Enumeration of Numerical Semigroups $\{0\} \cup [a, b] \cup [c, \infty)$ with Same Embedding Dimension

Chaiwat Namnak, Chonlada Phanmun, Ekkachai Laysirikul*

Department of Mathematics, Faculty of Sciences, Naresuan University

Abstract

We consider the set S defined as $\{0\} \cup [a, b] \cup [c, \infty)$ where $[a, b]$ be the set of all integers x such that $a \leq x \leq b$ and $[c, \infty)$ be the set of integers y such that $c \leq y$ when $a, b,$ and c are positive integers satisfying $2 \leq a \leq b < c - 1$. It is known that S is a numerical semigroup if and only if $c \leq 2a$. This research aims to characterize the minimal system of generators for numerical semigroups S and determine the count of numerical semigroups $\{0\} \cup [a, b] \cup [c, \infty)$ that share the same embedding dimension.

Keywords: numerical semigroup, minimal system of generators, embedding dimension

1. Introduction

Ferdinand Georg Frobenius (1849-1917) made significant contributions during the 19th century by introducing the Frobenius problem. This problem revolves around determining the largest positive integer that cannot be expressed as a solution to a linear equation with non-negative coefficients. Referred to as the Frobenius number, this value holds great importance in the field. Another concept related to this problem is the genus number, which represents the count of positive integers that cannot be expressed as a linear combination under the specified constraints.

In the case of two variables, Sylvester (Sylvester, 1884) proposed exact formulas for the Frobenius number and genus number. However, for three or more variables, Curtis (Curtis, 1990) proved that there is no formula in polynomial form for computing the Frobenius number.

In the 1950, the concept of numerical semigroups gained attention, particularly due to its applications in algebraic geometry. The Frobenius problem can be related to the coin problem, where the aim is to determine the largest amount of money that cannot be obtained

* *Corresponding author : ekkachail@nu.ac.th*

Received: 23 พ.ค. 66; Revised: 16 ต.ค. 66; Accepted: 1 ธ.ค. 66

using specific coins. This concept aligns with numerical semigroups, where every achievable number using the given coins belongs to a numerical semigroup.

A non-empty subset S of the set of non-negative integers $\mathbb{N}$ is referred to as a numerical semigroup if it satisfies the following conditions: it includes 0, it forms a semigroup under usual addition, and the complement set $\mathbb{N}\backslash S$ is finite. For a numerical semigroup S, a system of generators A is a finite subset of positive integers such that the linear combinations of the elements in A generate S, denoted as $\langle A \rangle = S$. Moreover, if for each $a \in A$ implies $a \notin \langle A \setminus \{a\} \rangle$, then A is considered the minimal system of generators for S. In this case, the cardinality of A represents the embedding dimension of S and denoted by $\mathbf{dim}(S)$. Rosales and Garcia-Sanchez (Rosales & Garcia-Sanchez, 2009) demonstrated that every numerical semigroup possesses a unique minimal system of generators.

The Frobenius number and genus number of a numerical semigroup S are denoted by $F(S)$ and $g(S)$, respectively. $F(S)$ represents the largest element in the complement set $\mathbb{N}\backslash S$, while $g(S)$ corresponds to the cardinality of $\mathbb{N}\backslash S$. The Frobenius problem can be formulated as finding the Frobenius number of a numerical semigroup $\langle A \rangle$ with $\mathbf{gcd}(A) = 1$.

In the case of embedding dimension two, Sylvester (Sylvester, 1884) provided a simple exact formula for the Frobenius number and genus number. However, it took nearly a century to develop methods and algorithms for finding the exact solution of the Frobenius number in case of embedding dimension three. Various formulas and techniques have been proposed, such as those by (Sylvester, 1884), (Rosales & Garcia-Sanchez, 2009), (Davison, 1994), and more recently, (Tripathi, 2017).

A few years ago, Chommi (Chommi, 2020) introduced a novel numerical semigroup, where the set S is defined as $\{0\} \cup [a, b] \cup [c, \infty)$. Here, $[a, b]$ represents the set of all integers x such that $a \leq x \leq b$, and $[c, \infty)$ represents the set of integer values y such that $c \leq y$. The author proved that a set S becomes a numerical semigroup if and only if $c \leq 2a$, with a, b and c being positive integers satisfying $2 \leq a \leq b < c - 1$. The author also explored the characterization of irreducible numerical semigroups $\{0\} \cup [a, b] \cup [c, \infty)$.

In the same year, Phosri (Phosri, 2020) conducted a study on the enumeration of numerical semigroups $\{0\} \cup [a,b] \cup [c,\infty)$ that share the same Frobenius number. The researcher also counted the number of k-symmetric numerical semigroup that share the same Frobenius number for k values of $3, 4$ and 5. Likewise, Kosasirisin (Kosasirisin, 2020) focused on k-symmetric numerical semigroups $\{0\} \cup [a,b] \cup [c,\infty)$ sharing the same Frobenius number, considering $k \geq 3$. Additionally, the author determined the count of 3-symmetric numerical semigroups $\{0\} \cup [a,b] \cup [c,\infty)$ with the same genus number.

Motivated by the aforementioned works (Chommi, 2020), (Phosri, 2020), and (Kosasirisin, 2020), this research is inspired to investigate the set $S = \{0\} \cup [a,b] \cup [c,\infty)$. Our objectives are twofold: first, we aim to identify the minimal system of generators for S, and secondly, we endeavor to determine the count of numerical semigroups $\{0\} \cup [a,b] \cup [c,\infty)$ sharing the same embedding dimension.

2. Main Results

In this research, we consider the set $S = \{0\} \cup [a,b] \cup [c,\infty)$ where a, b, c are positive integers satisfying $2 \leq a \leq b < c-1$. It is noteworthy that S is a numerical semigroup if and only if $c \leq 2a$. Our investigation begins with a characterization of the minimal system of generators for S. Additionally, we quantify the number of numerical semigroups $\{0\} \cup [a,b] \cup [c,\infty)$ that share the same embedding dimension.

Theorem 1. Let $S = \{0\} \cup [a,b] \cup [c,\infty)$ be a numerical semigroup. Then S is generated by the set $A = [a,b] \cup [c, c+a) \setminus [2a, 2b]$.

Proof. Clearly, $0 = 0 \cdot a \in \langle A \rangle$. We let $x \in S$ be such that $0 < x$ and $x \notin A$. Thus $x \notin [a,b]$ and $x \notin [c, c+a) \setminus [2a, 2b]$. From $x \in S$, we get $c \leq x$. There are 2 cases to consider.

Case 1 : $x \in [2a, 2b]$. Then $a \leq \frac{x}{2} \leq b$. By the division algorithm, there exist $k, r \in \mathbb{N}$ such that $x = 2k + r$ and $0 \leq r \leq 1$. If $r = 0$, then $x = 2k$ and $a \leq k \leq b$. That is $x \in \langle A \rangle$. Now, we suppose that $r = 1$. Since $a \leq \frac{x}{2} \leq b$, we obtain $a \leq k < b$. So $k + 1 \leq b$ and hence $k + 1 \in A$. Thus $x = 2k + 1 = (k+1) + k \in \langle A \rangle$.

Case 2 : $x \notin [2a, 2b]$. Since $x \notin [c, c+a)$, we get $c + a \leq x$. We note from $x \notin [2a, 2b]$ that $x < 2a$ or $2b < x$. If $x < 2a$, then $c + a \leq x < 2a$. This means that $c < a$ which is impossible. Hence $2b < x$. From $c + a \leq x$, we have $a \leq x - c$. By the division algorithm,

$$x - c = ak + r$$

where $k, r \in \mathbb{N}$ and $0 \leq r < a$. Since $x - c \geq a$, we have $1 \leq k$. Note that $x = ak + c + r$ and $c \leq c + r < c + a$. So $c + r \in [c, c + a)$. If $c + r \in [2a, 2b]$, then $c + r \notin A$. From Case 1, we get $c + r \in \langle A \rangle$. Hence $x = ak + c + r \in \langle A \rangle$. If $c + r \notin [2a, 2b]$, then $c + r \in [c, c + a) \setminus [2a, 2b] \subseteq A$. That is $x = ak + c + r \in \langle A \rangle$.

It follows from 2 cases that $S \subseteq \langle A \rangle$. Clearly, $A \subseteq S$ which implies $\langle A \rangle \subseteq S$. Therefore, $S = \langle A \rangle$. ■

Theorem 2. Let $S = \{0\} \cup [a, b] \cup [c, \infty)$ be a numerical semigroup and $A = [a, b] \cup [c, c + a\,) \setminus [2a, 2b]$. Then A is the minimal system of generators for S.

Proof. It follows from Theorem 1 that $S = \langle A \rangle$. To prove the minimality of A, we suppose that there exists $B \subsetneq A$ such that $\langle B \rangle = S$. Then we choose $x \in A \setminus B$. It is clear that $x \in \langle B \rangle$. So there exist $b_1, b_2, \ldots, b_k \in B \subseteq A$ and $n_1, n_2, \ldots, n_k \in \mathbb{N} \setminus \{0\}$ such that

$$x = n_1 b_1 + n_2 b_2 + \cdots + n_k b_k.$$

Note that $x \neq b_i$, for all $i = 1, \ldots, k$. From $b_i \in A$, we obtain that $a \leq b_i$ for all $i = 1, .., k$. There are 2 cases to consider.

Case 1 : $x \in [a, b]$. If $1 < k$, then $2a \leq b_1 + b_2 \leq n_1 b_1 + n_2 b_2 + \cdots + n_k b_k = x$ which is a contradiction with $x \leq b < c \leq 2a$. Hence $k = 1$ that means $x = n_1 b_1$. If $2 \leq n_1$, then $2a \leq n_1 a \leq n_1 b_1 = x$. It is also impossible. Therefore, we conclude that $n_1 = 1$ and then $x = b_1 \in B$. This contradicts the initial assumption, indicating that this case cannot occur.

Case 2 : $x \in [c, c + a) \setminus [2a, 2b]$.

Subcase 2.1 : $k = 1$. Then $x = n_1 b_1$. It follows that $n_1 \neq 1$ and so $2 \leq n_1$. If $b_1 \in [c, c + a)$, then

$$a + c \leq b_1 + b_1 = 2b_1 \leq x < c + a,$$

which is a contradiction. Thus $b_1 \in [a, b]$. If $n_1 = 2$, then $x = 2b_1 \in [2a, 2b]$, this is impossible. Thus $3 \leq n_1$. Consider

$$c + a \leq 2a + a \leq n_1 a \leq n_1 b_1 = x < c + a.$$

It is a contradiction.

Subcase 2.2 : $k \neq 1$. If there exists $i \in \mathbb{N}$ such that $b_i \in [c, c + a)$. Then

$$a + c \leq b_1 + b_i \leq n_1 b_1 + \cdots + n_k b_k = x < c + a$$

which is a contradiction. Thus $b_1, \ldots, b_k \in [a, b]$. Assume that $k = 2$. Then $x = n_1 b_1 + n_2 b_2$. Note from $x \notin [2a, 2b]$ that $n_1 \neq 1$ or $n_2 \neq 1$. This means that $2a \leq n_1 b_1$ or $2a \leq n_2 b_2$. Hence $2a + a \leq n_1 b_1 + n_2 b_2$. Then $c + a \leq 2a + a \leq n_1 b_1 + n_2 b_2 = x < c + a$, which is impossible and hence $3 \leq k$. Thus

$$c + a \leq 2a + a = 3a \leq b_1 + b_2 + b_3 \leq x < c + a.$$

It is a contradiction.

It follows from 2 cases that there is no proper subset B of A which can generate S. Therefore, A is the minimal system of generators. ∎

Theorem 3. Let $S = \{0\} \cup [a, a + n] \cup [c, \infty)$ be a numerical semigroup. Then $c < a + 2n + 2$ if and only if $S = \langle [a, a + n] \cup [c, 2a) \rangle$.

Proof. Suppose that $c < a + 2n + 2$. We note from Theorem 2 that the minimal system of generators for S is $A = [a, a + n] \cup [c, c + a) \setminus [2a, 2(a + n)]$. It is enough to prove that $A = [a, a + n] \cup [c, 2a)$. Clearly, $[a, a + n] \cup [c, 2a) \subseteq A$. Let $x \in A$ be such that $x \notin [a, a + n]$. Then $x \in [c, c + a) \setminus [2a, 2(a + n)]$. To show $x \in [c, 2a)$, we suppose $x \notin [c, 2a)$. Since $c \leq x$, we have $2a \leq x$. From $x \notin [2a, 2(a + n)]$, we get $2(a + n) < x$. It follows that $2a + 2n + 1 \leq x$ and $x < c + a$. Then $a + 2n + 2 \leq c$, which is a contradiction. Therefore, $x \in [c, 2a)$ and then $A \subseteq [a, a + n] \cup [c, 2a)$. Hence $A = [a, a + n] \cup [c, 2a)$.

Conversely, assume that $c \geq a + 2n + 2$. Let $b = a + n$, then we will show that $2b + 1 \in A$ and $2b + 1 \notin [a, a + n] \cup [c, 2a)$. Note that $c \leq 2a \leq 2b < 2b + 1$. By our assumption, we obtain $2a + 2n + 2 \leq c + a$. So, $2b + 1 < 2a + 2n + 2 \leq c + a$. Hence $c < 2b + 1 < c + a$. Note that $2b + 1 \notin [2a, 2b]$. Then $2b + 1 \in A$. Since $2b + 1 > 2a$, we obtain $2b + 1 \notin [a, a + n] \cup [c, 2a)$. Therefore, it follows from the minimality of A that $S \neq \langle [a, a + n] \cup [c, 2a) \rangle$. ∎

The following corollary is the consequence of Theorems 2 and Theorem 3 in the case where each interval is not empty.

Corollary 4. Let $S = \{0\} \cup [a, a + n] \cup [c, \infty)$ be a numerical semigroup. Then $a + 2n + 2 \leq c$ if and only if $S = \langle [a, a + n] \cup [c, 2a) \cup (2a + 2n, c + a) \rangle$.

Proof. Suppose that $a + 2n + 2 \leq c$. From Theorem 2 we note that the minimal system of generators for S is $A = [a, a + n] \cup [c, c + a) \setminus [2a, 2(a + n)]$. By assumption, we obtain $2(a + n) + 2 = 2a + 2n + 2 \leq c + a$. This implies that $[2a, 2(a + n)]$ is a proper subset of $[c, c + a)$. It follows that

$$A = [a, a + n] \cup [c, c + a) \setminus [2a, 2(a + n)] = [a, a + n] \cup [c, 2a) \cup (2a + 2n, c + a).$$

Conversely, assume that $S = \langle [a, a+n] \cup [c, 2a) \cup (2a+2n, c+a) \rangle$. Then there exists $x \in (2a+2n, c+a)$. This means that $2a+2b < x < c+a$ and hence $2a+2b+1 \leq x \leq c+a-1$. Therefore, $a+2n+2 \leq c$. ■

The formulation of the embedding dimension can be derived directly from Theorem 3 and Corollary 4 as follows:

Theorem 5. Let $S = \{0\} \cup [a, a+n] \cup [c, \infty)$ be a numerical semigroup. Then the following statements are true:

(i) If $c < a+2n+2$, then $\dim(S) = 2a+n-c+1$.

(ii) If $c \geq a+2n+2$, then $\dim(S) = a-n$.

Theorem 6. Let $S = \{0\} \cup [a, a+n] \cup [c, \infty)$ be such that $\dim(S) = d$ and $1 \leq n$. Then $a \in [d+1, 2d-1]$.

Proof. If $c \geq a+2n+2$, then from Theorem 5, $d = a-n$. It follows that $1+d \leq a$. Since S is a numerical semigroup, we have $c \leq 2a = 2d+2n$. By assumption, $a+2n+2 \leq 2d+2n$. Thus

$$d+1 \leq a < a+1 \leq 2d-1.$$

That is $a \in [d+1, 2d-1]$.

Now, we assume that $c < a+2n+2$. From Theorem 5, $d = 2a+n-c+1$. If $a < d+1$, then $a < 2a+n-c+2$, that is $c < a+n+2$. Recalled the condition of construction of S that $2 \leq a \leq a+n < c-1$. Thus $c \leq a+n+1 < c$ which is impossible. Hence $d+1 \leq a$. We will show that $a \leq 2d-1$. Suppose that $2d-1 < a$. Thus $2d \leq a$ and then $4a+2n-2c+2 \leq a$. This means that $2a+2n \leq 2c-a-2$. We note from $c < a+2n+2$ that $c+a-2 < 2a+2n$. Hence $c+a-1 \leq 2a+2n \leq 2c-a-2$ and so $c+a-1 \leq 2c-a-2$. Since S is a numerical semigroup, we obtain $c \leq 2a$. This means that $2a \leq c-1 < 2a$, which is a contradiction. Thus $a \leq 2d-1$. ■

In the remainder of this paper, we aim to determine the count of numerical semigroups of the form $\{0\} \cup [a, a+n] \cup [c, \infty)$ that share the same embedding dimension. We denote the class of these numerical semigroups with embedding dimension d as $\mathcal{A}$, i.e.,

$$\mathcal{A} = \{S : S = \{0\} \cup [a, a+n] \cup [c, \infty) \text{ is a numerical semigroup and } \dim(S) = d\}.$$

Furthermore, we divide the elements within class $\mathcal{A}$ into two distinct subclasses, as follows:

$$\mathcal{B} = \{S \in \mathcal{A} : S = \{0\} \cup [a, a+n] \cup [c, \infty), 1 \leq n \text{ and } c < a+2n+2\} \text{ and}$$

$$\mathcal{C} = \{S \in \mathcal{A} : S = \{0\} \cup [a, a+n] \cup [c, \infty), 1 \leq n \text{ and } c \geq a+2n+2\}.$$

The following remark arises as a consequence of Theorem 5.

Remark 7. Let $S \in \mathcal{A}$ be such that $S = \{0\} \cup [a, a] \cup [c, \infty)$. Then $a = d$ and $c \in [d + 2, 2d]$.

We now provide a characterization of subclasses $\mathcal{B}$ and $\mathcal{C}$ in Theorems 8 and 9, respectively.

Theorem 8. $\mathcal{B} = \{\{0\} \cup [a, a + n] \cup [2a + n + 1 - d, \infty) : n \in [a - d, d - 1],\ a \in [d + 1, 2d - 1]\}$.

Proof. Let $S \in \mathcal{B}$. Then $S = \{0\} \cup [a, a + n] \cup [c, \infty)$ where $c < a + 2n + 2$ and $1 \leq n$. From Theorem 5 (i), we obtain $d = \dim(S) = 2a + n - c + 1$. We note from $2a + n + 1 - d = c < a + 2n + 2$ that $a - d \leq n$. From Theorem 6, we get $a \in [d + 1, 2d - 1]$. Since $c \leq 2a$, we obtain $2a + n + 1 - d \leq 2a$. Thus $n \leq d - 1$.

Conversely, let $S = \{0\} \cup [a, a + n] \cup [2a + n + 1 - d, \infty)$ when $n \in [a - d, d - 1]$ and $a \in [d + 1, 2d - 1]$. Clearly, $1 \leq n$. We let $c = 2a + n + 1 - d$. From $a - d \leq n$, we obtain

$$c = 2a + n + 1 - d = a + n + (a - d) + 1 \leq a + n + n + 1 < a + 2n + 2.$$

It follows from Theorem 5 (i) that $\dim(S) = 2a + n - c + 1 = d$. Hence $S \in \mathcal{B}$. ■

Theorem 9. $\mathcal{C} = \{\{0\} \cup [d + n, d + 2n] \cup [c, \infty) : n \in [1, d - 2], c \in [d + 3n + 2, 2d + 2n]\}$.

Proof. Let $S \in \mathcal{C}$. Then $S = \{0\} \cup [a, a + n] \cup [c, \infty)$, $1 \leq n$ and $a + 2n + 2 \leq c$. From Theorem 5 (ii), we have $d = \dim(S) = a - n$, that is $a = d + n$. Clearly, $c \in [a + 2n + 2, 2a] = [d + 3n + 2, 2d + 2n]$. Since $a + 2n + 2 \leq c \leq 2a$, we then have $2n + 2 \leq a = d + n$. That is $n \leq d - 2$.

For the converse, let $S = \{0\} \cup [d + n, d + 2n] \cup [c, \infty)$ where $n \in [1, d - 2]$ and $c \in [d + 3n + 2, 2d + 2n]$. Let $d = a - n$. From Theorem 5 (ii) and $a + 2n + 2 \leq c$, we obtain $\dim(S) = d$. Hence $S \in \mathcal{C}$. ■

Theorem 10. Let $d \in \mathbb{N}$ be such that $2 \leq d$. Then the number of numerical semigroups of the form $\{0\} \cup [a, b] \cup [c, \infty)$ with an embedding dimension of d is given by $d(d - 1)$.

Proof. We will prove that the number of all elements in class $\mathcal{A}$ is equal to $d(d - 1)$. Firstly, we consider cardinality of subclasses $\mathcal{B}$ and $\mathcal{C}$. Clearly, $\mathcal{B} \cap \mathcal{C} = \emptyset$. From Theorem 8, we have

$$|\mathcal{B}| = |\{(a, n) : a \in [d + 1, 2d - 1] \text{ and } n \in [a - d, d - 1]\}| = \sum_{a=d+1}^{2d-1} (2d - a) = \frac{1}{2} d(d - 1).$$

Similarly, by Theorem 9, we obtain

$$|\mathcal{C}| = |\{(n, c) : n \in [1, d - 2], c \in [d + 3n + 2, 2(d + n)]\}| = \sum_{n=1}^{d-2} (d - n - 1) = \frac{(d - 2)(d - 1)}{2}.$$

It follows from Remark 7 that $|\mathcal{A}| = (d-1) + |\mathcal{B}| + |\mathcal{C}| = d(d-1)$. ■

3. Acknowledgement

We would like to express our sincere gratitude to the anonymous referees for their invaluable comments and suggestions, which greatly contributed to enhancing the presentation of this paper.

4. References

Chommi, N. (2020). *Irreducible numerical semigroups* $\{0\} \cup [a, b] \cup [c, \infty)$. (Undergraduate's thesis). Naresuan University.

Curtis, F. (1990). On formulas for the Frobenius number of a numerical semigroup. *Mathematica Scandinavica*, *67*, 190-193. https://www.jstor.org/stable/24492663

Davison, J.L. (1994). On the linear Diophantine problem of Frobenius. *Journal of Number Theory, 48*, 353–363. https://doi.org/10.1006/jnth.1994.1071

Johnson, S. M. (1960). A linear diophantine problem. *Canadian Journal of Mathematics*, *12*, 390-398. https://doi.org/10.4153/CJM-1960-033-6

Kosasirisin, P. (2020). *The number of* k*-symmetric numerical semigroup* $\{0\} \cup [a, b] \cup [c, \infty)$. (Undergraduate's thesis). Naresuan University.

Phosri, N. (2020). *The number of* k*-symmetric numerical semigroup* $\{0\} \cup [a, b] \cup [c, \infty)$ *for* $k = 3, 4, 5$. (Undergraduate's thesis). Naresuan University.

Rosales, J.C. & Garcia-Sanchez, P.A. (2009). *Numerical semigroups*. Springer: New York.

Sylvester, J.J. (1884). Mathematical questions with their solutions. *Educational Times*, *41*, 171-178.

Tripathi, A. (2017). Formulae for the Frobenius number in three variables. *Journal of Number Theory*, *170*, 368-389. https://doi.org/10.1016/j.jnt.2016.05.027