Regularity of Variants of Semigroups of Full Transformations with Restriction on Fixed set is Bijective

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Abstract

The variant of a semigroup S with respect to an element $a \in S$, is the semigroup with underlying set S and a new operation $*$ defined by $x * y = xay$ for $x, y \in S$. Let $T(X)$ be the full transformation semigroup of the nonempty set X and let

$$PG_Y(X) = \{\alpha \in T(X): \alpha|_Y \in G(Y)\}$$

where $Y \subseteq X$ and $G(Y)$ is the permutation group on Y. In this paper, we investigate regular, left regular and right regular elements for the variant of the semigroup $PG_Y(X)$.

Keywords: regular, left regular, right regular, completely regular

1. Introduction

Let S be a semigroup and a belong to S. We define a new binary operation $*$ on S by putting $x * y = xay$. The operation $*$ is clearly associative. Hence $(S,*)$ is a semigroup and it is called a *varian*t of S. We usually write (S, a) rather than $(S,*)$ to make the element an explicit.

Variants of abstract semigroups were first studied by Hickey (Hickey, 1983). Although variants of concrete semigroups of relations had earlier been considered by Magill (Magill,1967). The study of semigroup variants goes back to the 1960 monograph of Lyapin (Lyapin, 1960) and a 1967 paper of Magill (Magill, 1978) that considers semigroups of functions $X \to Y$ under an operation defined by $f \cdot g = f \circ \theta \circ g$, where θ is some fixed function $Y \to X$. In the case that $X = Y$, this provides an alternative product on the full transformation semigroup $T(X)$ (consisting of all functions $X \to X$).

For an element a of a semigroup S, a is called *regular* if there exists $x \in S$ such that $a = axa$. We call that a semigroup S is *regular* if every element of S is regular. Regular semigroups were introduced by Green (Green, 1951) in his influential 1951 paper "On the structure of semigroups". The concept of regularity in a semigroup which was adapted from an

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analogous condition for rings, already considered by Neumann (Neumann, 1936). It was Green's study of regular semigroups which led him to define his celebrated relations. According to a footnote in Green 1951, the suggestion that the notion of regularity be applied to semigroups was first made by Rees (Rees, 1940-41). This property of regular elements was first observed by Thierrin (Thierrin, 1952) in 1952.

Another important kind of the regularity was introduced by Clifford (Clifford, 1941) in 1941, who studied elements a of a semigroup S having the property that there exists $x \in S$ such that $a = axa$ and $ax = xa$, which we call now *a completely regular element*, and semigroups whose any element is completely regular, are called *completely regular semigroups*. The complete regularity was also investigated by Croisot (Croisot, 1953) in 1953, who also studied elements a of a semigroup S for which $a \in Sa^2$ (resp. $a \in a^2S$), called *left regular* (resp. *right regular*) *elements*, and semigroups whose every element is left regular (resp. right regular), called *left regular* (resp. *right regular*) *semigroups*.

In this paper, we focus the subsemigroup of $T(X)$ defined by

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$$

where Y is a nonempty subset of X and $G(Y)$ is the permutation group on Y. In 2016, Laysirikul (Laysirikul, 2016) proved that $PG_Y(X)$ is a regular semigroup and it is an inverse semigroup if and only if $|X| \leq 2$ or $Y = X$. Moreover, he also described characterizations of left regularity, right regularity, and completely regularity of elements of $PG_Y(X)$.

For a fixed element $\theta \in PG_Y(X)$, the variant semigroup of $PG_Y(X)$ with the sandwich function θ will be denoted by $PG_Y(X, \theta)$. The aim of this paper is to characterize the left regular, the right regular and the completely regular for elements of $PG_Y(X, \theta)$.

2. Main Results

In this section, we let X be an arbitrary set and Y a nonempty subset of X. Define a subset of $T(X)$ as follows:

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$$

where $G(Y)$ is the permutation group on Y. For a semigroup S, we denote

$Reg(S) = \{x \in S : x \text{ is regular}\}$

$LReg(S) = \{x \in S : x \text{ is left regular}\}$

$RReg(S) = \{x \in S : x \text{ is right regular}\}$

$CReg(S) = \{x \in S : x \text{ is completely regular}\}$.

We investigate the condition under which an element in $PG_Y(X,\theta)$ is left regular, right regular and completely regular, respectively. Throughout of the section, the symbol $\pi(\alpha)$ will denote the partition of X induced by $\alpha \in T(X)$ namely,

$$\pi(\alpha) = \{ y\alpha^{-1} : y \in X\alpha\}.$$

Then $\pi(\alpha) = X/\ker(\alpha)$ where $\ker(\alpha) = \{(x,y) \in X \times X : x\alpha = y\alpha\}$.

Theorem 1. Let $\alpha \in PG_Y(X,\theta)$. Then $\alpha \in Reg(PG_Y(X,\theta))$ if and only if $\ker(\alpha) = \ker(\alpha\theta)$ and $X\theta\alpha\theta = X\alpha\theta$.

Proof. Assume that $\alpha \in Reg(PG_Y(X,\theta))$. Then $\alpha = \alpha * \beta * \alpha$ where $\beta \in PG_Y(X,\theta)$ and so $\alpha = \alpha\theta\beta\theta\alpha$. Clearly, $\ker(\alpha) \subseteq \ker(\alpha\theta)$. On the other hand, let $x, y \in X$ be such that $x\alpha\theta = y\alpha\theta$. Then $x\alpha = x\alpha\theta\beta\theta\alpha = y\alpha\theta\beta\theta\alpha = y\alpha$. Hence $\ker(\alpha) = \ker(\alpha\theta)$. It is clear that $X\theta\alpha\theta \subseteq X\alpha\theta$ and noting that $X\alpha\theta = X\alpha\theta\beta\theta\alpha\theta \subseteq X\theta\alpha\theta$, therefore $X\theta\alpha\theta = X\alpha\theta$.

Conversely, suppose that $\ker(\alpha) = \ker(\alpha\theta)$ and $X\theta\alpha\theta = X\alpha\theta$. We obtain via $\alpha, \theta \in PG_Y(X,\theta)$ that $X\theta\alpha\theta = Y \cup (X\theta\alpha\theta \setminus Y)$. For any $y \in Y$, there exists only one $y' \in Y$ such that $y = y'\theta\alpha\theta$ since $(\theta\alpha\theta)|_Y$ is bijective. For any $x \in X\theta\alpha\theta \setminus Y$, we choose and fix $x' \in X$ such that $x = x'\theta\alpha\theta$. Define $\beta: X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in X\theta\alpha\theta, \\ x & \text{otherwise.} \end{cases}$$

It is obviously, $\beta|_Y \in G(Y)$, that is $\beta \in PG_Y(X,\theta)$. It remains to verify that $\alpha * \beta * \alpha = \alpha$. If $x \in X$, then $x\alpha\theta \in X\alpha\theta = X\theta\alpha\theta$ and $x\alpha\theta\beta\theta\alpha = (x\alpha\theta)'\theta\alpha$ with $(x\alpha\theta)'\theta\alpha\theta = x\alpha\theta$. Thus $((x\alpha\theta)'\theta, x) \in \ker(\alpha\theta) = \ker(\alpha)$. This implies that $x\alpha\theta\beta\theta\alpha = (x\alpha\theta)'\theta\alpha = x\alpha$ and therefore $\alpha = \alpha * \beta * \alpha$. Hence α is regular in $PG_Y(X,\theta)$. ■

Theorem 2. Let $\alpha \in PG_Y(X,\theta)$. Then $\alpha \in LReg\big(PG_Y(X,\theta)\big)$ if and only if $X\alpha = X(\theta\alpha)^2$.

Proof. Assume that $\alpha \in LReg\big(PG_Y(X,\theta)\big)$. Then $\alpha = \beta * \alpha * \alpha$ for some $\beta \in PG_Y(X,\theta)$, so that $\alpha = \beta\theta\alpha\theta\alpha$. Let $y \in X\alpha$. Then there exists $x \in X$ such that $y = x\alpha = x\beta\theta\alpha\theta\alpha \in X\theta\alpha\theta\alpha$. On the other hand, it is clear that $X\theta\alpha\theta\alpha \subseteq X\alpha$. Hence $X\alpha = X(\theta\alpha)^2$.

Conversely, assume that $X\alpha = X(\theta\alpha)^2$. We note that $(\theta\alpha\theta)|_Y$ is a bijection. Thus, for any $y \in Y$, there exists a unique $y' \in Y$ such that $y = y'\theta\alpha\theta$ and so $y\alpha = y'\theta\alpha\theta\alpha$. Let $x \in X \setminus Y$. Then by hypothesis, we choose $x' \in X$ such that $x\alpha = x'\theta\alpha\theta\alpha$. Define $\beta : X \to X$ by $x\beta = x'$ for all $x \in X$. We will show that $Y\beta = Y$. It is clear that $Y\beta \subseteq Y$, by the definition of β. For the inverse inclusion, let $y \in Y$. Since $Y\theta\alpha\theta = Y$, we obtain $y\theta\alpha\theta = x$ for some $x \in Y$, which is $y\theta\alpha\theta\alpha = x\alpha$. Noting that $x\beta = x'$ where $x\alpha = x'\theta\alpha\theta\alpha$. It follows by the uniqueness of x' that $y = x' = x\beta \in X\beta$ and hence $Y\beta = Y$. Let $x, y \in Y$ be such that $x\beta = y\beta$. Then $x' = y'$, which implies $x = x'\alpha = y'\alpha = y$. Therefore $\beta|_Y$ is injective. Hence $\beta \in PG_Y(X,\theta)$. Finally, for all $x \in X$, by the definition of x', we get that $x(\beta * \alpha * \alpha) = x\beta\theta\alpha\theta\alpha = x'\theta\alpha\theta\alpha = x\alpha$. So, we have the complete proof. ∎

Next, we give a characterization of right regular elements in $PG_Y(X,\theta)$.

Theorem 3. Let $\alpha \in PG_Y(X,\theta)$. Then $\alpha \in RReg\big(PG_Y(X,\theta)\big)$ if and only if $(\theta\alpha\theta)|_{X\alpha}$ is an injection.

Proof. Suppose that $\alpha \in RReg\big(PG_Y(X,\theta)\big)$. Then there exists $\beta \in PG_Y(X,\theta)$ such that $\alpha = \alpha * \alpha * \beta$ and so $\alpha = \alpha\theta\alpha\theta\beta$. Let $x, y \in X\alpha$ be such that $x = x'\alpha$ and $y = y'\alpha$ where $x', y' \in X$. If $x\theta\alpha\theta = y\theta\alpha\theta$, then $x = x'\alpha = x'\alpha\theta\alpha\theta\beta = x\theta\alpha\theta\beta = y\theta\alpha\theta\beta = y'\alpha\theta\alpha\theta\beta = y'\alpha = y$. It follows that $(\theta\alpha\theta)|_{X\alpha}$ is an injection.

Conversely, assume that $(\theta\alpha\theta)|_{X\alpha}$ is an injection. For any $x \in X\alpha\theta\alpha\theta$, there exists a unique element $x' \in X\alpha$ such that $x = x'\theta\alpha\theta$ by the condition $(\theta\alpha\theta)|_{X\alpha}$ is injective. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in X\alpha\theta\alpha\theta, \\ x & \text{otherwise.} \end{cases}$$

By the uniqueness of x', we get that $\beta|_Y$ is an injection. Next, we will show that $Y\beta = Y$, let $y \in Y$. Since $\alpha\theta\alpha\theta \in PG_Y(X,\theta)$, $X\alpha\theta\alpha\theta = Y \cup (X\alpha\theta\alpha\theta \setminus Y)$. Then by the assumption of y', we get that $y\beta = y' \in Y$. On the other hand, let $y \in Y$. Since $Y = Y\alpha\theta\alpha\theta = Y\theta\alpha\theta$, we obtain that $y = x\theta\alpha\theta$ for some $x \in Y$. By the definition of β, we get $x\beta = x'$ where $x = x'\theta\alpha\theta$. This implies that $x' = y$ by the uniqueness of x'. Consequently, $\beta \in PG_Y(X,\theta)$.

Finally, to show that $\alpha = \alpha\theta\alpha\theta\beta$, let $x \in X$, so $x\alpha\theta\alpha\theta \in X\alpha\theta\alpha\theta$. By the definition of β, $x\alpha\theta\alpha\theta\beta = (x\alpha\theta\alpha\theta)'$ where $(x\alpha\theta\alpha\theta)'\theta\alpha\theta = x\alpha\theta\alpha\theta = (x\alpha)\theta\alpha\theta$. Since $(x\alpha\theta\alpha\theta)'$ is unique, we get that $(x\alpha\theta\alpha\theta)' = x\alpha$. Thus $x\alpha\theta\alpha\theta\beta = (x\alpha\theta\alpha\theta)' = x\alpha$. Therefore $\alpha = \alpha * \alpha * \beta$. Hence $\alpha \in RReg\big(PG_Y(X,\theta)\big)$, as asserted. ∎

Corollary 4. Let $\alpha \in RReg\big(PG_Y(X,\theta)\big)$. If $P \cap X\alpha \neq \emptyset$ for some $P \in \pi(\theta\alpha\theta)$, then $|P \cap X\alpha| = 1$.

Proof. Let $P \in \pi(\theta\alpha\theta)$ be such that $|P \cap X\alpha| > 1$. Then there exist two distinct elements x and y in $P \cap X\alpha$. Thus $x\theta\alpha\theta = y\theta\alpha\theta$. By Theorem 3, we get that $x = y$. This is a contradiction, hence $|P \cap X\alpha| = 1$. ∎

Final of this section, we give a characterization of completely regular elements in $PG_Y(X,\theta)$. Recall that, an element a of a semigroup S is completely regular if and only if a is both left and right regular (Petrich, 1999). Hence, as an immediate consequence of Theorem 2 and Theorem 3, we have the following.

Theorem 5. Let $\alpha \in PG_Y(X,\theta)$. Then $\alpha \in CReg\big(PG_Y(X,\theta)\big)$ if and only if $X\alpha = X(\theta\alpha)^2$ and $(\theta\alpha\theta)|_{X\alpha}$ is an injection.

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