$\mathcal{pq}$-Continuous Mappings and $\mathcal{pq}$-Homeomorphisms

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Abstract

The purpose of the present paper to continuous mappings, namely $\mathcal{pq}$-continuous mappings and $\mathcal{pq}$-homeomorphisms in bi-quasi generalized weak spaces, and we establish some of their properties.

Keywords: bi-quasi generalized weak space, $\mathcal{pq}$-continuous mapping, $\mathcal{pq}$-homeomorphism

1. Introduction

Continuity and homeomorphism of mapping is one of the core concept for researchers interesting to study in any spaces. Especially, a topological space which is early space for most them to study and investigate. While, generalization topological concepts are important way to development in modern mathematics research for introduce and study.

Császár (Császár, 1997) introduced weak structure as generalization of general topology, and introduced closure and interior, also introduced some type of $\mathcal{w}$-open set. Lugojan (Lugojan, 1982) introduced a generalization of topology as follow a sub collection G_X on subset of a nonempty set X is called under arbitrary union. Maki (Maki, 1996) studied minimal structure as follow, collections of subsets of X containing the empty set and X, with no other restriction.

One of the generalizations of generalized topologies has been introduced by Kim and Min (Kim & Min, 2013). It is called a σ-structure, which is a nonempty sub collection S of subsets of X satisfying: if, for $i \in I \neq \emptyset, u_i \in S$ implies $\bigcup_{i \in I} u_i \in S$. Ávila and Molina (Ávila & Molina, 2012) introduced $\mathcal{q}$-open set, $\mathcal{q}$-closed set, $\mathcal{q}$-subspace and studied them on generalized weak structure.

Recently, there were researchers from Rajabhat Mahasarakham University introduced a generalized of this space namely, quasi generalized weak space. and bi-quasi generalized weak space. They introduced and studied on different topics as following. Janrongkam and Pongman (Janrongkam & Pongman, 2019) introduced quasi generalized weak space and $\mathcal{q}$-continuity and $\mathcal{q}^*$-continuity respectively. Ratkanok (Ratkanok,2019) introduced $\mathcal{q}$-compact on quasi generalized weak spaces. Furthermore,Thongpan (Thongpan, 2019) introduced a generalized space namely, bi-quasi generalized weak structure. Bandorn(Bandorn, 2019) introduced and studied $\mathcal{pq}$-interior and $\mathcal{pq}$-closed sets while, Auscharaporn (Auscharaporn, 2019) introduced and studied $\mathcal{pq}$-derived and $\mathcal{pq}$-boundary sets on bi-quasi generalized weak space.

All of these causes, its leads to our works. The purpose of this research is study and investigate the properties of $\mathcal{pq}$- continuous mapping and $\mathcal{pq}$- homeomorphism on bi- quasi generalized weak structure.

2. Preliminaries

In this section, we shall present theoretical background of our work. There are two main idea from Bandorn (2019) and Thongpan (2019). They introduced the concept of bi-quasi generalized weak space and $\mathcal{pq}$-interior sets and $\mathcal{pq}$-closed respective. All result presented by them.

Definition 2.1 A quasi generalized weak structures (briefly, QGWS space) an nonempty set X, is a nonempty collection $\mathcal{q}$ of subset of X satisfying the property : $U \cap V \in \mathcal{q}$ for any U,V∈ $\mathcal{q}$. A quasi generalized weak space consists of two object : a nonempty set X and is denoted by (X, $\mathcal{q}$). Each member of $\mathcal{q}$ is called a $\mathcal{q}$- open set in X (briefly $\mathcal{q}$-open set) and the complement of $\mathcal{q}$-open set is said to be $\mathcal{q}$-closed set in X (briefly $\mathcal{q}$-open set)

Example 2.2 Let X $=\{1,2,3\}$ and $\mathcal{q} = \{\{1\},\{1,2\}\}$. Then $\mathcal{q}$ is QGWS on X. We can see that $\{1\},\{1,2\}$ and all $\mathcal{q}$-open sets thus $\{1\}^c = \{2,3\}$ and $\{1,2\}^c = \{3\}$ are all $\mathcal{q}$-open set.

Note 2.3 Every topology is a QGWS, but converse is not true

Definition 2.4 Let X be a nonempty set and $\mathcal{p}$, $\mathcal{q}$ be quasi generalized weak structures on X. A triple (X, $\mathcal{p}$, $\mathcal{q}$) is called a bi-quasi generalized weak space (briefly, BQGW space).

Definition 2.5 Let $(X, \mathcal{p}, \mathcal{q})$ be a bi-quasi generalized weak space and $A \subseteq X$. Then A is called $\mathcal{pq}$-closed if $c_{\mathcal{p}}$ $(c_{\mathcal{q}}$(A)) = A. The complement of $\mathcal{pq}$-closed set is called a $\mathcal{pq}$-open set.

Example 2.6 Let X = {a, b, c}. Define the bi-quasi generalized weak spaces $\mathcal{p}$ and $\mathcal{q}$ on X as follows: $\mathcal{p}$ = {{a}, {a, b}} and $\mathcal{q}$ = {{b}, {a, b}}. Then $c_{\mathcal{q}}$ ({c}) = ⋂{{c}, {a, c}} = {c}. Consider $c_{\mathcal{q}}$ ($c_{\mathcal{q}}$ ({c}) = c_p ({c}) = ⋂{{c}, {b, c}} = {c}. Therefore $c_{\mathcal{p}}$ ($c_{\mathcal{q}}$ ({c})) = {c}.Then {c} is $\mathcal{pq}$-closed and {a, b} is $\mathcal{pq}$-open since $\{c\}^c$ = {a, b}.

Example 2.7 Let X = {a, b, c}. Define the bi-quasi generalized weak spaces $\mathcal{p}$ and $\mathcal{q}$ on X as follows: $\mathcal{p}$ = {{a}, {a, b}} and $\mathcal{q}$ = {{b}, {a, b}}. Then {a} is not $\mathcal{pq}$-closed. Because $c_{\mathcal{q}}$ ({c}) = ⋂{{a, c}} = {a, c}, we obtain $c_{\mathcal{p}}$ ($c_{\mathcal{q}}$ ({a})) = $c_{\mathcal{p}}$ ({a, c}) =$\cap \{F: \{a,c\} \subset F\}$ =⋂∅ = X. Therefore $c_{\mathcal{p}}$ ($c_{\mathcal{q}}$ ({a})) ≠ {a}.

Remark 2.8 Let (X, $\mathcal{p}$, $\mathcal{q}$) be a bi-quasi generalized weak space and A be a subset of X. Then A is a $\mathcal{pq}$- open if and only if A^C is a $\mathcal{pq}$-closed.

Theorem 2.9 Let (X, p, q) be a bi-quasi generalized weak space and A be a subset of X. Then A is pq-closed if and only if $c_p(A) = A$ and $c_q(A) = A$.

Corollary 2.10 Let (X, p, q) be a bi-quasi generalized weak space and A be a subset of X. Then A is pq-closed if and only if $c_q(c_p(A)) = A$.

Theorem 2.11 Let (x, p, q) be a bi-quasi generalized weak space and I be an index set. If A_i is pq-closed for all i ∈ I, then $\cap_{i\in I} A_i$ is pq-closed.

Theorem 2.12 Let (X, p, q) be a bi-quasi generalized weak space and A and B be subsets of X. If A and B are pq-closed, then A∪B is pq-closed.

Theorem 2.13 Let (X, p, q) be a bi-quasi generalized weak space and A is a subset of X. Then A is pq-open if and only if $A = i_p(i_q(A))$.

Theorem 2.14 Let (X, p, q) be a bi-quasi generalized weak space and I be an index set. If A_i be pq-open for all i ∈ I, then $\cup_{i\in I}\in A_i$ is pq-open.

Theorem 2.15 Let (X, p, q) be a bi-quasi generalized weak space and A and B are subsets of X. If A and B are pq-open, then A ∩ B is pq-open.

Definition 2.16 Let (X, p, q) be a BQGW space and A ⊆ X. The pq -interior of A, denoted by i_{pq} (A), is defined by $i_{pq}(A) = \bigcup\{U : U$ is a pq-open set and $U \subseteq A\}$.

Note 2.17 Let (X, p, q) be a BQGW space and A ⊆ X. Then $x \in i_{pq}(A)$ if and only if there exists a pq-open set U such that $X \in U \subseteq A$.

Example 2.18 Consider a BQGW space (X, p, q) where X ={1, 2, 3}, p = {{3}, {2, 3}} and q = $\{\{2\},\{2,3\}\}$. Then $\{2,3\}$ and ∅ are all pq-open sets because $i_q(i_p(\{2, 3\}) = \{2, 3\}$. Then $i_{pq}(X)$ = ⋃{U: U is a pq-open set and $U \subseteq X\} = \bigcup\{\{2,3\},\emptyset\} = \{2,3\}$.

Theorem 2.19 Let (X, p, q) be a BQGW space and A, B ⊆ X. Then

(1) $i_{pq}(\emptyset) = \emptyset$,

(2) $i_{pq}(A) \subseteq A$,

(3) If $A \subseteq B$, then $i_{pq}(A) \subseteq i_{pq}(B)$,

(4) i_{pq} (A) is a pq-open set,

(5) i_{pq} (A) is the largest pq-open set contained in A,

(6) A is a pq-open set if and only if $A = i_{pq}(A)$,

(7) $i_{pq}(i_{pq}(A)) = i_{pq}(A)$, and

(8) $i_{pq}(A) \subseteq i_q(i_p(A))$.

Corollary 2.20 Let (X, p, q) be a BQGW space and A ⊆ X. Then A is a pq-open set if and only if for every

x ∈ A, there exists a pq-open set U such that x ∈ U ⊆ A.

Theorem 2.21 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space and A, B ⊆ X. The following properties hold:

(1) $i_{\mathscr{pq}}$ (A) ∪ $i_{\mathscr{pq}}$ (B) ⊆ $i_{\mathscr{pq}}$ (A ∪ B).

(2) $i_{\mathscr{pq}}$ (A ∩ B) = $i_{\mathscr{pq}}$ (A) ∩ $i_{\mathscr{pq}}$ (B)

Definition 2.22 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space and A ⊆ X. The $\mathscr{pq}$ -closure of A, denoted by $c_{\mathscr{pq}}$ (A), is defined by $c_{\mathscr{pq}}$ (A) = ⋂{F : F is a $\mathscr{pq}$-closed set and A ⊆ F}.

Note 2.23 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space and A ⊆ X. Then x ∈ $c_{\mathscr{pq}}$ (A) if and only if x ∈ F for every $\mathscr{pq}$-closed set F with A ⊆ F.

Example 2.24 Consider a BQGW space $(X, \mathscr{p}, \mathscr{q})$ where X = {1, 2, 3, 4}, $\mathscr{p}$ = {{1}, {1, 2}} and $\mathscr{q}$ = {{1}, {1, 2, 3}}. Then {2, 3, 4} and X are all $\mathscr{pq}$-closed sets because $c_{\mathscr{p}}$ ($c_{\mathscr{q}}$ ({2, 3, 4}) = {2, 3, 4}. Let A = {2, 3}. Then $c_{\mathscr{pq}}$ (A) = ⋂{F : F is a $\mathscr{pq}$-closed set and A ⊆ F} = ⋂{{2, 3, 4}, X} = {2, 3, 4}.

Theorem 2.25 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space on X and A ⊆ X. Then

(1) $c_{\mathscr{pq}}$ (X) = X,

(2) A ⊆ $c_{\mathscr{pq}}$ (A),

(3) If A ⊆ B, then $c_{\mathscr{pq}}$ (A) ⊆ $c_{\mathscr{pq}}$ (B),

(4) $c_{\mathscr{pq}}$ (A) is a $\mathscr{pq}$-closed set,

(5) $c_{\mathscr{pq}}$ (A) is the smallest $\mathscr{pq}$-closed set containing A,

(6) A is a $\mathscr{pq}$-closed set if and only if $c_{\mathscr{pq}}$ (A) = A,

(7) $c_{\mathscr{pq}}$ ($c_{\mathscr{pq}}$ (A)) = $c_{\mathscr{pq}}$ (A), and

(8) $c_{\mathscr{q}}$ ($c_{\mathscr{p}}$ (A)) ⊆ $c_{\mathscr{pq}}$ (A).

Theorem 2.26 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space and A ⊆ X. Then x ∈ $c_{\mathscr{pq}}$ (A) if and only if U ∩ A ≠ ∅ for every $\mathscr{pq}$-open set U with x ∈ U.

Corollary 2.27 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space and A ⊆ X. Then $c_{\mathscr{pq}}$ (A) = {x ∈ X : U ∩ A ≠ ∅ for all $\mathscr{pq}$-open set U with x ∈ U}.

Theorem 2.28 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space and A and B be subsets of X. Then

(1) $c_{\mathscr{pq}}$ (A) ∪ $c_{\mathscr{pq}}$ (B) = $c_{\mathscr{pq}}$ (A ∪ B) and

(2) $c_{\mathscr{pq}}$ (A ∩ B) ⊆ $c_{\mathscr{pq}}$ (A) ∩ $c_{\mathscr{pq}}$ (B).

Theorem 2.29 Let $(X, \mathscr{p}, \mathscr{q})$ be a BQGW space and A ⊆ X. Then

(1) $[i_{\mathscr{pq}} (A)]^C = c_{\mathscr{pq}} (A^C)$,

(2) $i_{\mathscr{pq}}$ (A) = $[C_{\mathscr{pq}} (A^C)]^C$, and

(3) $i_{\mathscr{pq}} (A^C) = [C_{\mathscr{pq}} (A)]^C$.

Definition 2.30 (Auscharaporn, 2019) Let $(X, \mathscr{p}, \mathscr{q})$ be a bi-quasi generalized weak space and A ⊆ X. The $\mathscr{pq}$-derived set of A ,denoted by $d_{\mathscr{pq}}$(A) , is defined by

$$d_{\mathscr{pq}}(A) = \{x \in X : U \cap (A - \{x\}) \neq \emptyset \text{ for all } \mathscr{pq}\text{- open set } U \text{ with } x \in U\}$$

Example 2.31 Let $(X, \mathscr{p}, \mathscr{q})$ be a bi-quasi generalized weak space , where X = $\{a, b, c\}$, $\mathscr{p}$ = $\{\{a\}, \{a, c\}\}$ and $\mathscr{q}$ = $\{\{c\}, \{a, c\}\}$. Let A = $\{a, b\}$. Then $d_{\mathscr{pq}}(\{a, c\}) = \{a, b\}$.

Theorem 2.32 (Auscharaporn, 2019) Let $(X, \mathscr{p}, \mathscr{q})$ be a bi-quasi generalized weak space and A and B be subset of X. Then

(1). If A ⊆ X., Then $d_{\mathscr{pq}}$(A) ⊆ $d_{\mathscr{pq}}$(B)

(2). $d_{\mathscr{pq}}$(A) ⊆ $c_{\mathscr{pq}}$(A)

(3). A is a $\mathscr{pq}$-closed set if and only if $d_{\mathscr{pq}}$(A) ⊆ A

(4). $c_{\mathscr{pq}}$(A) = A ∪ $d_{\mathscr{pq}}$(A)

3. Main Results

In this section, we shall introduce and study the $\mathscr{pq}$ - continuous mapping, $\mathscr{pq}$ - homeomorphism and presentation its important properties as the following:

3.1 $\mathscr{pq}$ –continuous mappings

Definition 3.1.1 Let $(X, \mathscr{p}, \mathscr{q})$ and $(Y, \mathscr{r}, \mathscr{s})$ be bi-quasi generalized weak spaces. Then a mapping f : $(X, \mathscr{p}, \mathscr{q}) \rightarrow (Y, \mathscr{r}, \mathscr{s})$ is called $\mathscr{pq}$ -continuous on X if the inverse image of every $\mathscr{rs}$ -open set in Y is $\mathscr{pq}$ -open in X.

Example 3.1.2 Let X = $\{1,2,3\}$, $\mathscr{p} = \{\{1,2\}\}$ = $\mathscr{q}$

Y = $\{a, b\}$, $\mathscr{r} = \{\{a\}\}$, $\mathscr{s} = \{\{a\}, \{a, b\}\}$

Define f : $(X, \mathscr{p}, \mathscr{q}) \rightarrow (Y, \mathscr{r}, \mathscr{s})$ by f(1) = a = f(2) , f(3) = b. We have $\{1,2\}$ is $\mathscr{pq}$ -open in X and $\{a\}$ is $\mathscr{rs}$ -open in Y, so $f^{-1}\{a\} = \{1,2\}$.Therefore f is $\mathscr{pq}$ -continuous mapping on X.

The following theorem characterizes $\mathscr{pq}$ -continuous mapping in terms of $\mathscr{pq}$ -closed sets.

Theorem 3.1.3 A mapping f : $(X, \mathscr{p}, \mathscr{q}) \rightarrow (Y, \mathscr{r}, \mathscr{s})$ is $\mathscr{pq}$ -continuous if and only if the inverse image of every $\mathscr{rs}$ -closed sets in Y is $\mathscr{pq}$ -closed in X.

Proof : Let f be $\mathscr{pq}$ -continuous and F be $\mathscr{rs}$ -closed in Y. That is Y-F is $\mathscr{rs}$ -open in Y. Since f is $\mathscr{pq}$ -continuous, f^{-1}(Y-F) is $\mathscr{pq}$ -open in X. That is, X- f^{-1}(F) is $\mathscr{pq}$ -open in X. Therefore, f^{-1}(F) is $\mathscr{pq}$ -closed in X. Thus, the inverse image of every $\mathscr{rs}$ -closed set in Y is $\mathscr{pq}$ -closed in X, if f is $\mathscr{pq}$ -continuous on X.

Conversely, let the inverse image of every $\mathscr{rs}$ –closed in Y be $\mathscr{pq}$ -closed set in X. Let G be $\mathscr{rs}$ -open in Y. Then Y-G is $\mathscr{rs}$ -closed in Y. Then f^{-1}(Y-G) is $\mathscr{pq}$ -closed in X. That is X- f^{-1}(G) is $\mathscr{pq}$ -closed in X.Therefore, f^{-1}(G) is $\mathscr{pq}$ -open in X. Thus, the inverse image of every $\mathscr{rs}$ -open set in Y is $\mathscr{pq}$ -open in X. That is f is $\mathscr{pq}$ -continuous on X.

In the following theorem, we establish a characterization of $\mathscr{pq}$ -continuous mapping in terms of $\mathscr{pq}$ -closure.

Theorem 3.1.4 A mapping $f:(X,p,q)\to(Y,r,s)$ is pq-continuous if and only if $f:(X,p,q \subseteq c_{rs}(f(A))$ for every subset A of X.

Proof : Let f be pq-continuous and $A\subseteq X$. Then $f(A)\subseteq Y$, by theorem 2.29 (4), we have $c_{rs}(f(A))$ is rs-closed in Y. Since f is pq-continuous, $f^{-1}(c_{rs}(f(A)))$ is pq-closed in X. Since $f(A)\subseteq c_{rs}(f(A))$, $A\subseteq f^{-1}(c_{rs}(f(A)))$. Thus $f^{-1}(c_{rs}(f(A)))$ is a pq-closed set containing A. But $c_{pq}(A)$ is the smallest pq-closed set containing A. Therefore $c_{pq}(A)\subseteq f^{-1}(c_{rs}(f(A)))$. That is $f\left(c_{pq}(A)\right)\subseteq c_{rs}(f(A))$.

Conversely, let $f\left(c_{pq}(A)\right)\subseteq c_{rs}(f(A))$ for every subset A of X. If F is rs-closed in Y, since $f^{-1}(F)\subseteq X$, $f\left(c_{pq}\left(f^{-1}(F)\right)\right)\subseteq c_{rs}(f(f^{-1}(F)))\subseteq c_{rs}(F)$. That is $c_{pq}\left(f^{-1}(F)\right)\subseteq f^{-1}(c_{rs}(F))=f^{-1}(F)$, since F is rs-closed. Thus $c_{pq}\left(f^{-1}(F)\right)\subseteq f^{-1}(F)$. But $f^{-1}(F)\subseteq c_{pq}\left(f^{-1}(F)\right)$. By theorem 2.29 (2), Therefore, $f^{-1}(F)$ is
pq-closed in X for every rs-closed set F in Y. That is f is pq-continuous.

Remark 3.1.5 If $f:(X,p,q)\to(Y,r,s)$ is pq-continuous, then $f(c_{pq}(A))$ is not necessarily equal to $c_{rs}(f(A))$ where $A\subseteq X$ .See example : Let X=$\{1,2,3\}$,p=$\{\{1,2\}\}$,
q=$\{\{1\},\{1,2\}\}$,Y= $\{a,b,c\}$. r=$\{\{a\}\}$. s=$\{\{a\},\{a,b\}\}$. Let f:

Example 3.1.6 Let $X=\{1,2,3\}$, $p=\{\{1,2\}\}$, $q=\{\{1\},\{1,2\}\}$, $y=\{a,b,c\}$, $r=\{\{a\}\}$, $s=\{\{a\},\{a,b\}\}$.

Let $f:(X,p,q)\to(Y,r,s)$ by $f(1)=a=f(2), f(3)=b$. We have f is pq-continuous mapping. Let A = $\{3\}$,
then $c_{pq}\{3\}=\{3\}\Longrightarrow f(\{3\})=\{b\}$. But $c_{pq}f\{3\}=c_{pq}\{b\}=\{b,c\}\neq\{b\}$.

Theorem 3.1.7 A mapping $f:(X,p,q)\to(Y,r,s)$ is pq-continuous if and only if $c_{pq}\left(f^{-1}(B)\right)\subseteq f^{-1}\left(c_{rs}(B)\right)$ for every subset B of Y.

Proof : If f is pq-continuous and $B\subseteq Y$, $c_{rs}(B)$ is rs-closed in Y and hence $f^{-1}\left(c_{rs}(B)\right)$ is pq-closed in X. Therefore $c_{pq}(f^{-1}(c_{rs}(B)))=f^{-1}\left(c_{rs}(B)\right)$. Since $B\subseteq c_{rs}(B)$, $f^{-1}(B)\subseteq f^{-1}\left(c_{rs}(B)\right)$. Therefore $c_{pq}\left(f^{-1}(B)\right)\subseteq c_{pq}(f^{-1}(c_{rs}(B)))=f^{-1}\left(c_{rs}(B)\right)$. That is $c_{pq}\left(f^{-1}(B)\right)\subseteq f^{-1}\left(c_{rs}(B)\right)$.

Conversely, let $c_{pq}\left(f^{-1}(B)\right)\subseteq f^{-1}\left(c_{rs}(B)\right)$ for every $B\subseteq Y$. Let B be rs-closed in Y. By theorem 2.29 (6) Then $c_{rs}(B)=B$. By assumption, $c_{pq}\left(f^{-1}(B)\right)\subseteq f^{-1}\left(c_{rs}(B)\right)=f^{-1}(B)$. Thus, $c_{pq}\left(f^{-1}(B)\right)\subseteq f^{-1}(B)$. But $f^{-1}(B)\subseteq c_{pq}\left(f^{-1}(B)\right)$. Therefore $c_{pq}\left(f^{-1}(B)\right)=f^{-1}(B)$. That is, $f^{-1}(B)$ is pq-closed in X for every rs-closed set B in Y. Therefore, f is $pq-\text{continuous on } X$.

The following theorem establishes a criterion for pq-continuous maps in terms of inverse image of rs-interior of a subset of Y.

Theorem 3.1.8 A mapping $f:(X,p,q)\to(Y,r,s)$ is pq-continuous on X if and only if $f^{-1}(i_{rs}(B))\subseteq i_{pq}(f^{-1}(B))$ for every subset B of Y.

Proof : Let f be pq –continuous and B ⊆ Y. By theorem 2.23 (4) ,Then $i_{rs}(B)$ is rs-open in (Y, r, s). Therefore $f^{-1}(i_{rs}(B))$ is pq –open in (X, p, q). That is, $f^{-1}(i_{rs}(B)) = i_{pq}(f^{-1}(i_{rs}(B))$. Also, $i_{rs}(B) \subseteq B$ implies that $f^{-1}(i_{rs}(B)) \subseteq f^{-1}(B)$.Therefore $i_{pq}[f^{-1}(i_{rs}(B))] \subseteq i_{pq}(f^{-1}(B))$.

That is $f^{-1}(i_{rs}(B)) \subseteq i_{pq}(f^{-1}(B))$.

Conversely, let $f^{-1}(i_{rs}(B)) \subseteq i_{pq}(f^{-1}(B))$ for every subset B of Y. If B is rs-open in Y, by theorem 2.23 (4) $i_{rs}(B) = B$. Also, $f^{-1}(i_{rs}(B)) \subseteq i_{pq}(f^{-1}(B))$. That is, $f^{-1}(B) \subseteq i_{pq}(f^{-1}(B))$. But $i_{pq}(f^{-1}(B)) \subseteq f^{-1}(B)$. Therefore $f^{-1}(B) = i_{pq}(f^{-1}(B))$. Thus $f^{-1}(B)$ is pq –open in X for every rs-open set B in Y. Therefore f is pq –continuous.

Theorem 3.1.9 Let $f: (X, p, q) \rightarrow (Y, r, s)$, $g: (Y, r, s) \rightarrow (Z, m, n)$ are pq –continuous and rs-continuous maps respective. Then $gof: (X, p, q) \rightarrow (Z, m, n)$ is pq –continuous

Proof : Let G be an arbitrary mn-open set in Z. We shall prove gof is a pq –continuous maps, it is enough we shall prove that $(gof)^{-1}(G)$ is pq –open in X.

Since $(gof)^{-1}(G) = f^{-1}[g^{-1}(G)]$

$= f^{-1}(H)$ where $H = g^{-1}(G)$ …(*)

Since g is rs-continuous, G ⊆Z is rs-open set

$\Longrightarrow g^{-1}(G)$ is rs-open in Y.

$\Longrightarrow$ H is rs-open in Y.

H is rs-open in Y, f is pq –continuous maps

$\Longrightarrow f^{-1}(H)$ is pq-open in X

$\Longrightarrow (gof)^{-1}(G)$ is pq-open in X , by (*)

Definition 3.1.10 Let : (X, p, q) be a bi-quasi generalized weak space $A \subseteq X$. Then A is said to be a pq –dense set in X if $c_p\left(c_q(A)\right) = X$

Definition 3.1.11 A subset A of a bi-quasi generalized weak space (X, p, q) is said to be pq –dense in itself if $A \subseteq d_{pq}(A)$

Theorem 3.1.12 Let $f: (X, p, q) \rightarrow (Y, r, s)$ be an onto, pq –continuous maps. If A is pq –dense in X, then $f(A)$ is rs-dense in Y.

Proof : Since A is pq –dense in X, $c_{pq}(A) = X$. Then $f\left(c_{pq}(A)\right) = f(X)=Y$, Since f is onto. Since f is pq –continuous on X, $f(c_{pq}(A)) \subseteq c_{rs}(f(A))$. Therefore, $Y \subseteq c_{rs}(f(A))$. But $c_{rs}(f(A)) \subseteq Y$. Therefore, $c_{rs}(f(A))=Y$. That is, $f(A)$ is rs-dense in Y. Thus, a pq –continuous mapping maps pq –dense sets into rs-dense sets, provided it is onto.

Theorem 3.1.13 If $f: (X, p, q) \rightarrow (Y, r, s)$ be a one-one-continuous map, then f maps pq –dense in itself subset of X onto rs-dense in itself subset of Y.

Proof : Let A⊆ X be pq – dense in itself so that $A \subseteq d_{pq}(A)$ …(1).

We shall prove that $\mathrm{f(A)}$ is $\mathcal{rs}$-dense in itself. For this we prove $\mathrm{f(A)} \subseteq \mathrm{d}_{\mathcal{pq}}(\mathrm{f(A)})$ …(2).

Let y ∈ $\mathrm{f(A)}$ be arbitrary. Then $\exists$ x ∈ A such that f(x)=y, since f is one-to-one so that X=$\mathrm{f}^{-1}(\mathrm{y})$. We want to show that y is in $\mathcal{rs}$-derived of $\mathrm{f(A)}$. Let G be $\mathcal{rs}$-open in Y with y ∈ G

$\mathrm{y} \in \mathrm{G} \Rightarrow \mathrm{f}^{-1}(\mathrm{y}) \in \mathrm{f}^{-1}(\mathrm{G})$. Also f is $\mathcal{pq}$ –continuous.

$\Rightarrow \mathrm{x} \in \mathrm{f}^{-1}(\mathrm{G})$ be $\mathcal{pq}$ –open in X.

By (1), $\mathrm{x} \in \mathrm{A} \Rightarrow \mathrm{x} \in \mathrm{d}_{\mathcal{pq}}(\mathrm{A}) \Rightarrow (\mathrm{f}^{-1}(\mathrm{G}) - \{\mathrm{x}\}) \cap \mathrm{A} \neq \emptyset$

$\Rightarrow \exists\, \mathrm{z} \in (\mathrm{f}^{-1}(\mathrm{G}) - \{\mathrm{x}\}) \cap \mathrm{A}$

$\Rightarrow \exists\, \mathrm{z} \in \mathrm{A}, \mathrm{z} \in (\mathrm{f}^{-1}(\mathrm{G}) - \{\mathrm{x}\}), \mathrm{z} \neq \mathrm{x}$

$\Rightarrow \mathrm{f(z)} \in \mathrm{f(A)}, \mathrm{f(z)} \in \mathrm{G}, \mathrm{z} \neq \mathrm{x}$

Since f is one-to-one and $\mathrm{z} \neq \mathrm{x}$

$\Rightarrow \mathrm{f(z)} \neq \mathrm{f(x)} = \mathrm{y}$

$\Rightarrow \mathrm{y} \neq \mathrm{f(z)}$.

It means that $\mathrm{f(z)} \in \mathrm{f(A)} \cap ((\mathrm{G}) - \{\mathrm{y}\})$ or $\mathrm{f(A)} \cap ((\mathrm{G}) - \{\mathrm{y}\}) \neq \emptyset$.

Therefore y is in $\mathcal{rs}$-derived of $\mathrm{f(A)}$. Hence (2) is established.

3.2 $\mathcal{pq}$ –homeomorphisms

Definition 3.2.1 A mapping $\mathrm{f}: (\mathrm{X}, \mathcal{p}, \mathcal{q}) \rightarrow (\mathrm{Y}, \mathcal{r}, \mathcal{s})$ is a $\mathcal{pq}$ –open map if the image of every $\mathcal{pq}$ –open set in X is $\mathcal{rs}$-open in Y. The mapping f is said to be a $\mathcal{pq}$ –closed map if the image of every $\mathcal{pq}$ –closed set in X is $\mathcal{rs}$-closed in Y.

Example 3.2.2 Let X = $\{1,2,3\}$, p = $\{\{3\},\{1,3\}\}$, q = $\{\{3\},\{1,3\},\{2,3\}\}$. We have $\mathcal{pq}$ –open sets are $\{3\},\{1,3\},\emptyset$. Let y = $\{\mathrm{a},\mathrm{b},\mathrm{c}\}$, r = $\{\{\mathrm{c}\},\{\mathrm{a},\mathrm{c}\}\}$, s = $\{\{\mathrm{c}\},\{\mathrm{a},\mathrm{c}\},\{\mathrm{b},\mathrm{c}\}\}$. We have $\mathcal{rs}$-open sets are $\{\mathrm{c}\},\{\mathrm{a},\mathrm{c}\}$ and $\emptyset$.

Theorem 3.2.3 A mapping $\mathrm{f}: (\mathrm{X}, \mathcal{p}, \mathcal{q}) \rightarrow (\mathrm{Y}, \mathcal{r}, \mathcal{s})$ is a $\mathcal{pq}$ – closed map if and only if $\mathrm{c}_{\mathcal{rs}}(\mathrm{f(A)}) \subseteq \mathrm{f}(\mathrm{c}_{\mathcal{pq}}(\mathrm{A}))$ for every subset A of X.

Proof: If f is $\mathcal{pq}$ –closed, $\mathrm{f}(\mathrm{c}_{\mathcal{pq}}(\mathrm{A}))$ is $\mathcal{rs}$-closed in Y, since $\mathrm{c}_{\mathcal{pq}}(\mathrm{A})$ is $\mathcal{pq}$ –closed in X. Since A $\subseteq \mathrm{c}_{\mathcal{pq}}(\mathrm{A})$, by thorem 2.29 (2), so that $\mathrm{f(A)} \subseteq \mathrm{f}(\mathrm{c}_{\mathcal{pq}}(\mathrm{A}))$. Thus $\mathrm{f}(\mathrm{c}_{\mathcal{pq}}(\mathrm{A}))$ is $\mathcal{rs}$- closed containing $\mathrm{f(A)}$. Therefore, $\mathrm{c}_{\mathcal{rs}}(\mathrm{f(A)}) \subseteq \mathrm{f}(\mathrm{c}_{\mathcal{pq}}(\mathrm{A}))$.

Conversely, if $\mathrm{c}_{\mathcal{rs}}(\mathrm{f(A)}) \subseteq \mathrm{f}(\mathrm{c}_{\mathcal{pq}}(\mathrm{A}))$ for every subset A of X and if F is $\mathcal{pq}$ –closed in X, then $\mathrm{c}_{\mathcal{pq}}(\mathrm{F})$ =F and hence $\mathrm{f(F)} \subseteq \mathrm{c}_{\mathcal{rs}}(\mathrm{f(F)}) \subseteq \mathrm{f}\left(\mathrm{c}_{\mathcal{pq}}(\mathrm{F})\right) = \mathrm{f(F)}$. Thus $\mathrm{f(F)} = \mathrm{c}_{\mathcal{rs}}(\mathrm{f(F)})$. That is $\mathrm{f(F)}$ is $\mathcal{rs}$-closed in Y. Therefore, f is a $\mathcal{pq}$ –closed map.

Theorem 3.2.4 A mapping $\mathrm{f}: (\mathrm{X}, \mathcal{p}, \mathcal{q}) \rightarrow (\mathrm{Y}, \mathcal{r}, \mathcal{s})$ is $\mathcal{pq}$ – open map iff $\mathrm{f}(\mathrm{i}_{\mathcal{pq}}(\mathrm{A})) \subseteq \mathrm{i}_{\mathcal{rs}}(\mathrm{f(A)}), \forall \mathrm{A} \subseteq \mathrm{X}$

Proof : Let f be $\mathcal{pq}$ –open. Let $\mathrm{A} \subseteq \mathrm{X}$ be arbitrary. We shall prove $\mathrm{f}(\mathrm{i}_{\mathcal{pq}}(\mathrm{A})) \subseteq \mathrm{i}_{\mathcal{rs}}(\mathrm{f(A)}$

Since $\mathrm{i}_{\mathcal{pq}}(\mathrm{A}) \subseteq \mathrm{A}$ by theorem 2.23 (2) $\Rightarrow \mathrm{f}\left(\mathrm{i}_{\mathcal{pq}}(\mathrm{A})\right) \subseteq \mathrm{f(A)}$

$\Rightarrow \mathrm{i}_{\mathcal{rs}}(\mathrm{f}\left(\mathrm{i}_{\mathcal{pq}}(\mathrm{A})\right) \subseteq \mathrm{i}_{\mathcal{rs}}(\mathrm{f(A)}$

Since $\mathrm{i}_{\mathcal{pq}}(\mathrm{A})$ is $\mathcal{pq}$ –open $\Rightarrow \mathrm{f}\left(\mathrm{i}_{\mathcal{pq}}(\mathrm{A})\right)$ is $\mathcal{rs}$-open

$$\Rightarrow f\left(i_{pq}(A)\right) = i_{rs}(f\left(i_{pq}(A)\right)$$

$$\Rightarrow f\left(i_{pq}(A)\right) \subseteq i_{rs}(f(A)$$

Conversely, let $f(i_{pq}(A)) \subseteq i_{rs}(f(A))$ $\forall A \subseteq X$. We shall prove f is pq –open. Let G be pq –open so that $i_{pq}(G) = G \Rightarrow f\left(i_{pq}(G)\right) = f(G)$. But $f(i_{pq}(G)) \subseteq i_{rs}(f(G))$ by given. So, we have $f(G) \subset i_{rs}(f(G)$. Also $i_{rs}(f(G) \subset f(G)$ $\forall G$. Hence $f(G) = i_{rs}(f(G)$ is rs-open. Therefore f is pq –open.

Definition 3.2.5 A mapping $f: (X, p, q) \rightarrow (Y, r, s)$ is said to be a pq –homeomorphism if

(1). f is one-to-one and onto

(2). f is pq-continuous and

(3). f is pq –open

Theorem 3.2.6 Let $f: (X, p, q) \rightarrow (Y, r, s)$ be a one-one onto mapping. Then f is a pq –homeomorphism if and only if f is pq –closed and pq –continuous.

Proof : Let f be a pq –homeomorphism. Then f is pq –continuous. Let F be an arbitrary pq –closed set in (X, p, q). Then X-F is pq –open. Since f is pq –open , f(X-F) is rs-open in Y. That is, Y- $f(F)$ is rs-open in Y. Therefore $f(F)$ is rs-closed in Y. Thus the image of every pq –closed set in X is rs-closed in Y. That is f is pq –closed.

Conversely, let f be pq –closed and pq –continuous. Let G be pq –open in (X, p, q). Then X-G is pq –closed in X. Since f is pq –closed, $f(X - G) = Y - f(G)$ is rs-closed in Y. Therefore, $f(G)$ is rs-open in Y. Thus, f is pq –open and hence f is a pq –homeomorphism.

Theorem 3.2.7 A one-to-one map f of (X, p, q) onto (Y, r, s) is a pq –homeomorphism if and only if $f\left(c_{pq}(A)\right) = c_{rs}(f(A)$ for every subset A of X.

Proof : If f is a pq –homeomorphism, f is pq –continuous and pq –closed. If $\subseteq X$, $f(c_{pq}(A)) \subseteq c_{rs}(f(A)$, since f is pq –continuous. Since $c_{pq}(A)$ is pq –closed in X and f is pq –closed, $f(c_{pq}(A))$ is rs-closed in Y. $c_{rs}(f\left(c_{pq}(A)\right) = f(c_{pq}(A))$. Since $A \subseteq c_{pq}(A)$, $f(A) \subseteq f(c_{pq}(A))$ and hence $c_{rs}(f(A)) \subseteq c_{rs}(f\left(c_{pq}(A)\right) = f(c_{pq}(A))$. Therefore, $c_{rs}(f(A)) \subseteq f\left(c_{pq}(A)\right)$. Thus $f\left(c_{pq}(A)\right) = c_{rs}(f(A))$ if f is a pq–homeomorphism.

Conversely, if $f\left(c_{pq}(A)\right) = c_{rs}(f(A))$ for every subset A of X, then f is pq –continuous. If A is pq –closed in X, $c_{pq}(A) = A$ which implies $f(c_{pq}(A)) = f(A)$. Therefore, $c_{rs}(f(A)) = f(A)$. Thus, $f(A)$ is rs-closed in Y, for every pq –closed set A in X. That is f is pq –closed. Also f is pq –continuous. Thus, f is a pq –homeomorphism.

Theorem 3.2.8 An identity mapping is a pq –homeomorphism.

Proof : Let $f: (X, p, q) \rightarrow (X, p, q)$ be an identity map, given by $f(x) = x$ $\forall A \in X$

Let $G \subseteq X$ be an arbitrary pq –open set.

Then $f^{-1}(G) = \{x \in X: f(x) \in G\}$

$= \{x \in X: x \in G\}$
$=$ G is a $\mathcal{pq}$ –open.

Hence f is a $\mathcal{pq}$ –continuous.

Let $G \subseteq X$ be an arbitrary $\mathcal{pq}$ –open set.

Then $f(G) = \{f(x): x \in G\}$
$= \{x: x \in G\}$
= G is $\mathcal{pq}$ –open set.

Hence f^{-1} is $\mathcal{pq}$ –continuous map. Moreover f is one-to-one onto. This means that the identity map f is a $\mathcal{pq}$ –homeomorphism.

Theorem 3.2.9 A one-to-one onto map $f: (X, \mathcal{p}, \mathcal{q}) \rightarrow (Y, \mathcal{r}, \mathcal{s})$ is a $\mathcal{pq}$ –homeomorphism iff $f\left(i_{\mathcal{pq}}(A)\right) = i_{\mathcal{rs}}(f(A)), \forall A \subseteq X$

Proof : Suppose $f: (X, \mathcal{p}, \mathcal{q}) \rightarrow (Y, \mathcal{r}, \mathcal{s})$ is one-to-one onto map. Also suppose that $A \subseteq X$ is arbitrary. Suppose f is a $\mathcal{pq}$ –homeomorphism. We shall prove $f\left(i_{\mathcal{pq}}(A)\right) = i_{\mathcal{rs}}(f(A))$. Since f is a $\mathcal{pq}$ –homeomorphism, then f is a $\mathcal{pq}$ –continuous and f^{-1} = g (say) is $\mathcal{rs}$-continuous …(1)

Let B = f(A) then $B \subseteq Y$. Since $B \subseteq Y$ and f is $\mathcal{pq}$ – continuous then $f^{-1}(i_{\mathcal{rs}}(B)) \subseteq i_{\mathcal{pq}}(f^{-1}(B))$, Since B = f(A) so that $f^{-1}(i_{\mathcal{rs}}(f(A))) \subseteq i_{\mathcal{pq}}(f^{-1}(f(A))) = f^{-1}(f\left(i\right)_{\mathcal{pq}}(A)\right)$. Therefore $f^{-1}(i_{\mathcal{rs}}(f(A))) \subseteq f^{-1}(f(i_{\mathcal{pq}}(A)))$.Thus f : X ⟶ Y is a $\mathcal{pq}$ –continuous map and $A \subseteq X$.

Then $(i_{\mathcal{rs}}(f(A))\ )) \subseteq f\left(i_{\mathcal{pq}}(A)\right)$ …(2).

Similarly g : Y ⟶ X is $\mathcal{rs}$-continuous , $B \subseteq Y$. Then $i_{\mathcal{pq}}(g(B)) \subseteq g(i_{\mathcal{rs}}(B))$. Since $g = f^{-1}$ (by (1)) so that $i_{\mathcal{pq}}(f^{-1}(f(A))) \subseteq f^{-1}\left(i_{\mathcal{rs}}\left(f(A)\right)\right)$. Hence $f^{-1}(f(i_{\mathcal{pq}}(A)) \subseteq f^{-1}(i_{\mathcal{rs}}(f(A)))$.

Therefore $f(i_{\mathcal{pq}}(A)) \subseteq i_{\mathcal{rs}}(f(A))$ … (3).

By (2) and (3) , we have $f(i_{\mathcal{pq}}(A)) = i_{\mathcal{rs}}(f(A))$.

Conversely, suppose that $f(i_{\mathcal{pq}}(A)) = i_{\mathcal{rs}}(f(A))$. We shall prove that f is a $\mathcal{pq}$– homeomorphism, we have to show that

(i) f is one-to-one onto

(ii) f is $\mathcal{pq}$ –continuous

(iii) f^{-1} is $\mathcal{rs}$-continuous

By initial assumption, we found that f is one-to-one onto and

$i_{\mathcal{rs}}(f(A)) \subseteq f(i_{\mathcal{pq}}(A))$ … (4)
$f(i_{\mathcal{pq}}(A)) \subseteq i_{\mathcal{rs}}(f(A))$ … (5)

Let f^{-1}= g so that $f = g^{-1}$.By (5),we have $g^{-1}(i_{\mathcal{pq}}(A)) \subseteq i_{\mathcal{rs}}(g^{-1}(A))$

Hence g is $\mathcal{rs}$-continuous , by theorem 3.1.8.Therefore f^{-1} is $\mathcal{rs}$-continuous.

Let B = f(A) so that $A = f^{-1}(B)$.

By (4),we have $i_{\mathcal{rs}}(f(f^{-1}(B))) \subseteq f(i_{\mathcal{pq}}(f^{-1}(B)))$.

And $f(f^{-1}(i_{rs}(B))) \subseteq f(i_{pq}(f^{-1}(B)))$.
We have $f^{-1}(i_{rs}(B)) \subseteq i_{pq}(f^{-1}(B))$.

Therefore f is pq –continuous , by theorem 3.1.8.

Theorem 3.2.10 Let $f: (X, p, q) \to (Y, r, s)$ be one-to-one and onto map. Then the following statements are equivalent :

(i) f is pq –open and pq –continuous

(ii) f is a pq –homeomorphism

(iii) f is a pq –closed and pq –continuous.

Proof : Let $f: (X, p, q) \to (Y, r, s)$ be an one-to-one onto map. Let f be pq –open and pq –continuous. We shall prove (i) $\Longrightarrow$ (ii), prove that f is pq –homeomorphism, we have to prove that

(1) f is one-to-one onto

(2) f is pq –continuous

(3) f^{-1} is rs-continuous.

By assumption, (1) and (2) at once follow $f^{-1}: (Y, r, s) \to (X, p, q)$

Let G be a pq –open, then $(f^{-1})^{-1}(G) = f(G)$. Since f is pq –open. Hence $f(G)$ is rs-open in Y. It follows that f^{-1} is a rs-continuous.

We shall prove (ii) $\Longrightarrow$ (iii). Let f be a pq –homeomorphism. We shall prove that f is pq –closed and pq –continuous. Let F be a pq –closed subset of X so that X-F is an pq –open.

The f is a pq –homeomorphism $\Longrightarrow$ $f^{-1}: (Y, r, s) \to (X, p, q)$ is rs-continuous,

where X-F is pq –open so that $(f^{-1})(X - F) = f(X - F)$ is rs-open in Y.

$$= f(X) - f(F)$$

$$= Y - f(F).$$

Therefore $f(F)$ is rs-closed in Y. Therefore f is pq –closed map.

We shall prove that (iii) $\Longrightarrow$ (i).

Let f be a pq –closed and pq –continuous map. We shall prove that f is pq –open and pq –continuous.

By assumption, f is a pq –continuous.

Let G be pq –open subset of X so that X-G is a pq –closed subset of X. Consequently $f(X - G)$ is rs-closed subset of Y. For f is given to be pq –closed.

Now $f(X - G) = f(X) - f(G) = Y - f(G)$. Hence $Y - f(G)$ is rs-closed so that $f(G)$ is rs-open. Therefore f is pq –open.

4. Conclusion

The purpose of this research is to study and investigate the properties of a pq-continuous mapping and a pq-homeomorphism on bi-quasi generalized weak space. Characteristic of this mapping under the domain of set of pq-open and pq-closed sets ,also its taken for pq-open map and pq-closed map. Otherwise in some case we can use pq-interior and pq-closure determine to investigate some properties. In the future, we shall extend to study in a generalized continuity of this mapping and also homeomorphism.

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6. References

Auscharaporn, P. (2019).*pq –derived sets and pq –boundary sets in Bi-Quasi Generalized Weak Spaces*. Research in Mathematics, Rajabhat Mahasarakham University.

Ávila, J. and Molina, F. Generalized Weak Structures, *Int. Math. Forum, 7*(2012), no.52, 2589-2595.

Bandorn, D. (2019).*pq – interior sets and pq –closure sets in bi-quasi generalized weak spaces*, Research in Mathematics, Rajabhat Mahasarakham University.

Császár, A. Genaralized open sets, *Acta Math. Hungar. 75*(1997), 65-68.

Janrongkam, P. (2019). *Quasi generalized weak structure*, Research in Mathematics, Rajabhat Mahasarakham University.

Kelly, J.C. (1996). Bitopological spaces, *Prof. London Math. Soc., 3*(13), 71-79.

Kim Y.K. and Min, W.K. (2013). σ –structures and quasi-enlargingoperation, *Int. J. Pure and Appl. 86*(5), 791-798.

Lugojan, S. (1982).Generalized topology, *Stud.Cerc.Mat*; *34*, 348-360.

Maki, H. (1996). On generalizing semi-open and preopen sets, *Report for Meeting on Topological Space and its Applications*, Yatsushiro College of Technology, 13-18.

Pongman, J. (2019). *q –continuous on Quasi Generalized Weak Spaces*, Research in Mathematics, Rajabhat Mahasarakham University.

Ratkanok, N. (2019). *q_s –compact on Quasi Generalized Weak Space.* Research in Mathematics, Rajabhat Mahasarakham University.

Sharma, J.N. (2006). *Topology*, Krishna Prakashan Media (P). Ltd. Meerut. (U.P.).

Thongpan, J. (2019). *Bi-quasi generalize weak structures*, Research in Mathematics, Rajabhat Mahasarakham University.

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