Fixed Point of Ordered Banach Contractions in Partial b-Metric-Like Spaces

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Abstract

In this paper, some fixed point theorems in a partial b-metric-like space endowed with a partial order are proved. The results of this paper generalize and extend the Banach contraction principle and some other known results in partial b-metric-like spaces endowed with a partial order.

Keywords: Fixed point, Partial order, Partial b-metric-like space, Ordered Banach contraction

1. Introduction

Bakhtin (Bakhtin, 1989) and Czerwik (Czerwik, 1993) introduced b-metric spaces as a generalization of metric spaces. In these spaces, the triangular inequality of the usual metric function was replaced by a more general inequality consisting a constant $s \geq 1$ such that for we obtain the usual metric as a special case. They also obtained the generalized version of Banach contraction principle in such spaces. After this work, several interesting generalization in b-metric spaces have been obtained (see (Czerwik, 1993; Khamsi & Hussain, 2010; Jovanovic et al., 2010; Boriceanu, Bota & Petrusel, 2010; Bota et al., 2011; Roshan et al., 2013; Aghajani et al., 2014; Sarkar, 2020; Jain & Kaur, 2021) and the references therein).

Theorem 1.1 (Dine, Zoto & Ansari, 2018). Let (X, b_d) be a complete b-dislocated metric space with parameter $s \geq 1$ and S, T, F, G : $X \to X$ are self-mappings such that $F(X) \subset T(X)$, $G(X) \subset S(X)$ and satisfy generalized f (ψ, φ, s) weakly contractive condition. If one of F(X), S(X), T (X) or G(X) is closed, then (F, S) and (G, T) have a coincidence point in X. Moreover, if suppose that (F, S) and (G, T) are weakly compatible pairs, then F, G, S, T have a unique common fixed point.

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Received: 14 มิ.ย. 64; Revised: 7 ก.ย. 64; Accepted: 25 ต.ค. 64

Theorem 1.2 (Czerwik, 1993). Let (X, d) be a complete b-metric space and let $T : X \longrightarrow X$ satisfy

$$d(T(x), T(y)) \leq \phi[d(x, y)], \forall x, y \in X,$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is increasing function such that $\lim_{n\to\infty} \phi_n(t) = 0$ for each fixed $t > 0$. Then T has exactly one fixed point u and

$$\lim_{n\to\infty} d(T^n(x), u) = 0$$

for each $x \in X$.

In 1994, Matthews (Matthews, 1994) introduced the notion of partial metric spaces as a part of the study of denotational semantics of dataflow network. In these spaces, the usual metric was generalized by introducing the nonzero self-distance of points of space. Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification.

Theorem 1.3 (Matthews, 1994) (The partial metric contraction mapping theorem) For each complete pmetric $p : U^2 \longrightarrow \mathbb{R}$, and for each function $f : U \longrightarrow U$ such that, $\exists 0 \leq c < 1$, $\forall x, y \in U$,

$$p(f(x), f(y)) \leq cp(x, y)$$

called a contraction, firstly there exists a unique $a \in U$ such that $a = f(a)$, and secondly $p(a, a) =0$.

On the other hand, Ran and Reurings (Ran & Reurings, 2004) obtained the existence of fixed points of a self-mapping of a metric space equipped with a partial order. The fixed point results in spaces equipped with a partial order.

Theorem 1.4 (Ran & Reurings, 2004). Let T be a partially ordered set such that every pair $x, y \in T$ has a lower bound and an upper bound. Furthermore, let d be a metric on T such that (T, d) is a complete metric space. If F is a continuous, monotone (i.e., beither order-preserving or order-reversing) map from T into T such that

1. $\exists 0 < c < 1 : d(F(x), F(y)) \leq cd(x, y), \forall x \geq y$,
2. $\exists x_0 \in T : x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$

then F has a unique fixed point $\bar{x}$. Moreover, for every $x \in T$, $\lim_{n\to\infty} \phi\, F^n(x) = \bar{x}$

In 2014, Shukla (Shukla, 2014) generalized b-metric and partial metric spaces by introducing the notion of partial metric spaces and proved the Banach contraction principle in such spaces. Later in 2017, Shukla (Shukla, 2017) proved the generalization of the Banach contraction principle in partial b-metric space endowed with a partial order. In this paper, we prove a generalization of Banach contraction principle in partial b-metric-like space endowed with a partial order.

2. Preliminaries

First, we recall some definitions from b-metric, partial metric, partial b-metric spaces and partial b-metric-like spaces (see Jovanovic et al., 2014; Shukla, 2017)).

Definition 2.1 (Shukla, 2017). Let X be a nonempty set and the mapping $d : X \times X \longrightarrow \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) satisfies:

(bM1) $d(x, y) = 0$ if and only if $x = y$;

(bM2) $d(x, y) = d(y, x)$;

(bM3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$, for all $x, y \in X$.

Then d is called a b-metric on X and (X, d) is called a b-metric space with coefficient s.

Definition 2.2 (Shukla, 2017). A partial metric on a nonempty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) = p(y, x)$;

(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and b is a partial metric on X.

Definition 2.3 (Shukla, 2017). A partial b-metric on a nonempty set X is a function $b : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

(Pb1) $x = y$ if and only if $b(x, x) = b(x, y) = b(y, y)$;

(Pb2) $b(x, x) \leq b(x, y)$;

(Pb3) $b(x, y) = b(y, x)$;

(Pb4) there exists a real number $s \geq 1$ such that $b(x, y) \leq s[b(x, z) + b(z, y)] - b(z, z)$.

A partial b-metric space is a pair (X, b) such that X is a nonempty set and b is a partial b-metric on X. The number s is called the coefficient of (X, b).

Remark 2.4 (Shukla, 2017). In a partial b-metric space (X, b) if x, y $\in$ X and b(x, y) = 0, then x = y but converse may not be true.

Definition 2.5. A partial b-metric-like on a nonempty set X is a function $b : X \times X \to \mathbb{R}^+$ such that for all x, y, z $\in$ X:

(Pb1) if b(x, x) = b(x, y) = b(y, y), then x = y;

(Pb2) b(x, x) $\leq$ b(x, y);

(Pb3) b(x, y) = b(y, x);

(Pb4) there exists a real number s $\geq$ 1 such that b(x, y) $\leq$ s[b(x, z) + b(z, y)] - b(z, z).

A partial b-metric-like space is a pair (X, b) such that X is a nonempty set and b is a partial b-metric-like on X. The number s is called the coefficient of (X, b).

Definition 2.6. Let (X, b) be a partial b-metric-like space with coefficient s. Let $\{x_n\}$ be any sequence in X and x $\in$ X. Then

(i) the sequence $\{x_n\}$ is said to be convergent and converges to x, $\lim_{n\to\infty} b(x_n, x) = b(x, x)$,

(ii) the sequence $\{x_n\}$ is said to be Cauchy sequence in (X, b) if $\lim_{n,m\to\infty} b(x_n, x_m)$ exists and is finite,

(iii) (X, b) is said to be a complete partial b-metric space if for every Cauchy sequence $\{x_n\}$ in X, there exists x $\in$ X such that $\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x)$.

If a nonempty set X is equipped with a partial order "$\sqsubseteq$" such that (X, b) is a partial b-metric-like space with coefficient s $\geq$ 1, then the triple (X, b, $\sqsubseteq$) is called an ordered partial b-metric-like space. Elements x, y $\in$ X are called comparable, if x $\sqsubseteq$ y or y $\sqsubseteq$ x. A subset A of X is called well ordered if all the elements of A are comparable. A sequence $\{x_n\}$ in X is called nondecreasing with respect to $\sqsubseteq$, if $x_n \sqsubseteq x_{n+1}$ for all n $\in \mathbb{N}$. A mapping T : X $\to$ X is called non-decreasing with respect to $\sqsubseteq$, if x $\sqsubseteq$ y implies Tx $\sqsubseteq$ Ty. We denote the set of all fixed points of T by Fix (T), that is, Fix (T) = {x $\in$ X : Tx = x}.

Definition 2.7. Let $(X, b, \sqsubseteq)$ be an ordered partial b-metric-like space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping. Then T is called an ordered Banach contraction if the following condition holds: there exists $\lambda \in [0, 1)$ such that

$$x \sqsubseteq y \text{ implies } b(Tx, Ty) \leq \lambda b(x, y), \tag{1}$$

for all $x, y \in X$. The constant λ is called the contractive constant of T.

3. Results

The following lemma will be useful in the sequel.

Lemma 3.1 (Shukla, 2017). Let $(X, b, \sqsubseteq)$ be an ordered partial b-metric-like space and $T : X \to X$ be a mapping. If T is non-decreasing with respect to $\sqsubseteq$ and it is an ordered Banach contraction with contractive constant λ. Then for any $k \in \mathbb{N}$, the mapping $F : X \to X$ defined by $Fx = T^k x$, for all $x \in X$ is also non-decreasing with respect to $\sqsubseteq$ and it is an ordered Banach contraction with contractive constant λ^k.

Proof. Since T is nondecreasing with respect to $\sqsubseteq$, for $x, y \in X$ with $x \sqsubseteq y$, we have $Tx \sqsubseteq Ty$. Continuing in this manner, we obtain $T^k x \sqsubseteq T^k x$, that is, $Fx \sqsubseteq Fy$. Thus F is non-decreasing with respect to $\sqsubseteq$. For $x \sqsubseteq y$ since T is non-decreasing with respect to $\sqsubseteq$. we have $T^n x \sqsubseteq T^n y$ for all $n \in \mathbb{N}$, Using (1), get that

$$\begin{aligned} b\,(Fx, Fy) = b(T^k x, T^k y) &= b(TT^{k-1}x, TT^{k-1}y) \\ &\leq \lambda b(T^{k-1}x, T^{k-1}y) \\ &\ \vdots \\ &\leq \lambda^k b(x, y). \end{aligned}$$

Hence, F is an ordered Banach contraction with contractive constant λ^k. □

Now we state the ordered version of Banach contraction principle in partial b-metric-like spaces.

Theorem 3.2. Let $(X, b, \sqsubseteq)$ be an ordered and complete partial b-metric-like space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping such that the following conditions hold:

(I) T is an ordered Banach contraction with contractive constant λ;

(II) there exists $x_0 \in X$ such that $x \sqsubseteq Tx_0$;

(III) T is non-decreasing with respect to $\sqsubseteq$;

(IV) if $\{x_n\}$ is a non-decreasing sequence in X and converging to some z, then $x_n \sqsubseteq Z$ for all $n \in \mathbb{N}$.

Then T has a fixed point $u \in X$ and $b(u, u) = 0$. In addition, the set of fixed points of T, Fix (T) is well ordered if and only if the fixed point of T is unique.

Proof. As, $\lambda \in [0, 1)$ we can choose $n_0 \in \mathbb{N}$ such that, for given $0 < \varepsilon < 1$, we have $\lambda^{n0} < \frac{\varepsilon}{4s^2}$. Let $T^n{}_0 \equiv F$ and $F^k x_0 = x_k$ for all $k \in \mathbb{N}$. By Lemma 3.1., F is non-decreasing with respect to $\sqsubseteq$ and it is an ordered Banach contraction with contractive constant λ^n.

Since $x_0 \sqsubseteq Tx_0$ and T is non-decreasing with respect to $\sqsubseteq$, we have $Tx_0 \sqsubseteq TTx_0$ and $x_0 \sqsubseteq Tx_0 \sqsubseteq T^2x_0$. Continuing in this manner, we obtain

$$x_0 \sqsubseteq Tx_0 \sqsubseteq T^2x_0 \sqsubseteq \cdots \sqsubseteq T^nx_0 \sqsubseteq T^{n+1}x_0 \sqsubseteq \cdots, \text{ for all } n \in \mathbb{N}.$$

Therefore, $x_0 \sqsubseteq T^n{}_0x_0 \sqsubseteq T^{2n}{}_0x_0 \sqsubseteq \cdots \sqsubseteq T^{nn}{}_0x_0 \sqsubseteq \cdots$, for all $n \in \mathbb{N}$.

That is, $x_0 \sqsubseteq Fx_0 \sqsubseteq F^2x_0 \sqsubseteq \cdots \sqsubseteq F^nx_0 \sqsubseteq \cdots$, for all $n \in \mathbb{N}$.

So $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k \sqsubseteq x_{k+1} \sqsubseteq \cdots$, for all $k \in \mathbb{N}$.

Hence, the sequence $\{x_n\}$ is non-decreasing with respect to "$\sqsubseteq$".

Since $\{x_n\}$ is non-decreasing and F is an ordered Banach contraction with contractive constant $\lambda^n{}_0$, we obtain

$$\begin{aligned}
b(x_k, x_{k+1}) &= b(F^kx_0, F^{k+1}x_0) = b(F(F^{k-1}x_0), \\
F(F^kx_0)) &= b(Fx_{k-1}, Fx_k) \\
&\leq \lambda^{n0} b(x_{k-1}, x_k) \\
&\vdots \\
&\leq \lambda^{kn}{}_0 b(x_0, x_1) \rightarrow 0, \text{ as } k \rightarrow \infty.
\end{aligned}$$

We can choose $l \in \mathbb{N}$ such that $b(x_l, x_{l+1}) < \frac{\varepsilon}{8s^2}$.

Let $B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}] := \{y \in X : x_l \sqsubseteq y\}$,

$$b(x_l, y) \leq \frac{\varepsilon}{4s} + b(x_l, x_l).$$

We will show that F maps $B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$ into itself. Now it is obvious that $x_l \in B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$. Thus $B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}] \neq \emptyset$. Let $z \in B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$ be arbitrary, then $x_l \sqsubseteq z$. Since F is non-decreasing with respect to $\sqsubseteq$, we have $Fx_l \sqsubseteq Fz$. That is, $x_{l+1} \sqsubseteq Fz$. So $x_l \sqsubseteq x_{l+1} \sqsubseteq Fz$ which implies $x_l \sqsubseteq Fz$. Since F is an ordered Banach contraction with contractive constant λ^{n0}, we get

$$b(Fx_l, Fz) = \lambda^{n0} b(x_l, z) \leq \frac{\varepsilon}{4s^2}[\frac{\varepsilon}{4s} + b(x_l, x_l)].$$

Hence $\quad b(x_l, Fx_l) = b(x_l, x_{l+1}) < \frac{\varepsilon}{8s^2}$.

Therefore,

$$\begin{aligned}
b(x_l, Fz) &\leq s[b(x_l, Fx_l) + b(Fx_l, Fz)] - b(Fx_l, Fx_l) \\
&< s[\frac{\varepsilon}{8s^2} + \frac{\varepsilon}{4s^2}\{\frac{\varepsilon}{4s} + b(x_l, x_l)\}] - b(Fx_l, Fx_l) \\
&< s[\frac{\varepsilon}{8s^2} + \frac{\varepsilon}{4s^2}\{\frac{\varepsilon}{4s} + b(x_l, x_l)\}]
\end{aligned}$$

$= \frac{\varepsilon}{8s} + \frac{\varepsilon^2}{16s^2} + \frac{\varepsilon}{4s} b(x_l, x_l)$

$< \frac{\varepsilon}{4s} + b(x_l, x_l),$

Thus $Fz \in B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$. Hence, F maps $B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$ into itself.

Note that, $x_l \in B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$. Therefore $Fx_l \in B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$ and so $F^n x_l \in B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$ for all $n \in \mathbb{N}$.

That is, $x_m \in B_b^{\sqsubseteq}[x_l, \frac{\varepsilon}{4s}]$ for all $m \geq l$. For $n, m > l$,

$$\begin{aligned} b(x_n, x_m) &\leq s[b(x_n, x_l) + b(x_l, x_m)] - b(x_l, x_l) \\ &< s[\frac{\varepsilon}{4s} + b(x_l, x_l) + \frac{\varepsilon}{4s} + b(x_l, x_l)] - b(x_l, x_l) \\ &\leq \frac{\varepsilon}{2} + 2sb(x_l, x_l) \\ &\leq \frac{\varepsilon}{2} + 2sb(x_l, x_{l+1}) \\ &< \frac{\varepsilon}{2} + 2s\frac{\varepsilon}{8s^2} < \varepsilon. \end{aligned}$$

Hence, the sequence $\{x_n\}$ is a Cauchy sequence and $b(x_n, x_m) < \varepsilon$, for all $n, m > 1$.

By completeness of X, there exists $u \in X$ such that

$$\lim_{n\to\infty} b(x_n, u) = \lim_{n,m\to\infty} b(x_n, x_m) = b(u, u) = 0. \tag{2}$$

We will show that u is a fixed point of T.

By assumption **(IV)**, we have $x_n \sqsubseteq u$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, by (1), we have

$$\begin{aligned} b(u, Tu) &\leq s[b(u, Tx_n) + b(Tx_n, Tu)] - b(Tx_n, Tx_n) \\ &\leq s[[s\{b(u, x_n) + b(x_n, Tx_n)\} - b(x_n, x_n)] + \lambda b(x_n, u)] - b(Tx_n, Tx_n) \\ &\leq s^2(b(u, x_n)) + s^2(b(x_n, Tx_n)) - s^2(b(x_n, x_n)) + s\lambda b(x_n, u) \\ &= (s^2 + s\lambda)\, b(x_n, u) + s^2 b(x_n, Tx_n) \\ &= (s^2 + s\lambda)\, b(x_n, u) + s^2 b(F^n x_0, TF^n x_0) \\ &= (s^2 + s\lambda)\, b(x_n, u) + s^2 b(F^n x_0, F^n T x_0). \end{aligned} \tag{3}$$

Since $x_0 \sqsubseteq Tx_0$, $F^n x_0 \sqsubseteq F^n T x_0$ for all $n \in \mathbb{N}$. Since F is an ordered Banach contraction with contractive constant λ^{n0}, we have

$$\begin{aligned} b(F^n x_0, F^n T x_0) &= b(FF^{n-1}x_0, FF^{n-1}Tx_0) \\ &\leq \lambda^{n0} b(F^{n-1}x_0, F^{n-1}Tx_0) \\ &\vdots \\ &\leq \lambda^{nn}{}_0 b(x_0, Tx_0). \end{aligned}$$

From (3), we have

$$\begin{aligned} b(u, Tu) &\leq (s^2 + \lambda s)\, b(x_n, u) + S^2 b(F^n x_0, F^n T x_0) \\ &\leq (s^2 + \lambda s)\, b(x_n, u) + s^2 \lambda^{nn}{}_0 b(x_0, Tx_0), \end{aligned}$$

which together with (2) yields $b(u, Tu) = 0$. Therefore, u is a fixed point of T.

For uniqueness, suppose Fix (T) is well ordered and u, v $\in$ Fix (T), then Tu = u, Tv = v. Suppose that b(u, v) > 0. Since Fix (T) is well ordered, assume that u $\sqsubseteq$ v. Now it follows from (1) that b(u, v) = b(Tu, Tu) ≤ λb(u, v) < b(v, u). This contradiction shows that b(u, v) = 0. Thus u = v. Similarly, if v $\sqsubseteq$ u, then u = v. Therefore, fixed point of T is unique. Further, if T has a unique fixed point, then Fix (T) is singleton, and so well ordered. □

The following is a simple example which illustrates the above theorem and shows that the condition of well orderedness of Fix (T) for uniqueness of fixed point is not superfluous. Also it shows that the above theorem is a proper generalization of known results.

Definition 3.3. (Shukla, 2017). Let (X, b, $\sqsubseteq$) be an ordered and complete partial b-metric-like space with coefficient s ≥ 1 and f : X ⟶ X and T : X ⟶ X be two mappings. The mapping T is called an ordered f-contraction if there exists $\lambda \in [0, 1)$ such that

$$x \sqsubseteq y \text{ implies } b(fTx, fTy) \leq \lambda b(fx, fy) \quad (4)$$

for all x, y $\in$ X.

The constant λ is called the contractive constant of T.

Next, we prove a common fixed point result for two mappings as a consequence of Theorem 3.2.

Corollary 3.4 (Shukla, 2017). Let (X, b, $\sqsubseteq$) be an ordered and complete partial b-metric space with coefficient s ≥ 1. Let f : X ⟶ X and T : X ⟶ X be two mappings such that the following conditions hold:

(I) T is an ordered f-contraction;

(II) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx$;

(III) T is non-decreasing with respect to $\sqsubseteq$;

(IV) if $\{fx_n\}$ is a nondecreasing sequence in X and converging to some fz, then $x_n \sqsubseteq z$ for all $n \in \mathbb{N}$.

If f is continuous, injective and sequentially convergent, then T has a fixed point u $\in$ X and b(u, u) = 0. In addition, the set of fixed points of T, Fix (T) is well ordered if and only if the fixed point of T is unique.

Proof. Define $b_1 : X \times X \longrightarrow \mathbb{R}^+$ by $b_1(x, y) = b(fx,fy)$ for all $x, y \in X$.

To show that (X, b_1) is a complete partial b-metric space with same coefficient $s \geq 1$. For all x, y, $z \in X$, we have

(Pb1) $b_1(x, y) = b_1(x, x) = b_1(y, y)$ implies $b(fx, fy) = b(fx, fx) = b(fy, fy)$,

i.e., $fx = fy$ and f is injective, so $x = y$;

(Pb2) $b_1(x, x) = b(fx, fx) \leq b(fx, fy) = b_1(x, y)$;

(Pb3) $b_1(x, y) = b(fx,fy) = b(fy, fx) = b_1(y, x)$;

(Pb4) $b_1(x, y) = b(fx, f y) \leq s[b(fx, fz) + b(fz,fy)] - b(fz, fz) = s [b_1(x, z) + b_1(z, y)] - b_1(z, z)$.

Thus (X, b_1) is a partial b-metric space with coefficient $s \geq 1$.

Let $\{x_n\}$ be a Cauchy sequence in (X, b_1), then $\lim_{n,m\to\infty} b_1(x_n, x_m) = \lim_{n,m\to\infty} b(fx_n, fx_m)$ exists.

Hence, $\{fx_n\}$ is Cauchy sequence in (X, b). Since (X, b) is complete, so there exists $y \in X$ such that

$$\lim_{n\to\infty} b(fx_n, y) = b(y, y).$$

Thus $\{fx_n\}$ is convergent in (X, b). By the property of f, there exists $x \in X$ such that

$$\lim_{n\to\infty} b(x_n, x) = b(x, x).$$

By f, we have

$$\lim_{n\to\infty} b(fx_n, fx) = b(fx, fx) = b_1(x, x).$$

This is, $\lim_{n\to\infty} b_1(x_n, x)$. Thus $\{x_n\}$ converges in (X, b_1) and so it is complete.

Note that, the contractive conditions (a) is reduced into the following condition:

$$(a')\ x \sqsubseteq y \text{ implies } b_1(Tx, Ty) \leq \lambda b_1(x, y)$$

for all $x, y \in X$. Hence T is an ordered Banach contraction in (X, b_1).

By Theorem 3.2., we obtain which lead to T has a fixed point $u \in X$ and $b_1(u, u) = b(fu, fu) = 0$.

Again, the condition for uniqueness follows from Theorem 3.2. □

Remark 3.5 (Shukla, 2017). A pair (T, f) of self-maps of a nonempty set X is called a Banach pair if $Tfx = fTx$ for all $x \in Fix(T)$. If all the conditions of the above corollary are satisfied and in addition, (T, f) is a Banach pair, then T and f have a unique common fixed point. Indeed, if (T, f) is a Banach pair and if T has a unique fixed point $u \in X$ (which is ensured by (Corollary 3.4.), then $Tfu = fTu = fu$, and by uniqueness of fixed point of T, we have $fu = u$. Thus, the pair (T, f) has a unique common fixed point. In the above Corollary, for $u \in Fix(T)$ the self-distance $b_1(u, u) = 0$, but $b(u, u)$ need not be zero, also, when we consider the existence of common fixed point of the pair (T, f), then the condition that (T, f) is a Banach pair cannot be omitted, as shown in the following example.

Theorem 3.6. Let $(X, b, \sqsubseteq)$ be an ordered partial b-metric-like space and $T : X \longrightarrow X$ be a mapping satisfies the following condition:

$$x \sqsubseteq y \text{ implies } b(Tx, Ty) \leq \lambda b(x, y) \tag{5}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose, there exists $u \in X$ such that $u \sqsubseteq Tu$ and $b(u, Tu) \leq b(x, Tx)$ for all $x \in X$. Then u becomes a fixed point of T and $b(u, u) = 0$. In addition, the set of fixed points of T, Fix (T) is well ordered if and only if the fixed point of T is unique.

Proof. Let $F(x) = b(x, Tx)$ for all $x \in X$. By the assumption, we have

$$F(u) \leq F(x) \text{ for all } x \in X. \tag{6}$$

Suppose $F(u) > 0$. Since $u \sqsubseteq Tu$, it follows from (5) that

$$F(Tu) = b(Tu, TTu) \leq \lambda b(u, Tu) = \lambda F(u) < F(u).$$

Then, we have $F(Tu) < F(u)$, which contradicts the inequality (6).

Thus, we must have $F(u) = b(u,Tu) = 0$, That is, $Tu = u$. Hence u is a fixed point of T.

Now, for any fixed point $z \in X$ of T, if $b(z, z) > 0$, then from (5) we have

$$b(z, z) = b(Tz, Tz) \leq \lambda b(z, z) < b(z, z).$$

Lead to contradiction This implies that $b(z, z) = 0$.

For uniqueness, suppose Fix (T) is well ordered and $u, v \in$ Fix (T), $Tu = u$, $Tv = v$ and $b(u, u) = b(v, v) = 0$. Suppose $b(u, v) > 0$. That is since Fix (T) is well ordered,

assume that $u \sqsubseteq v$. From (5) that

$$b(u, v) = b(Tu, Tv) \leq \lambda b(u, v) < b(u, v).$$

Therefore, we must have $b(u, v) = 0$, that is, $u = v$. Similarly, if $v \sqsubseteq u$, then $u = v$. Hence fixed point of T is unique. Further, if fixed point of T is unique then Fix(T) is singleton, and so well ordered. That is

$$x \sqsubseteq y \text{ implies } b(Tx, Ty) \leq \lambda b(x, y) \tag{7}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$.

Suppose, there exists $u \in X$ such that $u \sqsubseteq Tu$ and $b(u, Tu) \leq b(x, Tx)$ for all $x \in X$. Then u becomes a fixed point of T and $b(u, u) = 0$. In addition, the set of fixed points of T, Fix (T) is well ordered if and only if the fixed point of T is unique.

4. Conclusions

Some fixed point theorems in a partial b-metric-like space endowed with a partial order are proved. The results of this paper generalize and extend the Banach contraction principle and some other known results in partial b-metric-like spaces endowed with a partial order.

5. References

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