Natural Partial Order on Semigroups of Transformations Preserving an Equivalence Relation and a Cross-Section with Fixed Sets

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Abstract

Let $T(X,\rho,R)$ be the semigroup consisting of all total transformations preserving an equivalence relation ρ and a cross-section R. The subsemigroup $T_Y(X,\rho,R)$ of $T(X,\rho,R)$ is defined as follows:

$$T_Y(X,\rho,R) = \{\alpha \in T(X,\rho,R) : x\alpha = x \text{ for all } x \in X \text{ in which } x\rho \cap Y \neq \emptyset\}.$$

In this paper, we characterize the natural partial order and find elements which are left compatible, right compatible and compatible.

Keywords: transformation semigroup, natural partial order, compatibility, equivalence relation, cross-section

1. Introduction

For a nonempty set X, denote by $T(X)$ a semigroup of total transformations on X which are functions from X into X itself. Generally, the semigroup $T(X)$ has been extensively studied since it can be viewed as a generalization of any semigroup. Several authors considered various subsemigroups of $T(X)$ and more prominent results were discovered. One of those subsemigroups we study in this paper is related to an equivalence relation and a cross-section. Let ρ be an equivalence relation on X and R a cross-section of a partition X/ρ such that each ρ-class contains exactly one element of R. A subsemigroup $T(X,\rho,R)$ of $T(X)$ is defined by

$$T(X,\rho,R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } (x,y) \in \rho \Longrightarrow (x\alpha, y\alpha) \in \rho\}.$$

Many interesting properties of $T(X,\rho,R)$ have been presented, widely. For example, in 2003, Araújo and Konieczny (Araújo & Konieczny, 2003) proved that $T(X,\rho,R)$ is a centralizer of idempotent transformations with kernel ρ and image R. Further, they investigated an automorphism group of $T(X,\rho,R)$. After that, in 2004, they (Araújo & Konieczny, 2004) continued

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to study on structures of $T(X,\rho,R)$ via Green's relations and regularity. In 2011, Sun et al. (Sun et al., 2011) constructed the natural partial order on $T(X,\rho,R)$ and characterized maximal elements, minimal elements and covering elements.

Another subsemigroup of $T(X,\rho,R)$ which corresponds to a fixed nonempty subset Y of X is defined as follows:

$$T_Y(X,\rho,R) = \{\alpha \in T(X,\rho,R) : x\alpha = x \text{ for all } x \in X \text{ in which } x\rho \cap Y \neq \emptyset\}.$$

The subsemigroup $T_Y(X,\rho,R)$ of $T(X,\rho,R)$ is called a semigroup of transformations preserving an equivalence relation ρ on X and a cross-section R of X/ρ with a fixed subset Y of X. An important of study the semigroup $T_Y(X,\rho,R)$ is that $T_Y(X,\rho,R)$ can be considered as a generalization of $Fix(X,Y)$, a subsemigroup of $T(X)$ which is introduced by Honyam and Sanwong (Honyam & Sanwong, 2013) in 2013. Recently, in 2021, Nupo and Pookpienlert (Nupo & Pookpienlert, 2021) presented the regularity on $T_Y(X,\rho,R)$ and enumerated the number of elements in $T_Y(X,\rho,R)$ corresponding to such regularity.

Let S be a semigroup and $E(S)$ a set of idempotents in S. The natural partial order on $E(S)$ is defined by

$$e \leq f \text{ if and only if } e = ef = fe.$$

In 1980, Hartwig (Hartwig, 1980) studied the natural partial order between regular elements in a semigroup S where a regular element $x \in S$ is an element satisfied that $x = xyx$ for some $y \in S$. In the same year, Nambooripad (Nambooripad, 1980) investigated the natural partial order on a regular semigroup S which is shown as follows:

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e,f \in E(S).$$

Later in 1986, Mitsch (Mitsch, 1986) presented the natural partial order on a semigroup S which is stated as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } xa = a \text{ for some } x,y \in S^1$$
$$\text{if and only if } a = xb = by \text{ and } ay = a \text{ for some } x,y \in S^1$$

where S^1 is a semigroup obtained by joining an identity to S if S does not have an identity and $S^1 = S$ if S already contains an identity.

Again in 1986, Kowol and Mitsch (Kowol & Mitsch, 1986) presented a characterization of the natural partial order on $T(X)$ in terms of images and kernels. Moreover, they also illustrated the maximality and minimality for elements in $T(X)$. In 2016, Chaiya et al. (Chaiya et al., 2016) provided the necessary and sufficient conditions for the natural partial order on $Fix(X,Y)$. Moreover, they investigated the compatibility, minimality and maximality of elements in $Fix(X, Y)$. For further references concerned with the natural partial order on semigroups, the readers are suggested to (Pei & Deng, 2013), (Sawatraksa et al., 2019) and (Sun & Sun, 2013).

In this paper, we focus on the semigroup $T_Y(X, \rho, R)$. We aim to present the characterization of the natural partial order on $T_Y(X, \rho, R)$ and investigate the compatibility for elements in $T_Y(X, \rho, R)$.

2. Preliminaries and Notations

We now provide relevant preliminaries in this section. Some basic mathematical terminologies and important notations used in what follows on semigroups are prescribed. Further, we refer to (Clifford & Preston, 1961) and (Howie, 1995) for more information about semigroups.

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. If for each set $A \in \mathcal{A}$, there is a set $B \in \mathcal{B}$ such that $A \subseteq B$, then we say that $\mathcal{A}$ refines $\mathcal{B}$ and is denoted by $A \hookrightarrow \mathrm{B}$. For each $\alpha \in T_Y(X, \rho, R)$, we denote by $\ker(\alpha)$, the kernel of α, the set of ordered pairs in $X \times X$ having the same image under α, that is,

$$\ker(\alpha) = \{(a, b) \in X \times X : a\alpha = b\alpha\}.$$

Moreover,

$$\blacktriangledown\alpha = \{(r\rho)\alpha : r \in R\} \text{ and } \blacktriangledown_Y\alpha = \{(r\rho)\alpha : r \in R \text{ and } r\rho \cap Y = \emptyset\}.$$

Let A be a nonempty subset of X. An equivalence relation ρ on X induces a partition A/ρ of A as follows:

$$A/\rho = \{x\rho \cap A : x \in X \text{ and } x\rho \cap A \neq \emptyset\}.$$

Let $\mathrm{A}_Y = \{x \in X : x\rho \cap Y = \emptyset\}$. Then an equivalence relation ρ on X induces an equivalence relation ρ^* on A_Y by

$$\rho^* = \rho \cap (\mathrm{A}_Y \times \mathrm{A}_Y).$$

Moreover, the notation

$$A/\rho^* = \{x\rho^* \cap A : x \in \mathrm{A}_Y \text{ and } x\rho^* \cap A \neq \emptyset\}$$

can be an empty set when $A \cap \mathrm{A}_Y = \emptyset$. On the other hand, if $A \subseteq \mathrm{A}_Y$, then an equivalence relation ρ^* on A_Y induces a partition A/ρ^* of A.

Obviously, the set A/ρ^* can be rewritten as

$$A/\rho^* = \{r\rho \cap A : r \in R \text{ and } r\rho \cap Y = \emptyset \neq r\rho \cap A\}.$$

We now introduce the following lemmas and theorem, which are useful for proving the natural partial order and compatibility on $T_Y(X, \rho, R)$.

Lemma 1. (Nupo & Pookpienlert, 2021) Let $\alpha, \beta \in T_Y(X, \rho, R)$. Then $\alpha = \gamma\beta$ for some $\gamma \in T_Y(X, \rho, R)$ if and only if $\blacktriangledown_Y \alpha \hookrightarrow \blacktriangledown \beta$.

Lemma 2. (Nupo & Pookpienlert, 2021) Let $\alpha, \beta \in T_Y(X, \rho, R)$. Then $\alpha = \beta\gamma$ for some $\gamma \in T_Y(X, \rho, R)$ if and only if $\ker(\beta) \subseteq \ker(\alpha)$.

Theorem 3. (Nupo & Pookpienlert, 2021) Let $\alpha \in T_Y(X, \rho, R)$. Then α is regular if and only if $X\alpha/\rho^* \subseteq \blacktriangledown \alpha$.

3. Main Results

The characterization of the natural partial order on $T_Y(X, \rho, R)$ is presented as follows.

Theorem 4. Let $\alpha, \beta \in T_Y(X, \rho, R)$. Then $\alpha \leq \beta$ if and only if the following statements hold:

(i) $\blacktriangledown_Y \alpha \hookrightarrow \blacktriangledown \beta$,

(ii) $\ker(\beta) \subseteq \ker(\alpha)$,

(iii) for each $a \in X$, if $a\beta \in X\alpha$, then $a\beta = a\alpha$.

Proof. Assume that $\alpha \leq \beta$. Then $\alpha = \gamma\beta = \beta\mu$ and $\alpha\mu = \alpha$ for some $\gamma, \mu \in T_Y(X, \rho, R)$. By Lemmas 1 and 2, we obtain that (i) and (ii) hold. Let $a \in X$ be such that $a\beta \in X\alpha$. Then $a\beta = b\alpha$ for some $b \in X$. It follows that $a\alpha = a\beta\mu = b\alpha\mu = b\alpha = a\beta$ and so (iii) holds.

Conversely, assume that conditions (i) – (iii) hold. Again by Lemmas 1 and 2, we have $\alpha = \gamma\beta = \beta\mu$ for some $\gamma, \mu \in T_Y(X, \rho, R)$. Let $a \in X$. Then $a\alpha = a\gamma\beta = b\beta$ for some $b \in X$. Since $b\beta \in X\alpha$, by (iii), we obtain that $b\beta = b\alpha$ and thus $a\alpha\mu = b\beta\mu = b\alpha = b\beta = a\alpha$. That is $\alpha\mu = \alpha$. Therefore, $\alpha \leq \beta$. ■

In order to describe the natural partial order on $T_Y(X, \rho, R)$ more clearly, the following figure is illustrated.

Figure 1 shows the Hasse diagram of $T_Y(X, \rho, R)$ when $X = \{1,2,3,4,5\}$, $Y = \{2\}$, $X/\rho = \{\{1,2\}, \{3\}, \{4,5\}\}$ and $R = \{1,3,4\}$. For convenience in drawing the Hasse diagram, the notation $(abcde)$ for $\alpha \in T_Y(X, \rho, R)$ means that $1\alpha = a, 2\alpha = b, 3\alpha = c, 4\alpha = d$ and $5\alpha = e$.

We can observe that if $\alpha = (12133)$ and $\beta = (12433)$, then $\blacktriangledown_Y \alpha = \{\{1\}, \{3\}\} \hookrightarrow \{\{1,2\}, \{3\}, \{4\}\} = \blacktriangledown \beta$ and $\ker(\beta) = \{(a, a) : a \in X\} \cup \{(4,5), (5,4)\} \subseteq \{(a, a) : a \in X\} \cup \{(1,3), (3,1), (4,5), (5,4)\} = \ker(\alpha)$. Moreover, for each $a \in X$, if $a\beta \in X\alpha = \{1,2,3\}$, then $a\beta = a\alpha$. By Theorem 4, we obtain that $\alpha \leq \beta$. On the other hand, we can see that $\blacktriangledown_Y \alpha$ does not refine $\blacktriangledown \gamma = \{\{1,2\}, \{4\}\}$ when $\gamma = (12412)$. Therefore, $\alpha \nleq \gamma$ by Theorem 4.

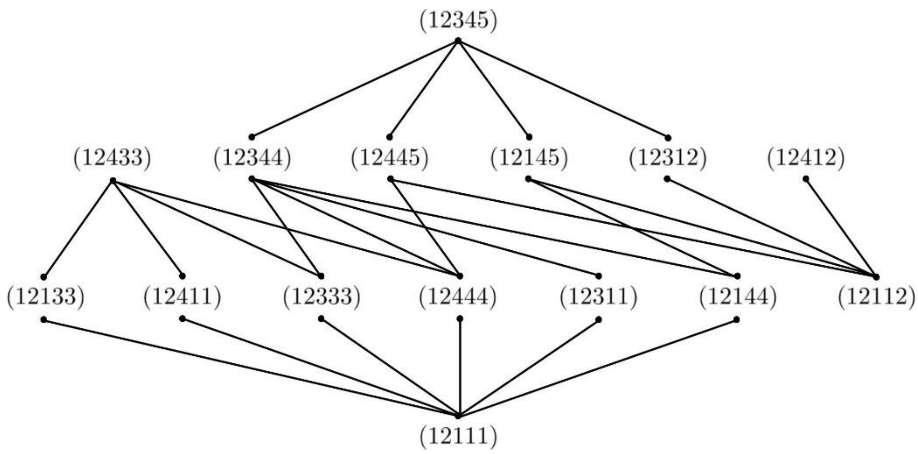

Figure 1 The Hasse diagram of $T_Y(X,\rho,R)$.

Let S be a semigroup together with a partial order $\leq$. An element $a \in S$ is said to be left (right) compatible if $b \leq c$, then $ab \leq ac$ ($ba \leq ca$). Moreover, $a \in S$ is said to be compatible if and only if a is both left and right compatible.

In order to determine elements which are left compatible, the following lemma is needed.

Lemma 5. Let $\alpha \in T_Y(X,\rho,R)$. Then α is regular and surjective if and only if for each $r \in R$, there exists $s \in R$ such that $(s\rho)\alpha = r\rho$.

Proof. Assume that α is regular and surjective. Let $r \in R$. If $r\rho \cap Y \neq \emptyset$, then $(r\rho)\alpha = r\rho$. If $r\rho \cap Y = \emptyset$, then $r\rho = r\rho \cap X = r\rho \cap X\alpha \in X\alpha/\rho^* \subseteq \blacktriangledown\alpha$ by Theorem 3. So $r\rho = (s\rho)\alpha$ for some $s \in R$.

Conversely, assume that the statement holds. Now, we first prove that α is surjective. Let $a \in X$. Then $a \in r\rho = (s\rho)\alpha$ for some $r, s \in R$. It follows that $a = b\alpha$ for some $b \in s\rho$ and so α is surjective. Let $r\rho \cap X\alpha \in X\alpha/\rho^*$. Since α is surjective, we obtain that $r\rho \cap X\alpha = r\rho = (s\rho)\alpha \in \blacktriangledown\alpha$ for some $s \in R$. Therefore, $X\alpha/\rho^* \subseteq \blacktriangledown\alpha$ which implies that α is regular by Theorem 3. ■

Theorem 6. Let $\gamma \in T_Y(X,\rho,R)$. If γ is regular and surjective, then γ is left compatible.

Proof. Assume that γ is regular and surjective. Let $\alpha, \beta \in T_Y(X,\rho,R)$ be such that $\alpha \leq \beta$. Now, we prove that $\gamma\alpha \leq \gamma\beta$ by applying Theorem 4 as follows.

(i) Let $(r\rho)\gamma\alpha \in \blacktriangledown_Y\gamma\alpha$. Then $(r\rho)\gamma\alpha \subseteq (s\rho)\alpha$ for some $s \in R$. If $s\rho \cap Y \neq \emptyset$, then $(r\rho)\gamma\alpha \subseteq (s\rho)\alpha = s\rho = (s\rho)\gamma\beta \in \blacktriangledown\gamma\beta$. If $s\rho \cap Y = \emptyset$, then $(r\rho)\gamma\alpha \subseteq (s\rho)\alpha \subseteq (t\rho)\beta =$

$((u\rho)\gamma)\beta = (u\rho)\gamma\beta \in \blacktriangledown\gamma\beta$ for some $t, u \in R$ by Theorem 4 (i) and Lemma 5. Thus $\blacktriangledown_Y\gamma\alpha \hookrightarrow \blacktriangledown\gamma\beta$.

(ii) Let $(a, b) \in \ker(\gamma\beta)$. Then $a\gamma\beta = b\gamma\beta$ and so $(a\gamma, b\gamma) \in \ker(\beta)$. By Theorem 4 (ii), we obtain that $(a\gamma, b\gamma) \in \ker(\alpha)$. It follows that $a\gamma\alpha = b\gamma\alpha$ and thus $(a, b) \in \ker(\gamma\alpha)$. Hence $\ker(\gamma\beta) \subseteq \ker(\gamma\alpha)$.

(iii) Let $a\gamma\beta \in X\gamma\alpha$. Then $(a\gamma)\beta = a\gamma\beta \in X\alpha$. By Theorem 4 (iii), we obtain that $a\gamma\beta = (a\gamma)\beta = (a\gamma)\alpha = a\gamma\alpha$.

Therefore, $\gamma\alpha \leq \gamma\beta$ which implies that γ is left compatible. ■

Next, we purpose the result of right compatibility of elements in $T_Y(X, \rho, R)$.

Theorem 7. Let $\gamma \in T_Y(X, \rho, R)$. If γ is injective, then γ is right compatible.

Proof. Assume that γ is injective. Let $\alpha, \beta \in T_Y(X, \rho, R)$ be such that $\alpha \leq \beta$. Now, we prove that $\alpha\gamma \leq \beta\gamma$ by applying Theorem 4 as follows.

(i) Let $(r\rho)\alpha\gamma \in \blacktriangledown_Y\alpha\gamma$. Since $\blacktriangledown_Y\alpha \hookrightarrow \blacktriangledown\beta$, we obtain that $(r\rho)\alpha \subseteq (s\rho)\beta$ for some $s \in R$. It follows that $(r\rho)\alpha\gamma = ((r\rho)\alpha)\gamma \subseteq ((s\rho)\beta)\gamma = (s\rho)\beta\gamma \in \blacktriangledown\beta\gamma$ and so $\blacktriangledown_Y\alpha\gamma \hookrightarrow \blacktriangledown\beta\gamma$.

(ii) Let $(a, b) \in \ker(\beta\gamma)$. Then $a\beta\gamma = b\beta\gamma$. From γ is injective, we obtain that $a\beta = b\beta$. Since $\ker(\beta) \subseteq \ker(\alpha)$, we obtain that $a\alpha = b\alpha$. It follows that $a\alpha\gamma = b\alpha\gamma$ and so $(a, b) \in \ker(\alpha\gamma)$. Hence $\ker(\beta\gamma) \subseteq \ker(\alpha\gamma)$.

(iii) Let $a\beta\gamma \in X\alpha\gamma$. Then $a\beta\gamma = b\alpha\gamma$ for some $b \in X$ which implies that $a\beta = b\alpha \in X\alpha$. By Theorem 4 (iii), we obtain that $a\beta = a\alpha$. Thus $a\beta\gamma = a\alpha\gamma$.

Therefore, $\alpha\gamma \leq \beta\gamma$ which implies that γ is right compatible. ■

Finally, we present the sufficient condition to obtain the compatibility of elements in $T_Y(X, \rho, R)$.

Theorem 8. Let $\gamma \in T_Y(X, \rho, R)$. If γ is bijective, then γ is compatible.

Proof. Assume that γ is bijective. Now, we prove that γ is regular. Let $r\rho \cap X\gamma \in X\gamma/\rho^*$. Since γ is a bijection and preserves a cross-section, there is exactly one element $s \in R$ such that $(s\rho)\gamma = r\rho$. It follows that $r\rho \cap X\gamma = r\rho \cap X = r\rho = (s\rho)\gamma \in \blacktriangledown\gamma$ and so $X\gamma/\rho^* \subseteq \blacktriangledown\gamma$. By Theorem 3, we obtain that γ is a regular element. Therefore, γ is compatible by Theorems 6 and 7. ■

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