Existence of Coexisting Between 5-cycle and Equilibrium Point on Piecewise Linear Map

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Abstract

A piecewise linear system of difference equations is one of the piecewise systems that has special characters like coexisting attracting sets. In this article, we also exhibit the coexisting attractors of 5-cycles and equilibrium point. We use some direct iterative calculations and an inductive statement to explain all behaviors of solutions belonging to the system with initial condition belonging to negative y-axis. We also found intervals of initial conditions that solutions become 5-cycles and equilibrium point.

Keywords: 5-cycle, Equilibrium point, Piecewise linear system, Difference equation

Introduction

The Lozi map (Lozi, 1978) and the Gingerbread-man map considered in (Devaney, 1984; Aharonov, Devaney & Ellas, 1987) are well-known two-dimensional piecewise smooth maps. Several families of piecewise linear map (a simplest case of piecewise smooth map) have been studied by many authors, see (Grove & Ladas, 2005; Tikjha, Lenbury & Lapierre, 2010; Tikjha, Lapierre & Sitthiwirattham, 2017) most of results dealing with classification possible families having a global attracting fixed point, suitable for applications in biology (Cannings, Hoppensteadt & Segel, 2005; Cull, 2006). However, two-dimensional piecewise linear map systems rarely succeed in such a simple dynamical behavior. Referring to the systems of the piecewise smooth maps, there are several articles showing coexisting of attracting sets (Simpson, 2010; Zhusubaliyev & Mosekilde, 2006; Zhusubaliyev, Mosekilde & Banerjee, 2008). In the case of coincidence of attracting sets coexist may lead to a fundamental source of uncertainty with respect to the applied meaning of the map (given a certain initial condition to which attracting set will converge the trajectory?). Both Lozi map and Gingerbread-man map are maps with absolute value which are non-differentiable at the cusp point. So stability theorems cannot apply to both maps. Ladas posted an open problem about the generalized Lozi map and Gingerbread-man maps to a system that were mentioned in (Grove, Lapierre & Tikjha, 2012):

$$x_{n+1} = |x_n| + ay_n + b, y_{n+1} = x_n + c|y_n| + d, n = 0,1,2,\dots \quad (1)$$

where parameters a, b, c and d are in $\{-1, 0, 1\}$ and the initial condition $(x_0, y_0) \in R^2$. There are many authors investigating the open problem, e.g., Tikjha, Lenbury & Lapierre (2010). They investigated special case of the system (1), when $a = b = c = -1$ and $d = 0$ and found that there are three attractors, 5-cycles and equilibrium point, and every solution is eventually either 5-cycles or equilibrium point. Grove, Lapierre & Tikjha (2012) found that every solution of a special case of the system (1), when $a = b = -1, c = 1$ and $d = 0$ is eventually prime period 3 solutions except for the unique equilibrium solution. Our ultimate goal to make generalized on parameter b of special case of the system (1):

$$x_{n+1} = |x_n| - y_n + b, y_{n+1} = x_n + |y_n| + 1, n = 0,1,2,\dots \quad (2)$$

where b is any real number. Krinket & Tikjha (2015) study a special case of the system (2), when $b = -1$, she found that solution is eventually prime period 4 with specific initial condition on y-axis. Tikjha & Piasu (2020) study the solution of special case of the system (2), when $b = -3$, with initial condition being a specific region in first quadrant is eventually equilibrium point or prime period 4. To predict global behavior of the system (2), we need to gather as much as results of special case to the system (2). So we continue to investigate special case of the system (2) for $b = -7$ as

$$x_{n+1} = |x_n| - y_n - 7, y_{n+1} = x_n - |y_n| + 1, n = 0,1,2,\dots \quad (3)$$

with initial condition on negative y-axis.

Objectives

The purpose of this article is to investigate the characters of solutions to piecewise linear system (3) and find the closed form of the solutions.

Materials and Methods

The following terminologies (Grove & Ladas, 2005) are used in this article: A *system of difference equations of the first order* is a system of the form

$$x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n), n = 0,1,2,\dots \quad (4)$$

where f and g are continuous functions which map R^2 into R. A *solution* of the system of (4) is a sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ which satisfies the system for all $n \ge 0$. If we prescribe an *initial condition* (x_0, y_0) in R^2 then $(x_1, y_1) = (f(x_0, y_0), g(x_0, y_0))$, $(x_2, y_2) = (f(x_1, y_1), g(x_1, y_1))$, $\cdots$ and so the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of the system of (4) exists for all $n \ge 0$ and is uniquely determined by the initial condition (x_0, y_0). A solution of the system of difference equations (4) which is constant for all $n \ge 0$ is called an *equilibrium solution*. If $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \ge 0$ is an equilibrium solution of the system (3), then $(\bar{x}, \bar{y})$ is called an *equilibrium point* of the system (4). A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called *eventually periodic with prime period* p or *eventually prime period* p *solution* if there exists an integer $N > 0$ and p is the smallest positive integer such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is periodic with period p; that is, $(x_{n+p}, y_{n+p}) = (x_n, y_n)$ for all $n \ge N$. The *5* consecutive point of the periodic solution is called a *5-cycle* of the system (3). We denote

$$\begin{pmatrix} a, & b \\ c, & d \\ e, & f \\ g, & h \\ i, & j \end{pmatrix}$$

as 5-cycle which consists of 5 consecutive points: when $(a,b), (c,d), (e,f), (g,h)$ and (i,j) in xy plane. It is worth noting that solution is eventually periodic with period p when *orbit* (forward iterations) contains a member of the cycle. An *attracting set* $A \subseteq I$ of a map f is a closed invariant set for which a neighborhood $U(A)$ exist such that $f(U(A)) \subseteq U(A)$ and $\bigcap_{i=0}^{\infty} f^i(U(A)) = A$. We will show that every solution of the system (3) with negative y-axis initial condition is eventually either equilibrium point or 5-cycle by finding pattern of solutions and then verifying that the closed form is true by using mathematical induction.

Results

By solving the system of equations: $\bar{x} = |\bar{x}| - \bar{y} - 7, \bar{y} = \bar{x} - |\bar{y}| + 1$, we obtain a unique equilibrium point (-1,-5). By brute force calculations, we have two 5-cycles as:

$P_{5.1} = \begin{pmatrix} -5 & -7 \\ 5 & -11 \\ 9 & -5 \\ 7 & 5 \\ -5 & 3 \end{pmatrix}$ and $P_{5.2} = \begin{pmatrix} 15/7 & -57/7 \\ 23/7 & -35/7 \\ 9/7 & -5/7 \\ -35/7 & 11/7 \\ -25/7 & -39/7 \end{pmatrix}$. We will show that every solution with specific initial condition $(x_0, y_0) \in L = \{(x,y) \in R^2 \mid x = 0, y < 0\}$ is eventually either 5-cycle or equilibrium point as following result.

Theorem Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a solution of the system (3) with the initial condition (x_0, y_0) is in L. Then the solution is eventually either 5-cycle or equilibrium point (-1,-5).

Proof. By the assumption (x_0, y_0) in L, we have $x_0 = 0$ and $y_0 < 0$. Then $x_1 = |x_0| - y_0 - 7 = |0| - y_0 - 7 = -y_0 - 7$, $y_1 = x_0 - |y_0| + 1 = 0 + y_0 + 1 = y_0 + 1$. To calculate the next iteration, we need to know that x_1, y_1 are negative or nonnegative. By assuming initial condition y_0 in proper interval, we could summarize the following results. If $y_0 \in [-7,-1)$, $y_0 \in [-1,-1/2)$, $y_0 \in [-1/4, 0)$, $y_0 \in [-1/2, -1/4)$, $y_0 \in [-15/2, -7)$ and $y_0 \in [-17/2, -15/2)$ then the solutions are equilibrium point in 2,4,7,6,4,5 iterations respectively. If $y_0 \in (-\infty, -19/2]$ then the solution is eventually 5-cycle in 9 iterations. The remain interval is $(-19/2, -17/2)$. Now we suppose $y_0 \in (-19/2, -17/2)$. By direct calculations, we obtain $x_3 = -2y_0 - 17 > 0$ and $y_3 = -2y_0 - 19 < 0$. We shall use an inductive statement to show that solutions with initial condition $y_0 \in (-19/2, -17/2)$ are eventually either 5-cycle or equilibrium point. The following sequences:

$$u_n = \frac{9 - 64 \cdot 2^{3n-2}}{7 \cdot 2^{3n-2}}, a_n = \frac{11 - 64 \cdot 2^{3n-1}}{7 \cdot 2^{3n-1}}, l_n = \frac{-5 - 64 \cdot 2^{3n-2}}{7 \cdot 2^{3n-2}}, b_n = \frac{15 - 64 \cdot 2^{3n}}{7 \cdot 2^{3n}},$$

$c_n = \dfrac{23 - 64 \cdot 2^{3n+1}}{7 \cdot 2^{3n+1}}$, and $\delta_n = \dfrac{11 - 64 \cdot 2^{3n-1}}{7}$ are used in the inductive statement.

Let $P(n)$ be the following statement:

" for $x_0 \in (l_n, u_n)$, $x_{5n-1} = -5$, $y_{5n-1} = -2^{3n-1} y_0 + \delta_n$;

if $y_0 \in [a_n, u_n)$ then $y_{5n-1} \le 0$ and so

$$x_{5n} = 2^{3n-1}y_0 - (\delta_n + 2) < 0, \qquad y_{5n} = -2^{3n-1}y_0 + (\delta_n - 4) < 0;$$
$$x_{5n+1} = -1, \qquad y_{5n+1} = -5;$$

if $y_0 \in (l_{\mathrm{n}}, a_n)$ then $y_{5n-1} > 0$ and so

$$x_{5n} = 2^{3n-1}y_0 - (\delta_n + 2) < 0, \qquad y_{5n} = 2^{3n-1}y_0 - (\delta_n + 4) < 0;$$
$$x_{5n+1} = -2^{3n}y_0 + (2\delta_n - 1), \qquad y_{5n+1} = 2^{3n}y_0 - (2\delta_n + 5) < 0;$$

if $y_0 \in [b_{\mathrm{n}}, a_n)$ then $x_{5n+1} \le 0$ and so

$$x_{5n+2} = -1, \qquad y_{5n+2} = -5;$$

if $y_0 \in (l_{\mathrm{n}}, b_n)$ then $x_{5n+1} > 0$ and so

$$x_{5n+2} = -2^{3n+1}y_0 + (4\delta_n - 3), \qquad y_{5n+2} = -5;$$

if $y_0 \in [c_{\mathrm{n}}, b_n)$ then $x_{5n+2} \le 0$ and so

$$x_{5n+3} = 2^{3n+1}y_0 - (4\delta_n - 1) < 0, \qquad y_{5n+3} = -2^{3n+1}y_0 + (4\delta_n - 7) < 0;$$
$$x_{5n+4} = -1, \qquad y_{5n+4} = -5;$$

if $y_0 \in (l_{\mathrm{n}}, c_n)$ then $x_{5n+2} > 0$ and so

$$x_{5n+3} = -2^{3n+1}y_0 + (4\delta_n - 5), \qquad y_{5n+3} = -2^{3n+1}y_0 + (4\delta_n - 7);$$

if $y_0 \in (l_{\mathrm{n}}, l_{n+1}]$ then $x_{5n+3} > 0$ and $y_{5n+3} \ge 0$ and so

$$x_{5n+4} = -5, \qquad y_{5n+4} = 3;$$

If $y_0 \in [u_{\mathrm{n+1}}, c_n)$ then $x_{5n+3} \le 0$ and $y_{5n+3} < 0$ and so

$$x_{5n+4} = 2^{3n+2}y_0 - (8\delta_n - 5) < 0, \qquad y_{5n+4} = -2^{3n+2}y_0 + (8\delta_n - 11) < 0;$$
$$x_{5n+5} = -1, \qquad y_{5n+5} = -5;$$

if $y_0 \in (l_{n+1}, u_{\mathrm{n+1}})$ then $x_{5n+3} > 0$ and $y_{5n+3} < 0$”.

Firstly, we shall show that $P(1)$ is true. We let $y_0 \in (l_1, u_1) = (-19/2, -17/2)$ and we know that $x_3 = -2y_0 - 17 > 0$ and $y_3 = -2y_0 - 19 < 0$. Then $x_4 = -5, y_4 = -4y_0 - 35 = -2^{3(1)-1}y_0 + \delta_1$. If $y_0 \in [a_1, u_1) = [-35/4, -17/2)$ then $y_4 = -4y_0 - 35 \le 0$. Thus $x_5 = 4y_0 + 33 = 2^{3(1)-1}y_0 - (\delta_1 + 2) < 0$ and $y_5 = -4y_0 - 39 = -2^{3(1)-1}y_0 + (\delta_1 - 4) < 0$. So $(x_6, y_6) = (-1, -5)$. If $y_0 \in (l_1, a_1) = (-19/2, -35/4)$ then $y_4 = -4y_0 - 35 > 0$. So $x_5 = 4y_0 + 33 = 2^{3(1)-1}y_0 - (\delta_1 + 2) < 0$ and $y_5 = 4y_0 + 31 = 2^{3(1)-1}y_0 - (\delta_1 + 4) < 0$. We have $x_6 = -8y_0 - 71 = -2^{3(1)}y_0 + (2\delta_1 - 1)$ and $y_6 = 8y_0 + 65 = 2^{3(1)}y_0 - (2\delta_1 + 5) < 0$.

If $y_0 \in [b_1, a_1) = [-71/8, -35/4)$ then $x_6 = -8y_0 - 71 \le 0$. So $(x_7, y_7) = (-1, -5)$.
If $y_0 \in (l_1, b_1) = [-19/2, -71/8)$ then $x_6 = -8y_0 - 71 > 0$.
So $x_7 = -16y_0 - 143 = -2^{3(1)+1} y_0 + (4\delta_1 - 3)$ and $y_7 = -5 < 0$.
If $y_0 \in [c_1, b_1) = [-143/16, -71/8)$ then $x_7 = -16y_0 - 143 \le 0$. Thus
$x_8 = 16y_0 + 141 = 2^{3(1)+1} y_0 - (4\delta_1 - 1)$ and $y_8 = -16y_0 - 147 = -2^{3(1)+1} + (4\delta_1 - 7) < 0$.
We have $(x_9, y_9) = (-1, -5)$. If $y_0 \in (l_1, c_1) = (-19/2, -143/16)$
then $x_7 = -16y_0 - 143 > 0$. Thus $x_8 = -16y_0 - 145 = -2^{3(1)+1} y_0 + (4\delta_1 - 5)$
and $y_8 = -16y_0 - 147 = -2^{3(1)+1} + (4\delta_1 - 7)$. If $y_0 \in (l_1, l_2] = (-19/2, -143/16]$ then
$x_8 = -16y_0 - 145 > 0$ and $y_8 = -16y_0 - 147 \ge 0$. So $(x_9, y_9) = (-5, 3)$. If
$y_0 \in [u_2, c_1) = [-145/16, -143/16)$ then $x_8 = -16y_0 - 145 \le 0$ and
$y_8 = -16y_0 - 147 < 0$. Thus $x_9 = 32y_0 + 285 = 2^{3(1)+2} y_0 - (8\delta_1 - 5) < 0$
and $y_9 = -32y_0 - 291 = -2^{3(1)+2} + (8\delta_1 - 11) < 0$. We have $(x_{10}, y_{10}) = (-1, -5)$.
If $y_0 \in (l_2, u_2) = (-147/16, -145/16)$ then $x_8 = -16y_0 - 145 > 0$ and
$y_8 = -16y_0 - 147 < 0$. Hence $P(1)$ is true. Now we suppose further that $P(k)$ is true. We have $x_{5k+3} = -2^{3k+1} y_0 + (4\delta_k - 5) > 0$ and $y_{5k+3} = -2^{3k+1} y_0 + (4\delta_k - 7) < 0$ for

$$y_0 \in (l_{k+1}, u_{k+1}) = \left(\frac{-5 - 64 \cdot 2^{3k+1}}{7 \cdot 2^{3k+1}}, \frac{9 - 64 \cdot 2^{3k+1}}{7 \cdot 2^{3k+1}} \right)$$

Then $x_{5k+4} = |x_{5k+3}| - y_{5k+3} - 7 = -5$ and $y_{5k+4} = x_{5k+3} - |y_{5k+3}| + 1 = -2^{3k+2} y_0 + 8\delta_k - 11 = -2^{3k+2} y_0 + \delta_{k+1}$. We note that

$$8\delta_k - 11 = 8\left(\frac{11 - 64 \cdot 2^{3k-1}}{7} \right) - 11 = \frac{8(11) - 64 \cdot 2^{3k+2} - 11(7)}{7} = \frac{11 - 64 \cdot 2^{3k+2}}{7} = \delta_{k+1}.$$

If $y_0 \in [a_{k+1}, u_{k+1}) = \left[\frac{11 - 64 \cdot 2^{3k+2}}{7 \cdot 2^{3k+2}}, \frac{9 - 64 \cdot 2^{3k+1}}{7 \cdot 2^{3k+1}} \right)$ then $y_{5k+4} = -2^{3k+2} y_0 + \delta_{k+1} \le 0$. Thus $x_{5k+5} = 2^{3k+2} y_0 - (\delta_{k+1} + 2) < 0$ and $y_{5k+5} = -2^{3k+2} y_0 + (\delta_{k+1} - 4) < 0$. We determine the sign of x_{5k+5} and y_{5k+5}, negative sign, by substituting a_{k+1} and u_{k+1} into y_0 of linear functions x_{5k+5} and y_{5k+5}. From now on, we will determine the sign of solutions by using this method. Then $(x_{5k+6}, y_{5k+6}) = (-1, -5)$ as require.

If $y_0 \in (l_{k+1}, a_{k+1}) = \left(\frac{-5-64\cdot 2^{3k+1}}{7\cdot 2^{3k+1}}, \frac{11-64\cdot 2^{3k+2}}{7\cdot 2^{3k+2}} \right)$ then $y_{5k+4} = -2^{3k+2} y_0 + \delta_{k+1} > 0$.

Thus $x_{5k+5} = 2^{3k+2} y_0 - (\delta_{k+1} + 2) < 0$ and $y_{5k+5} = 2^{3k+2} y_0 - (\delta_{k+1} + 4) < 0$. The signs of x_{5k+5} and y_{5k+5} are also determine by the boundaries l_{k+1} and a_{k+1}. So

$x_{5k+6} = -2^{3k+3} y_0 + 2\delta_{k+1} - 1$ and $y_{5k+6} = 2^{3k+3} y_0 - 2\delta_{k+1} - 5 < 0$.

If $y_0 \in [b_{k+1}, a_{k+1}) = \left[\frac{15-64\cdot 2^{3k+3}}{7\cdot 2^{3k+3}}, \frac{11-64\cdot 2^{3k+2}}{7\cdot 2^{3k+2}} \right)$ then $x_{5k+6} \le 0$. We have

$(x_{5k+7}, y_{5k+7}) = (-1,-5)$ as require. If $y_0 \in (l_{k+1}, b_{k+1}) = \left(\frac{-5-64\cdot 2^{3k+1}}{7\cdot 2^{3k+1}}, \frac{15-64\cdot 2^{3k+3}}{7\cdot 2^{3k+3}} \right)$

then $x_{5k+6} > 0$. Thus we have $x_{5k+7} = -2^{3(k+1)+1} y_0 + 4\delta_{k+1} - 3$ and $y_{5k+7} = -5$.

If $y_0 \in [c_{k+1}, b_{k+1}) = \left[\frac{23-64\cdot 2^{3k+4}}{7\cdot 2^{3k+4}}, \frac{15-64\cdot 2^{3k+3}}{7\cdot 2^{3k+3}} \right)$ then $x_{5k+7} \le 0$. Then

$x_{5k+8} = 2^{3k+4} y_0 - 4\delta_{k+1} + 1 < 0$ and $y_{5k+8} = -2^{3k+4} y_0 + 4\delta_{k+1} - 7 < 0$. We have

$(x_{5k+9}, y_{5k+9}) = (-1,-5)$ as require. If $y_0 \in (l_{k+1}, c_{k+1}) = \left(\frac{-5-64\cdot 2^{3k+1}}{7\cdot 2^{3k+1}}, \frac{23-64\cdot 2^{3k+4}}{7\cdot 2^{3k+4}} \right)$

then $x_{5k+7} > 0$. $x_{5k+8} = -2^{3k+4} y_0 + 4\delta_{k+1} - 5$ and $y_{5k+8} = -2^{3k+4} y_0 + 4\delta_{k+1} - 7$.

If $y_0 \in (l_{k+1}, l_{k+2}) = \left(\frac{-5-64\cdot 2^{3k+1}}{7\cdot 2^{3k+1}}, \frac{-5-64\cdot 2^{3k+4}}{7\cdot 2^{3k+4}} \right)$ then $x_{5k+8} > 0, y_{5k+8} > 0$. Thus

$(x_{5k+9}, y_{5k+9}) = (-5,3)$ as require. If $y_0 \in [u_{k+2}, c_k) = \left[\frac{9-64\cdot 2^{3k+4}}{7\cdot 2^{3k+4}}, \frac{23-64\cdot 2^{3k+4}}{7\cdot 2^{3k+4}} \right)$

then $x_{5k+8} \le 0$ and $y_{5k+8} < 0$. Then $x_{5k+9} = 2^{3k+5} y_0 - 8\delta_{k+1} + 5 < 0$

and $y_{5k+9} = -2^{3k+5} y_0 + 8\delta_{k+1} - 11 < 0$. Then $(x_{5k+10}, y_{5k+10}) = (-1,-5)$ as require.

If $y_0 \in (l_{k+2}, u_{k+2}) = \left(\frac{-5-64\cdot 2^{3k+4}}{7\cdot 2^{3k+4}}, \frac{9-64\cdot 2^{3k+4}}{7\cdot 2^{3k+4}} \right)$ then $x_{5(k+1)+3} > 0$ and $y_{5(k+1)+3} < 0$.

Hence $P(k+1)$ is true. We use the induction to conclude that $P(n)$ is true for every positive integer n. By the inductive statement $P(n)$, we can conclude as follows:

1. If $y_0 \in [a_{\mathrm{n}}, u_n)$ for some *n*, then the solution is eventually equilibrium point.
2. If $y_0 \in [b_{\mathrm{n}}, a_n)$ for some *n*, then the solution is eventually equilibrium point.
3. If $y_0 \in [c_{\mathrm{n}}, b_n)$ for some *n*, then the solution is eventually equilibrium point.

4. If $y_0 \in (l_n, l_{n+1}]$ for some *n*, then the solution is eventually 5-cycle $P_{5.1}$.
5. If $y_0 \in [u_{n+1}, c_n)$ for some *n*, then the solution is eventually equilibrium point.

We note that sequence l_n is in the left side of its limit and sequences a_n, b_n, c_n, u_n are in the right side of their limit. We also note that the sequences a_n, b_n, c_n, u_n, l_n have the same limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} u_n = \lim_{n\to\infty} l_n = -\frac{64}{7}.$$

If we exactly choose the initial condition at $(x_0, y_0) = (0, -64/7)$, then after one iteration $(x_1, y_1) = (15/7, -57/7) \in P_{5.2}$. Therefore we can conclude that every solution of the system (3) with initial condition in L is eventually equilibrium point or 5-cycles. □

Discussion and Conclusion

The result showed the three coexisting attractors that are equilibrium point, $P_{5.1}$, and $P_{5.2}$ with specific initial condition on negative y-axis. We used direct iterative calculations to investigate behaviors of solution to the system (3) (the detail of proof in the beginning of Theorem can be found in the link, shorturl.at/dhEO7), and we found a pattern of solutions with initial condition $x_0 = 0$ and $y_0 \in (-19/2, -17/2)$. Then we formulate inductive statement to describe all behaviors of solutions. The attractors in this article is the same as Tikjha, Lenbury & Lapierre, (2010), only 5-cycles and equilibrium point in any initial condition in R^2, but this article only focus on initial condition on negative y-axis. It is possible to have other attracting sets. Moreover the family of the system (2) there exist the 4-cycle in Krinket & Tikjha (2015) and Tikjha & Piasu (2020). For further investigations are necessary to change initial condition into another region of R^2 which are left for future works.

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