Some Properties of Edge-intersection Graph of 3-uniform Hypergraphs of Order 6n and Size 4n

Supawan Nanta[1]* and Krittawit Limkul[1]

[1]Mathematics Program, Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun

Abstract

Let $H = (V, E)$ be a 3-uniform, 2-regular, connected hypergraph of order $6n$ and size $4n$ for some $n \in \mathbb{N}$. For $e \in E$, let $E_e = \{f \in E \setminus \{e\}: e \cap f \neq \emptyset\}$, the transversal number of a hypergraph H in which $|E_e| = 2$ for all $e \in E$ are investigated. In this paper, we are interested in studying some properties of an edge-intersection graph $L(H)$ of a hypergraph H such as the vertex cover, the matching, and the independent set. We prove that $L(H)$ is a bipartite graph and the transversal number of H is equal to the vertex covering number of $L(H)$.

Keywords: Hypergraph, Edge-intersection graph, Transversal, Bipartite graph

1. Introduction

A graph is a pair $G = (V, E)$ of sets satisfying $E \subseteq V^2$; thus, the elements of E are 2-element subsets of V. The elements of V are the *vertices* of the graph G, the elements of E are its *edges*. The vertex set and the edge set of a graph G is referred to as $V(G)$ and $E(G)$, respectively. The number of vertices of graph is its *order*, written as $|V(G)|$; its number of edges is called *size* of G, denoted by $|E(G)|$. Two vertices x, y of G are *adjacent*, if $\{x, y\}$ is an edge of G.

A vertex v is *incident* with an edge e if $v \in e$; then e is an *edge at* v. The set of all the edges in E at a vertex v is denoted by $E(v)$. The degree $d(v)$ of a vertex v is the number $|E(v)|$ of edges at v. A graph G is called *r-regular*, if all the vertices of G have the same degree r. A *path* is a non- empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, where the x_i are all distinct. The vertices x_0 and x_k are *linked* by P and are called its *ends*. The number of edges of a path is its *length*. We refer to a path by the sequence of its vertices, $P = x_0, x_1, \ldots, x_k$ and calling P a path from x_0 to x_k. A path $P = x_0, x_1, \ldots, x_k$ where $k \geq 3$ is called *cycle* if $x_0 = x_k$. A non-empty graph G is called *connected*

* *Corresponding author : nanta.supawan@gmail.com*

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if any two vertices are linked by a path in G. A graph $G = (V, E)$ is called *bipartite* if V admits a partition into 2 classes such that every edge has its ends in different classes: vertices in same partition classes must not be adjacent.

A subset U of vertex set V of a graph G is a *vertex cover* of G if every edge of G is incident with a vertex in U. A minimum cardinality of a vertex cover of G is called a *vertex covering number* of G, denoted by $\beta(G)$.

Pairwise non-adjacent vertices or edges are called *independent*. More formally, a set of vertices or of edges is *independent set* if no two of its elements are adjacent. A maximum cardinality of a vertex independent set of a graph G is called *independent number* of G, denoted by $\alpha(G)$. A set M of independent edges in a graph $G = (V, E)$ is called a *matching*. A maximal cardinality of a matching of G is called *matching number* of G, denoted by $\mu(G)$. The independent number and the matching number are well studied in the literature (see, for example [Bouchou, A. & Blidia, M. (2014)], [Harant, J. & Rautenbach, D. (2011)], [Henning, M. A., Lowenstein, C. & Rautenbach, D. (2012)], [Zhang, Z. & Lou, D. (2010)]).

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A hypergraph $H = (V, E)$ is a nonempty finite set V of elements, called *vertices*, together with a family of finite subsets E of V , called *hyperedges* or simply edges. We shall use the notation $n_H = |V|$ and $m_H = |E|$, and sometimes simply n and m without subscript if actual H need not be emphasized, to denote the order and size of H, respectively. The edge of set E is often allowed to be a multiset in the literature, but in this paper, we exclude multiple edges.

A k-*edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. For $2 \leq r \leq n$, we define the *complete r-uniform hypergraph* to be a hypergraph $K_n^r = (V, E)$ for which $|V| = n$ and E is the family of all subset of V of size r.

Two vertices x and y of H are *adjacent* if there is an edge $e \in E(H)$ such that $\{x, y\} \subseteq e$. The *degree* of a vertex v in H, denoted by $d_H(v)$ or simply by $d(v)$ is the number of edges of H which contain v. A hypergraph with all vertices has the same degree r is called *r-regular hypergraph*. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2, \dots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1,2, \dots, k$. A hypergraph H is said to be *connected hypergraph* if every pair of vertices are connected (see Fig1 for example).

A subset T of vertices in hypergraph H is a *transversal* (also called vertex cover or hitting set) if T intersects every edge of H. The *transversal number* $\tau(H)$ of H is the minimum cardinality of transversal in H. Transversals in hypergraphs are well studied in the literature (see, for example [Bujtas, Cs., Henning, M. A. & Tuza, Zs. (2012)], [Chvatal, V. & McDiarmid, C. (1992)], [Cockayne, E. J., Hedetniemi, S. T. & Slater, P. J. (1979)], [Henning, M. A. & Yeo, A. (2008)]).

Let $H = (V, E)$ be a hypergraph. The *edge-intersection graph* $L(H)$ of H is a graph where $E(H)$ is a vertex set and any two vertices of $L(H)$ are adjacent if and only if the corresponding edges have a non-empty intersection (see Fig2 for example).

We are willing to refer to [Diestel, R. (2000)] and [Alain, B. (2013)] for more information about graph and hypergraph, respectively.

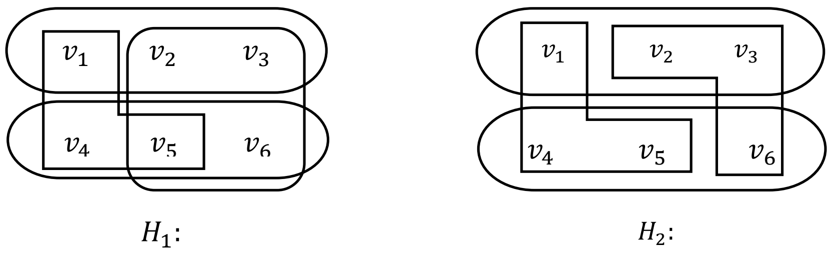

Fig. 1. The hypergraph H_1 consists 4 edges of size 3 and 4. The right hand, a hypergraph H_2 consists 4 edges of size 3. Thus H_2 is a 3-uniform hypergraph. Moreover, H_2 is 2-regular.

There have been many graphs whose properties have been investigated. $L(H)$ constructed from a hypergraph H is also a graph, so we are interested in its properties and some relations with H. Since the characteristic of $L(H)$ depends on the properties of H, it is necessary to provide some properties for H in each study and the desired hypergraph H in this research is a 3-uniform Hypergraph of Order $6n$ and Size $4n$.

2. Main Results

On Transversal number of a hypergraph H

For each $e \in E$, denoted by E_e, a set of all edge of H that intersects the edge e i.e. $E_e = \{f \in E \setminus \{e\}: e \cap f \neq \emptyset\}$. In this section, we present a transversal number of the hypergraph H where H is a 3-uniform, 2-regular, connected of order $6n$ and size $4n$ for some $n \in \mathbb{N}$ such that $|E_e| = 2$ for all $e \in E$. Clearly that, for each $e \in E(H)$ there is a unique $f \in E(H)$ such that $|e \cap f| = 1$, since H is 2-regular and $|E_e| = 2$ for all $e \in E$.

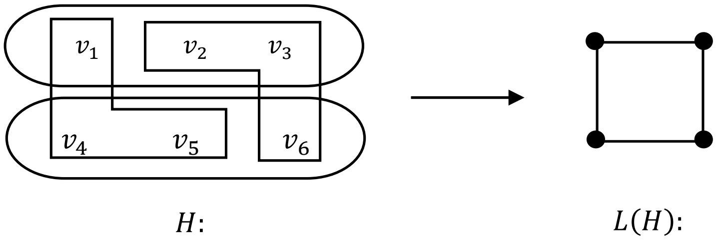

Fig. 2. The edge-intersection graph $L(H)$ of a 3-uniform, 2-regular hypergraph H.

Let $V^{'}$ be the set of all vertices of H such that, for all $v \in V^{'}$, $v \in e \cap f$ where $|e \cap f| = 1$ for some $e, f \in E$. The following lemma, we show that $|V^{'}| = 2n$.

Lemma 1 Let H be a 3-uniform, 2-regular, connected of order $6n$, size $4n$. If $|E_e| = 2$ for all $e \in E(H)$, then $|V^{'}| = 2n$.

Proof: Let $|E_e| = 2$ for all $e \in E(H)$. Since H is a 3-uniform and $|E_e| = 2$ for all $e \in E(H)$, for each $e \in E$ there is a unique $f \in E$ such that $|e \cap f| = 1$. Since $|E(H)| = 4n$, there are $2n$ pairs of edges of H such that $|e \cap f| = 1$. By the definition of V', we get that $|V^{'}| = 2n$. □

Theorem 2 Let H be a 3-uniform, 2-regular, connected of order $6n$ and size $4n$. If $|E_e| = 2$ for all $e \in E(H)$, then $\tau(H) = 2n$.

Proof: Let $|E_e| = 2$ for all $e \in E(H)$. By Lemma 1, we get that V' is a transversal of H. Thus $\tau(H) \leq 2n$, since $\tau(H)$ is a minimum cardinality. Next, we will show that $\tau(H) \geq 2n$. Assume to the contrary that there exists a transversal T of H such that $|T| < 2n$. Thus $T \cap e \neq \emptyset$ for all $e \in E$. Hence $\sum_{v \in T} d(v) < 4n$, since H is 2-regular. Therefore T intersects at most $4n - 2$ edges of H. Because $|E| = 4n$, this contradicts the hypothesis that T is a transversal. □

Edge-intersection graph $L(H)$ of the hypergraph H

In this part, we investigate a vertex covering number of an edge-intersection graph $L(H)$ of a hypergraph H in which $|E_e| = 2$ for all $e \in E(H)$. Throughout this part, we assume that H is a 3-uniform, 2-regular, connected hypergraph of order $6n$ and size $4n$ in which $|E_e| = 2$ for all $e \in E(H)$. Moreover, $L(H)$ is a 2-regular graph, since $E_e = \{f \in E \setminus \{e\}: e \cap f \neq \emptyset\}$.

Lemma 3 If H is a connected hypergraph, then $L(H)$ is connected graph.

Proof: Let H be a connected hypergraph and $e, f \in V(L(H))$ where $e \neq f$. By the definition of the edge-intersection graph of H, we get $e, f \in E(H)$. Suppose that $x, y \in V(H)$ such that $x \in e$ and $y \in f$. Since H is connected, there is a sequence $x = v_0, v_1, \dots, v_k = y$ of vertices of H where v_{i-1} is adjacent to v_i for $i = 1,2, \dots, k$. Thus $\{v_{i-1}, v_i\} \subseteq e_i$ where $e_i \in E(H)$ and $i = 1,2, \dots, k$. Consequently, $e_i \cap e_{i+1} \neq \emptyset$ for all i. Since $x = v_0 \in e$ and $y = v_k \in f$, $e \cap e_1 \neq \emptyset$ and $e_k \cap f \neq \emptyset$. By the definition of the edge-intersection graph of H, we get $e, e_1, e_2, \dots, e_k, f$ is a path from e to f in $L(H)$. Therefore $L(H)$ is connected. □

Lemma 4 If $|E_e| = 2$ for all $e \in E(H)$ then $L(H)$ is a bipartite graph. Moreover, $V(L(H)) = V_1 \dot{\cup} V_2$ where $|V_1| = |V_2| = 2n$.

Proof: Let $|E_e| = 2$ for all $e \in E(H)$. Then $L(H)$ is a 2-regular graph and $|E(L(H))| = \frac{\sum_{v \in V(L(H))} d(v)}{2} = \frac{2(4n)}{2} = 4n$. Assume that, $L(H)$ contains an odd cycle C. Thus $|V(C)| < 4n$. Since $L(H)$ is a 2-regular graph and every vertex in the cycle C has degree 2, C is a component of $L(H)$. Hence $L(H)$ is a disconnected graph, which is contradicts to the connectivity of $L(H)$ by Lemma 3. Therefore $L(H)$ contains no odd cycle and we conclude that $L(H)$ is a bipartite graph.

Next, let $V(L(H)) = V_1 \cup V_2$. Consider the number of edges associated with vertex in V_1 and V_2, we get $2|V_1|$ and $2|V_2|$ are numbers of edges that have endpoints in V_1 and V_2, respectively. Since $L(H)$ is bipartite, $2|V_1| = 2|V_2|$. Therefore $|V_1| = |V_2| = 2n$. □

Theorem 5 If $|E_e| = 2$ for all $e \in E(H)$, then $\beta(L(H)) = \tau(H) = 2n$.

Proof: Let $|E_e| = 2$ for all $e \in E(H)$. Since $L(H)$ is a bipartite graph, every edge has endpoint in V_1. Thus V_1 is a vertex cover of $L(H)$. Therefore $\beta(L(H)) \leq |V_1| = 2n$. Clearly that, if U is a vertex cover of $L(H)$, then $|U| \geq 2n$, because $|E(L(H))| = 4n$ and $L(H)$ is 2-regular. Hence $\beta(L(H)) \geq 2n$. Therefore $\beta(L(H)) = 2n = \tau(H)$. □

In 1931, Dénes Kónig described the relation between the minimum cardinality of a vertex covering and the maximum cardinality of a matching in a bipartite graph as follow:

Theorem 6 (Kónig 1931) [Diestel, R.(2000)] The maximum cardinality of a matching in a bipartite graph G is equal to the minimum cardinality of a vertex cover.

It is easy to check that, X is a minimum cardinality vertex cover of a graph G if and only if $V(G) \setminus X$ is a maximum cardinality independent set. By using the Kónig theorem and because $L(H)$ is a bipartite, we conclude the following,

Corollary 7 If $|E_e| = 2$ for all $e \in E(H)$, then $\mu(L(H)) = \beta(L(H)) = 2n$.

Proof: Clearly, by Kónig theorem. □

Corollary 8 If $|E_e| = 2$ for all $e \in E(H)$, then $\alpha(L(H)) = 2n$.

Proof: Since $|V(G)| = \beta(L(H)) + \alpha(L(H))$, $\alpha(L(H)) = |V(G)| - \beta(L(H))$. Since $|E(L(H))| = 4n$ and $\beta(L(H)) = 2n$, $\alpha(L(H)) = 2n$. □

By the assumption of the hypergraph H that H is a 3-uniform, 2-regular, connected of order $6n$ and size $4n$ for some $n \in \mathbb{N}$ such that $|E_e| = 2$ for all $e \in E$. We obtain a relation between a vertex covering number $\beta(L(H))$ and the transversal number $\tau(H)$. Moreover, we get $\beta(L(H)) = \mu(L(H)) = \tau(H) = 2n$, by using Kónig theorem.

In this part, we will give a relation between a chromatic index of a complete r-uniform hypergraph K_n^r of order n and a chromatic number of an edge-intersection graph $L(K_n^r)$. Clearly that, a complete r-uniform hypergraph is a hypergraph of size $\binom{n}{r}$.

Recall that, a vertex coloring of a graph $G = (V, E)$ is a map $c: V \to \{1,2, \dots, k\}$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The element of $\{1,2, \dots, k\}$ is called the available *colors*. If k is a smallest integer of a vertex coloring $c: V \to \{1,2, \dots, k\}$ of G, then k is the *chromatic number* of G, denoted by $\chi(G)$.

The *chromatic index* $\chi'(H)$ of a hypergraph H is the least number of colors necessary to color the edges of H in such a way that, any two intersecting edges have distinct colors. It is easy to notice that an edge-coloring of a hypergraph is equivalent to a vertex coloring of it edge-intersection graph.

The following theorem describe the chromatic index of a complete r-uniform hypergraph K_n^r. It is the useful theorem to obtain the relation between a chromatic index of a complete r-uniform hypergraph K_n^r and a chromatic number of an edge-intersection graph $L(K_n^r)$.

Theorem 9 [Baranyai, Zs. (1975)] If n is a multiple of r, then $\chi'(K_n^r) = \binom{n-1}{r-1}$.

Theorem 10 Let K_n^r be a complete r-uniform hypergraph of order n, where n is a multiple of r, and $L(K_n^r)$ be an edge-intersection graph of K_n^r. Then $\chi\big(L(K_n^r)\big) = \frac{n}{r}\chi'(K_n^r)$.

Proof: Since $L(K_n^r)$ is an edge-intersection graph of a complete hypergraph K_n^r, $L(K_n^r)$ is a complete graph with $\binom{n}{r}$ vertices. Thus $d(v) = \binom{n}{r} - 1$ for all $v \in V\big(L(K_n^r)\big)$ and $\chi\big(L(K_n^r)\big) = \binom{n}{r}$. Hence

$$\begin{aligned}
\chi\big(L(K_n^r)\big) &= \frac{n!}{r!(n-r)!} \\
&= \frac{n(n-1)(n-2)\dots(2)(1)}{[r(r-1)(r-2)\dots(2)(1)](n-r)!} \\
&= \left(\frac{n}{r}\right)\left(\frac{(n-1)(n-2)\dots(2)(1)}{[(r-1)(r-2)\dots(2)(1)](n-r)!}\right) \\
&= \left(\frac{n}{r}\right)\left(\frac{(n-1)!}{(r-1)!(n-r)!}\right) \\
&= \left(\frac{n}{r}\right)\binom{n-1}{r-1} \\
&= \left(\frac{n}{r}\right)\chi'(K_n^r)
\end{aligned}$$

Therefore $\chi\big(L(K_n^r)\big) = \frac{n}{r}\chi'(K_n^r)$.

3. Conclusion

Our main results have been separated into three parts. The first one, we obtain the transversal number of H. The second part, the independent number, the vertex covering number and the matching number of $L(H)$ are described. In addition, we obtain that $\tau(H) = \alpha\big(L(H)\big) = \beta\big(L(H)\big) = \mu\big(L(H)\big) = 2n$. In the last part, we also show that $\chi\big(L(K_n^r)\big) = \frac{n}{r}\chi'(K_n^r)$.

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