \cdot \frac{1}{(\theta t)^{1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} t^{-1/2}\exp\{-t/2\} \quad ,$$

which is a density function of the $\chi^2(1)$ distribution with one degree of freedom. □

The theorem proved above gives as a way to generate Inverse Gaussian $IG(\lambda,\theta)$ random numbers. There is no difficulty to generate a random variable Z with Chi–square distribution with one degree of freedom. The problem is that when we solve the equation $Z = \frac{(Y-\lambda\theta)^2}{\theta Y}$ for Y, then there are two roots:

$u_1 = \lambda\theta + \frac{\theta}{2}\left[Z - \sqrt{Z^2 + 4\lambda Z}\right]$ (smaller root) and

$u_2 = \frac{\lambda^2\theta^2}{u_1}$ (larger root).

Which root to choose, smaller or larger? The next theorem answers this question and it is based on an argument presented in Michael, Schucany, and Haas (1976).

Theorem 3 *Let Z be a random variable with Chi–square distribution with one degree of freedom. Consider a random variable Y that takes value $u_1 = \lambda\theta + \frac{\theta}{2}\left[Z - \sqrt{Z^2 + 4\lambda Z}\right]$ with probability $p_1 = \frac{\lambda\theta}{\lambda\theta + u_1}$ and the value $u_2 = \frac{\lambda^2\theta^2}{u_1}$ with probability $p_2 = 1 - p_1(Z) = \frac{u_1}{\lambda\theta + u_1}$. Then Y has $IG(\lambda,\theta)$ distribution.*

Proof: Consider the interval $(t-h, t+h)$, where $h > 0$. According to the inverse function theorem, for h sufficiently small, the inverse image g^{-1} of the interval $(t-h, t+h)$ is comprised of two disjoint intervals about roots u_1 and u_2. Denote them by (v_{11}, v_{12}) (contains u_1) and (v_{21}, v_{22}) (contains u_2). If $p_1(t)$ denotes the probability with which an observation of Y should be chosen from the first interval (v_{11}, v_{12}) given that Z is in the interval $(t-h, t+h)$, then

$$p_1^h(t) = P(v_{11} < Y < v_{12} \mid t-h < Z < t+h)$$

$$= \frac{P(v_{11} < Y < v_{12} \ and \ t-h < Z < t+h)}{P(t-h < Z < t+h)}$$

$$= \frac{P(v_{11} < Y < v_{12})}{P(v_{11} < Y < v_{12}) + P(v_{21} < Y < v_{22})}$$

$$= \frac{F_{IG}(v_{12}) - F_{IG}(v_{11})}{F_{IG}(v_{12}) - F_{IG}(v_{11}) + F_{IG}(v_{22}) - F_{IG}(v_{21})}$$

$$= \left(1 + \frac{F_{IG}(v_{22}) - F_{IG}(v_{21})}{F_{IG}(v_{12}) - F_{IG}(v_{11})}\right)^{-1}.$$

Note that $\lim_{h\to 0}(t-h, t+h) = t$ and $\lim_{h\to 0}(v_{11}, v_{12}) = u_1$, and so $p_1(t) = \lim_{h\to 0} p_1^h(t)$ will yield the conditional probability with which the first root u_1 should be selected. Hence,

$$\begin{aligned} p_1(t) &= \lim_{h\to 0} p_1^h(t) \\ &= \left(1 + \lim_{h\to 0}\frac{F_{IG}(v_{22}) - F_{IG}(v_{21})}{F_{IG}(v_{12}) - F_{IG}(v_{11})}\right)^{-1} \\ &= \left(1 + \lim_{h\to 0}\frac{(v_{22}-v_{21})/h}{(v_{12}-v_{11})/h}\cdot\frac{(F_{IG}(v_{22}) - F_{IG}(v_{21}))/(v_{22}-v_{21})}{(F_{IG}(v_{12}) - F_{IG}(v_{11}))/(v_{12}-v_{11})}\right)^{-1} \\ &= \left(1 + \left|\frac{g'(u_1)}{g'(u_2)}\right|\frac{f_{IG}(u_2)}{f_{IG}(u_1)}\right)^{-1}. \end{aligned}$$

The absolute value in the expression $\left|\frac{g'(u_1)}{g'(u_2)}\right|$ appears because $\frac{v_{22}-v_{21}}{v_{12}-v_{11}} > 0$ always. Note that $g'(x) = \frac{x^2 - \lambda^2\theta^2}{\theta x^2}$, and using the relations $u_1 u_2 = \lambda^2\theta^2$ and

$$\exp\left\{-\frac{(u_2-\lambda\theta)^2}{2u_2\theta}\right\} = \exp\left\{-\frac{(u_1-\lambda\theta)^2}{2u_1\theta}\right\}$$

established above, we obtain

$$\frac{g'(u_1)}{g'(u_2)} = \frac{(u_2^2 - \lambda^2\theta^2)u_1^2}{(u_1^2 - \lambda^2\theta^2)u_2^2} = \frac{u_1(u_1-u_2)u_2^2}{u_2(u_2-u_1)u_1^2} = -\frac{u_2}{u_1} = -\frac{\lambda^2\theta^2}{u_1^2}$$

and

$$\frac{f_{IG}(u_2)}{f_{IG}(u_1)} = \frac{u_2^{-3/2}}{u_1^{-3/2}} = \frac{u_1^3}{\lambda^3\theta^3}.$$

Hence, the smaller root u_1 should be chosen with probability $p_1(t) = \frac{\lambda\theta}{\lambda\theta + u_1}$ and the larger root $u_2 = \lambda^2\theta^2/u_1$ should be chosen with probability $p_2(t) = 1 - p_1(t) = \frac{u_1}{\lambda\theta + u_1}$. □

The above theorem is the main driver to generate IG random number. Therefore, the

IG random number generator process follows the steps below.

The Inverse Gaussian $IG(\lambda,\theta)$ random number generation procedure.

The following is an IG(λ,θ) random number generator procedure.

1. Generate a random number a uniform [0,1] and independently a standard normal number α .
2. Calculate $u=\lambda\theta+\frac{\theta}{2}\left[\alpha^2-\sqrt{\alpha^4+4\lambda\alpha^2}\right]$.
3. If $a<\frac{\lambda\theta}{(\lambda\theta+u)}$, then take $IG=u$, otherwise $IG=\frac{\lambda^2\theta^2}{u}$.

Now we can easily obtain LB random number generator process.

The Length Biased Inverse Gaussian $LB(\lambda,\theta)$ random number generation procedure.

The following is an LB(λ,θ) random number generator procedure.

1. Generate a random number a uniform [0,1] and independently a two independent standard normal numbers α and α_1 .
2. Calculate $u=\lambda\theta+\frac{\theta}{2}\left[\alpha^2-\sqrt{\alpha^4+4\lambda\alpha^2}\right]$.
3. If $a<\frac{\lambda\theta}{(\lambda\theta+u)}$, then take $IG=u$, otherwise $IG=\frac{\lambda^2\theta^2}{u}$.
4. Take $LB=IG+\theta\cdot\alpha_1^2$.

The Crack random number generation procedure.

According to the composition method the following is the Crack random number generation procedure.

1. Fix λ,θ,p , the parameters of the Crack distribution.
2. Generate a random number b uniform [0,1].
3. If $b<p$, then generate a random number with $IG(\lambda,\theta)$ distribution. Otherwise, generate a random number with $LB(\lambda,\theta)$ distribution.

Algorithm for Crack random number generation

Fix initial values for p,λ and θ .

Select two random numbers uniformly distributed between 0 and 1, call them a and b .

Let $u=\lambda\theta+\frac{\theta}{2}(\alpha^2\sqrt{\alpha^4+4\lambda\alpha^2})$

if $a < \left(\dfrac{\lambda\theta}{\lambda\theta + u} \right)$ then $CR = u$ else $CR = \dfrac{\lambda^2\theta^2}{u}$.

if $b > p$ then $CR = CR + \theta r^2$ where r is random number from a standard normal distribution.

CR is a random number from the Crack distribution.

Results of Monte Carlo simulations Crack–distribution and Conclusion

For computer simulations we consider the following values of λ, θ and p.

$\lambda = 2, 5, 10, 20, 50$, $\theta = 1, 5, 10, 50$ and $p = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$

For each fixed values of three parameters, We run simulations of corresponding random numbers independently. Simulations are repeated 1,000 times.

We presented the research result. The histogram from generating the random numbers that follow three-parameter Crack distribution by the Composition method and density function of the Crack distribution shown in Figure 1-15.

Figure 1 Histogram of Crack when λ = 2, θ = 1 and p = 0.2.

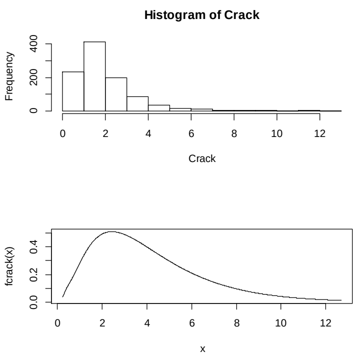


Figure 2 Histogram of Crack when λ = 5, θ = 1 and p = 0.2.

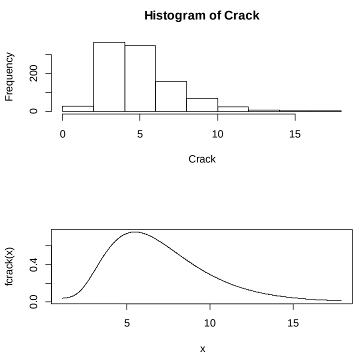


Figure 3 Histogram of Crack when λ = 10, θ = 1 and p = 0.2.

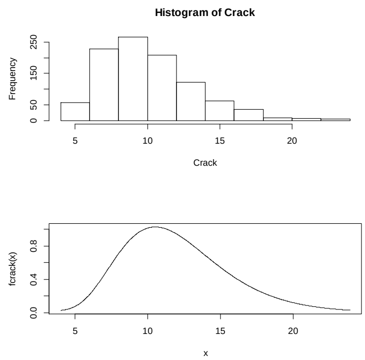


Figure 4 Histogram of Crack when λ = 20, θ = 1 and p = 0.2.

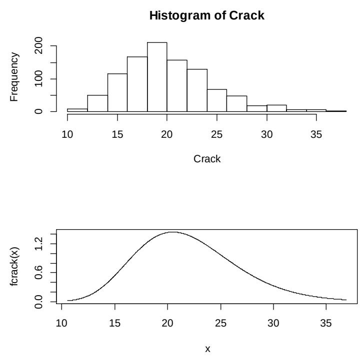
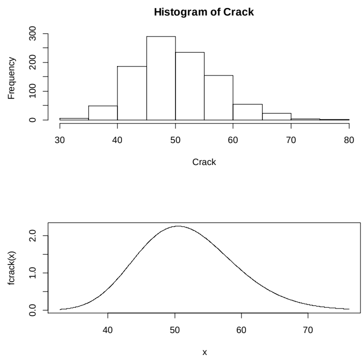


Figure 5 Histogram of Crack when λ = 50, θ = 1 and p = 0.2.



From Figure 1-5, we show the histogram of Crack when parameter θ and p are fixed but λ varies, we found that in each case, the histogram shape based on the random numbers that follow three-parameter Crack distribution generation by the Composition method and the shape of the density function graph of crack distribution were similar. Moreover, we found that the histogram shape and the shape of the density function graph of crack distribution will change the shape from skewed to the right to the bell shape when the value of parameter λ increase.

Figure 6 Histogram of Crack when λ = 2, θ = 1 and p = 0.4.

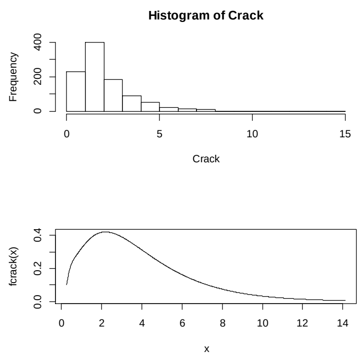


Figure 7 Histogram of Crack when λ = 2, θ = 5 and p = 0.4.

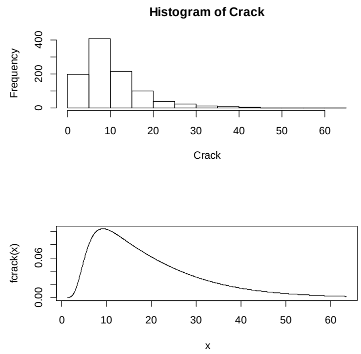


Figure 8 Histogram of Crack when λ = 2, θ = 10 and p = 0.4.

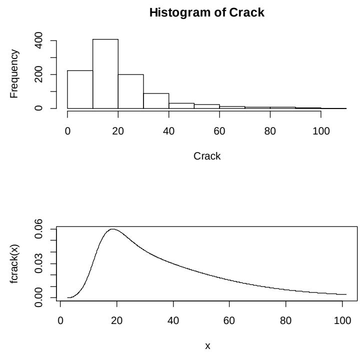
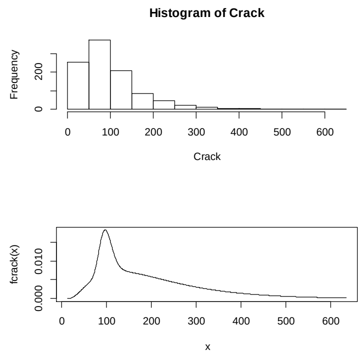


Figure 9 Histogram of Crack when λ = 2, θ = 50 and p = 0.4.



From Figure 6-9, we show the histogram of Crack when parameter λ and p are fixed but θ varies, we found that in each case, the histogram shape based on the random numbers that follow three-parameter Crack distribution generation by the Composition method and the shape of the density function graph of crack distribution were similar. If the value of parameter θ increasing, the graph shape will not change obviously.

Figure 10 Histogram of Crack when λ = 50, θ = 5 and p = 0.0.

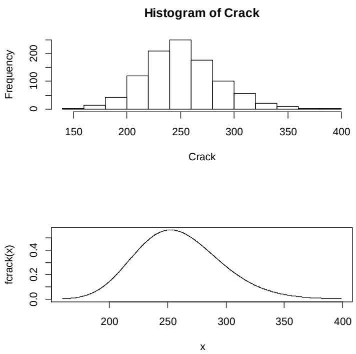


Figure 11 Histogram of Crack when λ = 50, θ = 5 and p = 0.2.

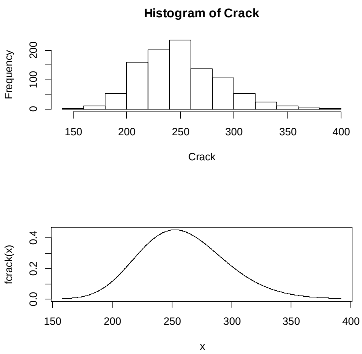


Figure 12 Histogram of Crack when λ = 50, θ = 5 and p = 0.4.

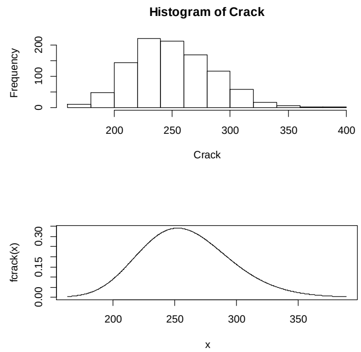


Figure 13 Histogram of Crack when λ = 50, θ = 5 and p = 0.6.

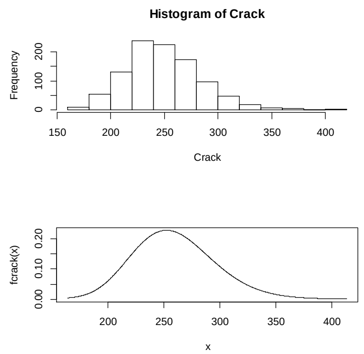


Figure 14 Histogram of Crack when λ = 50, θ = 5 and p = 0.8.

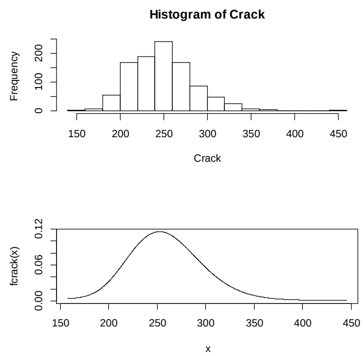
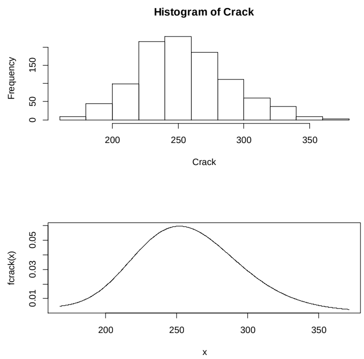


Figure 15 Histogram of Crack when λ = 50, θ = 5 and p = 1.0.



From Figure 10-15, we show the histogram of Crack when parameter λ and θ are fixed but p varies, the histogram shape based on the random numbers that follow three-parameter Crack distribution generation by the Composition method and the shape of the density function graph of crack distribution were similar. When the value of parameter p increase, the graph shape will not change obviously.

Discussions

When we consider all cases found that all of histogram shapes based on the random numbers that follow three-parameter Crack distribution generation by the Composition method and the shape of the density function graph of crack distribution were both similar. This shows that the Composition method can be used to generate the random numbers that follow Crack distribution for all values of parameters.

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