The Attractive Points and Strong Convergence Theorems for Generalized Hybrid Mappings (α, β) in CAT(κ) Space

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Abstract

This research paper proves the strong convergence theorems of the Ishikawa iterative process to the set of attractive points for generalized hybrid mappings (α, β) in CAT(κ) space with $\kappa > 0$.

Keywords: Attractive point; CAT(κ) space, Ishikawa iterative process

Introduction

Let X be a metric space and C be a nonempty subset of X. Let T be a mapping from C into X. A point $y \in X$ is called an *attractive point* of T if for each $x \in C$,

$$\rho(Tx, y) \leq \rho(x, y).$$

Let $A(T)$ be the set of all attractive points of T. Then

$$A(T) = \{z \in X : \rho(z, Ty) \leq \rho(z, y),\ \forall y \in C\}.$$

A mapping $T : C \to C$ is called *nonexpansive* if

$$\rho(Tx, Ty) \leq \rho(x, y), \ \ \forall x,\ y \in C.$$

In 2008, Kohsaka and Takahashi (Kohsaka et al., 2008), introduced the class of nonspreading mappings in a Hilbert space H. A mapping $T : C \to H$ is called *nonspreading* if

$$2\rho^2(Tx, Ty) \leq \rho^2(x, Ty) + \rho^2(Tx, y), \ \ \forall x, y \in C.$$

In 2010, Takahashi (Takahashi, 2010) and Kocourek, Takahashi and Yao (Kocourek et al., 2010), introduced wider classes of nonspreading mappings in Hilbert spaces as follows: A mapping $T:C\to H$ is called *hybrid* (Takahashi, 2010) if

$$3\rho^2(Tx,Ty)\le \rho^2(x,Ty)+\rho^2(Tx,y)+\rho^2(x,y),\quad \forall x,y\in C.$$

A mapping $T:C\to X$ is called (α,β)*-generalized hybrid* (Kocourek et al., 2010) if there exist $\alpha,\beta\in\mathbb{R}$ such that

$$\alpha\rho^2(Tx,Ty)+(1-\alpha)\rho^2(x,Ty)\le \beta\rho^2(Tx,y)+(1-\beta)\rho^2(x,y),\quad \forall x,y\in C.$$

We can see that $(0,1)$, $(2,1)$ and $\left(\frac{3}{2},\frac{1}{2}\right)$-generalized hybrid mappings are nonexpansive mappings, nonspreading mappings and hybrid mappings, respectively.

In 2015, Zheng (Zheng, 2015) proved the following result.

Theorem 1.1 (Zheng, 2015) Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and let $T:C\to C$ be an (α,β)-generalized hybrid mapping with $A(T)\neq\varnothing$ and satisfies Condition I. Suppose that the sequence $\{x_n\}$ is defined by the Ishikawa iteration

$$\begin{cases} y_n=\beta_n x_n+(1-\beta_n)Tx_n, \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)Ty, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that

$$\liminf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n)>0.$$

Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.

In this paper, motivated by Zheng (Zheng, 2015), we consider the concept of attractive points in CAT(κ) spaces with $\kappa>0$ and prove strong convergence theorems of the Ishikawa iteration for (α,β)-generalized hybrid mappings in such spaces.

Preliminaries

Let (X,ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, morebriefly, a *geodesic* from x to y) is a map $c:[0,l] \subseteq \mathbb{R} \to X$ such that $c(0)=x,\ c(l)=y$ and $\rho(c(p),\ c(q))=|\ p-q\ |$ for all $p,\ q \in [0,l]$. In particular, c is an isometry and $\rho(x,y)=l$. The image of a geodesic path is called a *geodesic segment*. This geodesic segment is denoted by $[x,y]$ if it is unique. We write that $w \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that $\rho(x,w)=(1-\alpha)\rho(x,y)$ and $\rho(y,w)=\alpha\rho(x,y)$.

In this case, we will write $w=\alpha x \oplus (1-\alpha)y$ for simplicity. Let D be a positive constant. A metric space (X,ρ) is said to be a *geodesic space* (D*-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be a *uniquely geodesic* (D*-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x,y \in X$ (for $x,y \in X$ with $\rho(x,y)<D$).

For a real number κ, let M_κ^2 denote the following metric spaces:

(i) if $\kappa=0$, then M_κ^2 is the Euclidean space $\mathbb{R}^2$;

(ii) if $\kappa>0$, then M_κ^2 is obtained from the spherical space $\mathbb{S}^2$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$;

(iii) if $\kappa<0$, then M_κ^2 is obtained from the hyperbolic space $\mathbb{H}^2$ by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

The diameter of M_κ^2 is denoted by D_κ. Thus $D_\kappa=\infty$ if $\kappa \le 0$ and $D_\kappa=\pi/\sqrt{\kappa}$ if $\kappa<0$. A *geodesic triangle* $\Delta(x,y,z)$ in a geodesic space (X,ρ) consists of three points x,y,z in X (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ).

A *comparison triangle* for a geodesic triangle $\Delta(x,y,z)$ in (X,ρ) is a triangle $\overline{\Delta}(\overline{x},\ \overline{y},\ \overline{z})$ in M_κ^2 such that $\rho(x,y)=\rho_{M_\kappa^2}(\overline{x},\overline{y})$, $\rho(y,z)=\rho_{M_\kappa^2}(\overline{y},\overline{z})$ and $\rho(z,x)=\rho_{M_\kappa^2}(\overline{z},\overline{x})$. If $\kappa>0$, then such a triangle exists whenever $\rho(x,y)+\rho(y,z)+\rho(z,x)<2D_\kappa$, where $D_\kappa=\pi/\sqrt{\kappa}$. A point $\overline{p} \in [\overline{x},\overline{y}]$ is called a *comparison point* for $p \in [x,y]$ if $\rho(x,p)=\rho_{M_\kappa^2}(\overline{x},\overline{p})$.

A geodesic triangle $\Delta(x,y,z)$ in X is said to satisfy the *CAT(κ) inequality* if for any $p,\, q \in \Delta(x,y,z)$ and for their comparison points $\overline{p},\, \overline{q} \in \overline{\Delta}(\overline{x},\overline{y},\overline{z}),$ we have $\rho(p,q) \le \rho_{M_\kappa^2}(\overline{p},\overline{q}).$ Given $\kappa > 0,$ a metric space X is called a *CAT(κ) space* if X is D_κ-geodesic and all geodesic triangles in X of perimeter less than $2D_\kappa$ satisfy the CAT(κ) inequality. For example, the n-dimensional unit sphere $\mathbb{S}^n$ is a CAT(1) space.

Lemma 2. 3 (Ohta, 2007) Let $\kappa > 0$ and (X,ρ) be a CAT(κ) space with $\operatorname{diam}(X) \le \dfrac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Then for any three points $x,\ y,\ z \in X$ and $\alpha \in [0,1],$ we have

$$\rho^2\left(x,(1-\alpha)y \oplus \alpha z\right) \le (1-\alpha)\rho^2(x,y) + \alpha\rho^2(x,z) - \frac{K}{2}\alpha(1-\alpha)\rho^2(y,z),$$

where $K = (\pi - 2\varepsilon)\tan(\varepsilon).$

Lemma 2.4 (Bridson and Haeiger, 1999) Let $\kappa > 0$ and (X,ρ) be a complete CAT(κ) space with

$\operatorname{diam}(X) \le \dfrac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Then

$$\rho(x,\alpha y \oplus (1-\alpha)z) \le \alpha\rho(x,y) + (1-\alpha)\rho(x,z),$$

for all $x,y,z \in X$ and $\alpha \in [0,1]$.

Main Results

We begin this section by proving the following lemmas.

Lemma 3.1 Let $\kappa > 0$ and (X,ρ) be a complete CAT(κ) space with $\operatorname{diam}(X) \le \dfrac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let C be a nonempty closed convex subset of X. Let $T : C \to C$ be an (α,β)-generalized hybrid mapping with $A(T) \ne \varnothing.$ Define a sequence $\{x_n\}$ in C by

$$\begin{cases} y_n = \beta_n x_n \oplus (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Ty_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. Then $\lim_{n\to\infty}\rho(x_n,z)$ exists for each $z \in A(T).$

Proof. Let $z \in A(T)$. Then

$$\begin{aligned}\rho(x_{n+1}, z) &= \rho(\alpha_n x_n \oplus (1-\alpha_n)Ty_n, z)\\ &\le \alpha_n \rho(x_n, z) + (1-\alpha_n)\rho(Ty_n, z)\\ &\le \alpha_n \rho(x_n, z) + (1-\alpha_n)\rho(y_n, z)\\ &\le \alpha_n \rho(x_n, z) + (1-\alpha_n)\rho(\beta_n x_n \oplus (1-\beta_n)Tx_n, z)\\ &\le (\alpha_n + (1-\alpha_n)(\beta_n + (1-\beta_n)))\rho(x_n, z)\\ &\le \rho(x_n, z).\end{aligned}$$

This implies that $\{\rho(x_n, z)\}$ is bounded below and nonincreasing for all $z \in A(T)$. Hence $\lim_{n\to\infty} \rho(x_n, z)$ exists.

Lemma 3.2 Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with $\mathrm{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X and $T : C \to C$ be an (α, β)-generalized hybrid mapping with $A(T) \ne \emptyset$. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} y_n = \beta_n x_n \oplus (1-\beta_n)Tx_n,\\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Ty_n,\end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that

$$\liminf_{n\to\infty} \beta_n (1-\alpha_n)(1-\beta_n) > 0.$$

Then $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0.$

Proof. Let $z \in A(T)$. Then by Lemma 3.1, we have $\lim_{n\to\infty} \rho(x_n, z)$ exists. From Lemma 2.3, we obtain

$$\begin{aligned}\rho^2(y_n, z) &= \rho^2(\beta_n x_n \oplus (1-\beta_n)Tx_n, z)\\ &\le \beta_n \rho^2(x_n, z) + (1-\beta_n)\rho^2(Tx_n, z) - \frac{K}{2}\beta_n(1-\beta_n)\rho^2(x_n, Tx_n)\\ &\le \beta_n \rho^2(x_n, z) + (1-\beta_n)\rho^2(x_n, z) - \frac{K}{2}\beta_n(1-\beta_n)\rho^2(x_n, Tx_n)\\ &\le \rho^2(x_n, z) - \frac{K}{2}\beta_n(1-\beta_n)\rho^2(x_n, Tx_n)\end{aligned}$$

and

$$\begin{aligned}\rho^2(x_{n+1},z) &= \rho^2(\alpha_n x_n \oplus (1-\alpha_n)Ty_n, z)\\ &\le \alpha_n\rho^2(x_n,z)+(1-\alpha_n)\rho^2(Ty_n,z)-\frac{K}{2}\alpha_n(1-\alpha_n)\rho^2(x_n,Ty_n)\\ &\le \alpha_n\rho^2(x_n,z)+(1-\alpha_n)\rho^2(Ty_n,z)\\ &\le \alpha_n\rho^2(x_n,z)+(1-\alpha_n)\rho^2(y_n,z)\\ &\le \alpha_n\rho^2(x_n,z)+(1-\alpha_n)\left(\rho^2(x_n,z)-\frac{K}{2}\beta_n(1-\beta_n)\rho^2(x_n,Tx_n)\right)\\ &\le \alpha_n\rho^2(x_n,z)+(1-\alpha_n)\rho^2(x_n,z)-\frac{K}{2}(1-\alpha_n)\beta_n(1-\beta_n)\rho^2(x_n,Tx_n).\end{aligned}$$

Hence

$$\frac{K}{2}(1-\alpha_n)\beta_n(1-\beta_n)\rho^2(x_n,Tx_n)\le \rho^2(x_n,z)-\rho^2(x_{n+1},z).$$

By taking $n\to\infty$ in the above inequality, we obtain that $\lim_{n\to\infty}\rho(x_n,Tx_n)=0.$

Strong Convergent Theorems

Let C be a nonempty closed convex subset of a metric space X. A mapping $T:C\to C$ is said to satisfy *Condition I* (Senter et al., 1974) if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0,\ f(r)>0$ for $r\in(0,\infty)$ such that $\rho(x,Tx)\ge f\left(\rho(x,A(T))\right)$ for all $x\in C,$ where $\rho(x,A(T))=\inf\{\rho(x,y): y\in A(T)\}.$

Theorem 4.1 Let $\kappa>0$ and (X,ρ) be a complete CAT(κ) space with $\mathrm{diam(X)}\le\frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon\in(0,\pi/2)$. Let C be a nonempty closed convex subset of X. Let $T:C\to C$ be an (α,β)-generalized hybrid mapping with $A(T)\neq\varnothing$ and satisfying Condition I. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} y_n=\beta_n x_n\oplus(1-\beta_n)Tx_n,\\ x_{n+1}=\alpha_n x_n\oplus(1-\alpha_n)Ty_n,\end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that

$$\liminf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n)>0.$$

Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.

Proof. Let $p \in A(T)$. It follows from Lemma 3.1 that the sequence $\{x_n\}$ is bounded and

$$\rho(x_{n+1}, p) \le \rho(x_n, p) \text{ for each } p \in A(T). \tag{4.1}$$

By Lemma 3.2, we have

$$\lim_{n\to\infty} \rho(x_n, Tx_n) = 0. \tag{4.2}$$

By condition I, we obtain that $\lim_{n\to\infty} f\left(\rho\left(x_n, A(T)\right)\right) = 0$ and hence

$$\lim_{n\to\infty} \rho(x_n, A(T)) = 0. \tag{4.3}$$

Next we show that the sequence $\{x_n\}$ is a Cauchy sequence of C. In fact, for any $n, m \in \mathbb{N}$
without loss of generality, we may set $m > n$. Then $\rho(x_m, p) \le \rho(x_n, p)$ for each $p \in A(T)$ by (4.1), and so

$$\rho(x_n, x_m) \le \rho(x_n, p) + \rho(p, x_m) \le 2\rho(x_n, p). \tag{4.4}$$

Since p is arbitrary, then we may take the infimum for p in above,

$$\rho(x_n, x_m) \le 2\inf\{\rho(x_n, p) : p \in A(T)\} = 2\rho(x_n, A(T)).$$

From (4.3), it follows that $\lim_{n\to\infty} \rho(x_n, x_m) = 0,$ which means that $\{x_n\}$ is a Cauchy sequence. So there exists $z \in C$ such that

$$\lim_{n\to\infty} \rho(x_n, z) = 0.$$

By (4.2), we have

$$\lim_{n\to\infty} \rho(Tx_n, z) = 0.$$

Now, we prove $z \in A(T)$. In fact, it follows from the definition of (α, β)-generalized hybrid mapping that for all $x \in C,$ we get

$$\alpha\rho^2(Tx_n, Tx) + (1-\alpha)\rho^2(x_n, Tx) \le \beta\rho^2(Tx_n, x) + (1-\beta)\rho^2(x_n, x). \tag{4.5}$$

Let $n \to \infty$ in (4.5). Then

$$\alpha\rho^2(z, Tx) + (1-\alpha)\rho^2(z, Tx) \le \beta\rho^2(z, x) + (1-\beta)\rho^2(z, x),$$

and hence

$$\rho(z, Tx) \le \rho(z, x)$$

for all $x \in C$. So $z \in A(T)$ and $\lim_{n\to\infty} \rho(x_n, z) = 0.$ The proof is completed.

A mapping $T : C \to C$ is said to be *demicompact* (Petryshyn, 1966) provided whenever a sequence $\{x_n\} \subset C$ is bounded and $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0,$ then there is a subsequence $\{x_{n_j}\}$ which strongly converges.

As a consequence of Theorem 4.1, we obtain the following.

Corollary 4.2 Let (X,ρ) be a complete CAT(0) space and C be a nonempty bounded closed convex subset of X. Let $T:C\to C$ be an (α,β)-generalized hybrid mapping with $A(T)\neq\varnothing$ and satisfying Condition I. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} y_n=\beta_n x_n\oplus(1-\beta_n)Tx_n, \\ x_{n+1}=\alpha_n x_n\oplus(1-\alpha_n)Ty_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that

$$\liminf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n)>0.$$

Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.

Theorem 4.3 Let $\kappa>0$ and (X,ρ) be a complete CAT(κ) space with $\mathrm{diam(X)}\leq\frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon\in(0,\pi/2)$. Let C be a nonempty closed convex subset of X. Let $T:C\to C$ be an (α,β)-generalized hybrid and demicompact with $A(T)\neq\varnothing$. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} y_n=\beta_n x_n\oplus(1-\beta_n)Tx_n, \\ x_{n+1}=\alpha_n x_n\oplus(1-\alpha_n)Ty_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that

$$\liminf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n)>0.$$

Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.

Proof. By Lemma 3.2, we have

$$\lim_{n\to\infty}\rho(x_n,Tx_n)=0. \tag{4.6}$$

By the demicompactness of T, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z\in C$ such that

$$\lim_{j\to\infty}\rho(x_{n_j},z)=0. \tag{4.7}$$

By (4.6), we get that $\lim_{j\to\infty}\rho(Tx_{n_j},z)=0$. It follows from the definition of (α,β)-generalized hybrid mapping that for all $x\in C$,

$$\alpha\rho^2(Tx_{n_j},Tx)+(1-\alpha)\rho^2(x_{n_j},Tx)\le \beta\rho^2(Tx_{n_j},x)+(1-\beta)\rho^2(x_{n_j},x). \quad (4.8)$$

Let $j\to\infty$ in (4.8). Then

$$\alpha\rho^2(z,Tx)+(1-\alpha)\rho^2(z,Tx)\le \beta\rho^2(z,x)+(1-\beta)\rho^2(z,x)$$

and hence

$$\rho(z,Tx)\le\rho(z,x)$$

for all $x\in C$. So $z\in A(T)$. Since $\lim_{n\to\infty}\rho(x_n,z)$ exists for each $z\in A(T)$ by Lemma 3.1, then we have $\lim_{n\to\infty}\rho(x_n,z)=0$. The proof is completed.

By Theorem 4.3, we have the following corollary.

Corollary 4.4 Let (X,ρ) be a complete CAT(0) space and C be a nonempty bounded closed convex subset of X. Let $T:C\to C$ be an (α,β)-generalized hybrid and demicompact with $A(T)\ne\emptyset$. Define a sequence $\{x_n\}$ in C by

$$\begin{cases} y_n=\beta_n x_n\oplus(1-\beta_n)Tx_n, \\ x_{n+1}=\alpha_n x_n\oplus(1-\alpha_n)Ty_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that

$$\liminf_{n\to\infty}\beta_n(1-\alpha_n)(1-\beta_n)>0.$$

Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.

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