A Note on Maximal and Minimal Elements in Semigroups of Partial Transformation Preserving Order and Contraction

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Abstract

For a positive integer n, let $[n] = \{1, 2, 3, \dots, n\}$, consider the semigroup COP_n consisting of all partial transformations α from $[n]$ to $[n]$ such that α preserves both a natural partial order $\leq$ (if $x \leq y$ then $x\alpha \leq y\alpha$) and contraction $(|x\alpha - y\alpha| \leq |x - y|)$. In this paper, we give a characterization of maximal and minimal elements of COP_n with respect to its natural partial order.

Keywords: partial order, maximal (minimal) elements, transformation semigroup

1. Introduction and Preliminaries

For an arbitrary nonempty set X, let PT_X denotes the set of all partial transformations of X, that is, all mappings α whose domain, $dom\ \alpha$, and range α, $ran\ \alpha$, are subsets of X. The empty transformation which is a partial transformation of X with empty domain is denoted by $\emptyset$. Then PT_X becomes a semigroup under composition of mappings, that is, for every $\alpha, \beta \in PT_X$, $\alpha\beta \in PT_X$ is defined by

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{for all } x \in dom\ \alpha\beta.$$

We also have $dom\ \alpha\beta = (ran\ \alpha \cap dom\ \beta)\alpha^{-1}$ and $ran\ \alpha\beta = (ran\ \alpha \cap dom\ \beta)\beta$.

Let $[n] = \{1, 2, 3, \dots, n\}$ for a positive integer n. We will use the notation PT_n in place of $PT_{[n]}$ to highlight that the set $[n]$ has cardinal n and has ordered in the standard way. Then PT_n is a semigroup of all partial transformations of $[n]$. For $\alpha \in PT_n$, α is called an *order-preserving mapping* if for every $x, y \in dom\ \alpha$, $x \leq y$ implies $x\alpha \leq y\alpha$, whereas α is called a *contraction* if for every $x, y \in dom\ \alpha$, $|x\alpha - y\alpha| \leq |x - y|$. Let OP_n be the semigroup of all partial order-preserving transformations of $[n]$ and COP_n be the subsemigroup

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of OP_n consisting of all contraction transformations on $[n]$. The semigroup COP_n was first studied by Zhao and Yang in 2012 which is called the *semigroup of partial transformation that preserving order and contraction on* $[n]$. They also described Green's relations on COP_n and investigated regularity of elements in COP_n.

In 1986, Mitsch (1986) gave a characterization of the natural partial order on any semigroup S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1$$

where S^1 is a monoid obtained from S by adjoining an identity if necessary. The natural partial order on various special subsemigroups of partial transformation semigroups have been studied by many researchers examples are (Kowol, & Mitsch, 1986), (Marques-Smith, & Sullivan, 2003), (Sun, Pei, & Cheng, 2008), (Sun, Deng, & Pei, 2011), (Sangkhanan, & Sanwong, 2012), (Sun, & Sun, 2013), (Sun, & Wang, 2013), (Sun, & Sun, 2016), (Han, & Sun, 2018) and (Sangkhanan, 2021).

Next, we introduce some definitions and notations that will be used in the sequel.

Let $(X, \leq)$ be a partially ordered set. An element $a \in X$ is called *maximal* (*minimal*) if $a \leq x$ $(x \leq a)$ and $x \in X$ implies $a = x$, and $b \in X$ is called *maximum* (*minimum*) if $x \leq b$ $(b \leq x)$ for all $b \in X$. We let $\max X$ $(\min X)$ denote the maximum (minimum) element of X.

A subset C of $[n]$ is said to be *convex* if C has the form $[i, i+t] = \{x \in [n] \mid i \leq x \leq i+t\}$ for some $i \in [n]$ and $0 \leq t \leq n-1$.

Let A and B be any two subsets of $[n]$. We say that A is *less than* B and write $A < B$, if $a < b$ for all $a \in A$,$b \in B$.

For $\alpha, \beta \in PT_n$, let

$$\alpha\beta^{-1} = \{(x, y) \in [n] \times [n] \mid x\alpha = y\beta\}.$$

For $\alpha \in COP_n$, we expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix}$$

where $dom\, \alpha = A_1 \cup A_2 \cup \cdots \cup A_s, ran\, \alpha = \{a_1, a_2, \dots, a_s\}, a_1 < a_2 < \cdots < a_s$, $A_1 < A_2 < \cdots < A_s$,
$a_i - a_{i-1} \leq \min A_i - \max A_{i-1}$ for $i = 2, 3, \dots, s$ and $a_i\alpha^{-1} = A_i$ for $i = 1, 2, \dots, s$.

In this introductory section, we present a number of Theorems most of which will be indispensable for our research.

Propostion 1. (Kemprasite, & Changphas, 2000) Let $\alpha \in OP_n$ and $x, y \in ran\ \alpha$ be such that $x < y$.
Then $x\alpha^{-1} < y\alpha^{-1}$.

Theorem 2. (Marques-Smith, & Sullivan, 2003) Let $\alpha, \beta \in PT_n$. Then $\alpha \leq \beta$ on PT_n if and only if

(i) $dom\ \alpha \subseteq dom\ \beta$

(ii) $ran\ \alpha \subseteq ran\ \beta$

(iii) $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and

(iv) $\beta\beta^{-1} \cap (dom\ \beta \times dom\ \alpha) \subseteq \alpha\alpha^{-1}$.

Theorem 3. (Namnak, & Sawatraksa, 2015) Let $\alpha, \beta \in COP_n$ be such that $\alpha = \begin{pmatrix} A_1 \\ a_1 \end{pmatrix}$. Then $\alpha \leq \beta$ if and only if

(i) $A_1 \subseteq dom\ \beta$

(ii) $a_1 \in ran\ \beta$

(iii) $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and

(iv) $\beta\beta^{-1} \cap (dom\ \beta \times dom\ \alpha) \subseteq \alpha\alpha^{-1}$.

Theorem 4. (Namnak, & Sawatraksa, 2015) Let $\alpha, \beta \in COP_n$ be such that $\alpha = \begin{pmatrix} A_1 & A_2 \\ a_1 & a_2 \end{pmatrix}$. Then $\alpha \leq \beta$ if and only if

(i) $dom\ \alpha \subseteq dom\ \beta$

(ii) $ran\ \alpha \subseteq ran\ \beta$

(iii) for every $i \in \{1, 2\}$ and $y \in ran\ \beta$, if $y\beta^{-1} \cap A_i \neq \emptyset$, then $y\beta^{-1} \subseteq A_i$ and

(iv) $(\max A_1)\beta = a_1$ and $(\min A_2)\beta = a_2$.

Theorem 5. (Namnak, & Sawatraksa, 2015) Let $\alpha, \beta \in COP_n$ be such that $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix}$ where $s \geq 3$. Then $\alpha \leq \beta$ if and only if

(i) $dom\ \alpha \subseteq dom\ \beta$

(ii) $ran\ \alpha \subseteq ran\ \beta$

(iii) $a_i\beta^{-1} = A_i$ for all $i \in [2, s-1]$

(iv) for every $i \in \{1, s\}$ and $y \in ran\ \beta$, if $y\beta^{-1} \cap A_i \neq \emptyset$, then $y\beta^{-1} \subseteq A_i$ and

(v) $(\max A_1)\beta = a_1$ and $(\min A_s)\beta = a_s$.

The purpose of this article is to investigate the conditions under which elements in COP_n are maximal and minimal with respect to the natural partial order.

2. Main Results

Note that, if $n = 1$, we get $COP_1 = \{\emptyset, \begin{pmatrix} \{1\} \\ 1 \end{pmatrix}\}$ and then $\begin{pmatrix} \{1\} \\ 1 \end{pmatrix}$ is the maximum element of COP_1. If $n = 2$, then $COP_2 =$ $\left\{\emptyset, \begin{pmatrix} \{1\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{1\} \\ 2 \end{pmatrix}, \begin{pmatrix} \{2\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{2\} \\ 2 \end{pmatrix}, \begin{pmatrix} \{1,2\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{1,2\} \\ 2 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2\} \\ 1 & 2 \end{pmatrix}\right\}$. It is easily verified that $\left\{\begin{pmatrix} \{1\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{1\} \\ 2 \end{pmatrix}, \begin{pmatrix} \{2\} \\ 1 \end{pmatrix}, \begin{pmatrix} \{2\} \\ 2 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2\} \\ 1 & 2 \end{pmatrix}\right\}$ is the set of all maximal elements of COP_2.

In order to prove our main results, the following lemmas will be needed later.

Lemma 6. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix} \in COP_n$. If α is maximal, then $ran\ \alpha$ is a convex subset of $[n]$.

Proof. If $s = 1$ is trivial. Assume that $s \geq 2$, suppose that $ran\ \alpha$ is not convex. Then there exists an element $y \in [a_1, a_s]$ where $y \notin ran\ \alpha$. Thus, $a_{i-1} < y < a_i$ for some $i \in [2, s]$. Hence, $a_i - a_{i-1} > 1$. By contraction of α, we get $\min A_i - \max A_{i-1} > 1$.

Let $k = y - a_{i-1}$ and $l = a_i - y$. Then $k + l = a_i - a_{i-1} \leq \min A_i - \max A_{i-1}$.

Let $z \in [\max A_{i-1} + k, \min A_i - l]$. Then $\max A_{i-1} < z < \min A_i$. Define $\beta \in OP_n$ by

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{i-1} & \{z\} & A_i & \dots & A_s \\ a_1 & a_2 & \dots & a_{i-1} & y & a_i & \dots & a_s \end{pmatrix}.$$

Obviously, $\alpha \neq \beta$. If $x \in A_{i-1}$, then

$$|x\beta - z\beta| = |a_{i-1} - y| = y - a_{i-1} = k = (\max A_{i-1} + k) - \max A_{i-1} \leq z - \max A_{i-1} \leq |x - z|.$$

If $x \in A_i$, then

$$|x\beta - z\beta| = |a_i - y| = a_i - y = l = \min A_i - (\min A_i - l) \leq \min A_i - z \leq |x - z|.$$

We deduce that $\beta \in COP_n$. It can be easily checked that $\alpha \leq \beta$ from Theorem 4 or 5. This proves that α is not maximal. This completes the proof. ❑

Lemma 7. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix} \in COP_n$. If α is maximal and $n \in ran\ \alpha$, then $A_1 = \{1\}$.

Proof. Let α be a maximal element of COP_n and $n \in ran\ \alpha$. Then $a_s = n$. Suppose that $A_1 \neq \{1\}$. Then $A_1 \setminus \{1\} \neq \emptyset$. If $a_1 = 1$, choose $x \in A_1 \setminus \{1\}$. Hence, $x > 1$ and $x\alpha = 1$. By contraction of α, we have

$$n - 1 = |\min A_s\alpha - x\alpha| \leq \min A_s - x \leq n - x$$

which is impossible. Thus, $a_1 > 1$. Define $\beta \in OP_n$ by

$$\beta = \begin{pmatrix} \{1\} & A_1 \setminus \{1\} & A_2 & \dots & A_s \\ a_1 - 1 & a_1 & a_2 & \dots & n \end{pmatrix}.$$

For each $x \in A_1 \setminus \{1\}$,

$$|x\beta - 1\beta| = a_1 - (a_1 - 1) = 1 \leq x - 1 = |x - 1|,$$

so $\beta \in COP_n$. It can be easily checked that $\alpha \leq \beta$ from Theorem 4 or 5. By property of α, we have $\alpha = \beta$ which is a contradiction. Hence, $A_1 = \{1\}$. ❑

Lemma 8. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix} \in COP_n$. If α is maximal and $n \notin ran\ \alpha$, then $A_s = \{n\}$.

Proof. Let α be a maximal element of COP_n and $n \notin ran\ \alpha$. Suppose that $A_s \neq \{n\}$. Then $A_s \setminus \{n\} \neq \emptyset$. Since $n \notin ran\ \alpha$, it follows that $a_s < n$. Define $\beta \in OP_n$ by

$$\beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \setminus \{n\} & \{n\} \\ a_1 & a_2 & \cdots & a_s & a_s + 1 \end{pmatrix}.$$

For each $x \in A_s \setminus \{n\}$,

$$|x\beta - n\beta| = |a_s - (a_s + 1)| = 1 \leq |x - n|,$$

thus $\beta \in COP_n$. It can be easily checked that $\alpha \leq \beta$ from Theorem 4 or 5. By property of α, we have $\alpha = \beta$ which is a contradiction. Hence, $A_s = \{n\}$. ❑

Lemma 9. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix} \in COP_n$. If α is maximal, then $1 \in ran\ \alpha$ or $A_1 = \{1\}$.

Proof. Suppose that $1 \notin ran\ \alpha$ and $A_1 \neq \{1\}$. Then $a_1 \neq 1$ and $A_1 \setminus \{1\} \neq \emptyset$. Define $\beta \in OP_n$ by

$$\beta = \begin{pmatrix} \{1\} & A_1 \setminus \{1\} & A_2 & \cdots & A_s \\ a_1 - 1 & a_1 & a_2 & \cdots & a_s \end{pmatrix}.$$

For each $x \in A_1 \setminus \{1\}$,

$$|1\beta - x\beta| = |a_1 - (a_1 + 1)| = 1 \leq x - 1,$$

so, $\beta \in COP_n$. It can be easily checked that $\alpha \leq \beta$ from Theorem 4 or 5. Thus α is not a maximal element of COP_n. ❑

Lemma 10. Let $\alpha, \beta \in COP_n$ be such that $\alpha \leq \beta$. If $x \in dom\ \beta \setminus dom\ \alpha$, then $x\beta \notin ran\ \alpha$.

Proof. Let $x \in dom\ \beta \setminus dom\ \alpha$. Suppose that $x\beta \in ran\ \alpha$. Then there exists $z \in dom\ \alpha$ such that $z\alpha = x\beta$. Hence, $(z, x) \in \alpha\beta^{-1}$. Since $\alpha \leq \beta$, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ by Theorem 2(iii). This implies that $(z, x) \in \alpha\alpha^{-1}$, so $x\alpha = z\alpha$ which contradicts to $x \notin dom\ \alpha$. Therefore, $x\beta \notin ran\ \alpha$. ❑

Lemma 11. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix}, \beta \in COP_n$ be such that $\alpha \leq \beta$. If $dom\ \alpha = dom\ \beta, A_1 = \{1\}$ and $a_s = n$, then $\alpha = \beta$.

Proof. Let $dom\ \alpha = dom\ \beta, A_1 = \{1\}$ and $a_s = n$. We show that $\alpha = \beta$, let $x \in dom\ \alpha$. We consider three possible cases.

Case 1: $s = 1$. Then $dom\ \alpha = \{1\}$ and hence $x = 1$ and $x\alpha = a_1$. Since $\alpha \leq \beta$, $a_1 \in ran\ \beta$ by Theorem 3(ii). By assumption, we have $dom\ \beta = \{1\}$, it then follows that $x\beta = 1\beta = a_1 = x\alpha$.

Case 2: $s = 2$. Then $dom\ \alpha = A_1 \cup A_2 = \{1\} \cup A_2$, and $a_2 = n$.

If $x \in A_1$, then $x\alpha = a_1$. By Theorem 4(iv), $x\alpha = a_1 = (\max A_1)\beta = 1\beta = x\beta$.

If $x \in A_2$, then $x\alpha = n$ and $\min A_2 \leq x$. Since $\beta \in OP_n$, $(\min A_2)\beta \leq x\beta$. By Theorem 4(iv),

$(\min A_2)\beta = a_2 = n$. This implies that $x\beta = x\alpha$.

Case 3: $s \geq 3$. Then $dom\ \alpha = A_1 \cup A_2 \cup \dots \cup A_s$ and $a_s = n$.

If $x \in A_1 = \{1\}$, then $x\alpha = a_1 = (\max A_1)\beta = 1\beta = x\beta$ by Theorem 5(v).

If $x \in A_i$ for some $i \in [2, s-1]$, then $x\alpha = a_i$. By Theorem 5(iii), we have that $x \in a_i\beta^{-1}$ which implies that $x\beta = a_i$. Thus, $x\beta = x\alpha$.

If $x \in A_s$, then $x\alpha = n$ and $\min A_s \leq x$. Since $\beta \in OP_n$ and by Theorem 5(v), we have $n = (\min A_s)\beta \leq x\beta$. This implies that $x\beta = n = x\alpha$.

From above three cases we obtain $\alpha = \beta$, as required. ❑

Lemma 12. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix}, \beta \in COP_n$ be such that $\alpha \leq \beta$. If $dom\ \alpha = dom\ \beta, 1 \in ran\ \alpha$ and $A_s = \{n\}$, then $\alpha = \beta$.

Proof. Let $dom\ \alpha = dom\ \beta, 1 \in ran\ \alpha$ and $A_s = \{n\}$. Then $a_1 = 1$. To show that $\alpha = \beta$, we consider three possible cases.

Case 1: $s = 1$. Clearly, $n\alpha = 1 = n\beta$.

Case 2: $s = 2$. Then $dom\ \alpha = A_1 \cup \{n\}$.

If $x = n$, then $x\alpha = a_2$. By Theorem 2(iv), $n\beta = (\min A_2)\beta = a_2$. Hence, $x\alpha = x\beta$.

If $x \in A_1$, then $x\alpha = 1$. By Theorem 2(iv), $(\max A_1)\beta = 1$. Since $x \leq \max A_1$ and $\beta \in OP_n$, $x\beta \leq (\max A_1)\beta$. This implies that $x\beta = 1$, hence, $x\alpha = x\beta$.

Case 3: $s \geq 3$. Then $dom\ \alpha = A_1 \cup A_2 \cup \dots \cup \{n\}$.

If $x \in A_1$, then $x\alpha = 1$. By Theorem 2(iv), $a_1 = (\max A_1)\beta$. Since $\beta \in OP_n$ and $x \leq \max A_1$, we obtain $x\beta \leq (\max A_1)\beta$. It follows that $x\beta = 1$. Thus, $x\alpha = x\beta$.

If $x \in A_i$ for some $i \in [2, s-1]$, then $x\alpha = a_i$. By Theorem 2(iv) we have that $x \in a_i\beta^{-1}$ which implies that $x\beta = a_i$. Thus, $x\beta = x\alpha$.

If $x \in A_s = \{n\}$, then $x = n$ and $x\alpha = a_s$. By Theorem 2(iv), $a_s = (\min A_s)\beta = n\beta = x\beta$. Hence, $x\alpha = x\beta$.

From above three cases we obtain $\alpha = \beta$, as required. ❑

Lemma 13. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix}, \beta \in COP_n$ be such that $\alpha \leq \beta$. If $dom\ \alpha = dom\ \beta, A_1 = \{1\}$ and $A_s = \{n\}$, then $\alpha = \beta$.

Proof. Let $dom\ \alpha = dom\ \beta, A_1 = \{1\}$ and $A_s = \{n\}$. To show that $\alpha = \beta$, we consider three possible cases.

Case 1: $s = 1$. Obviously, $1\alpha = 1\beta$.

Case 2: $s = 2$. Then $A_2 = \{n\}$, hence $dom\ \alpha = \{1, n\}$. By Theorem 4(iv),

$$1\alpha = a_1 = (\max A_1)\beta = 1\beta \text{ and } n\alpha = a_2 = (\min A_2)\beta = n\beta.$$

Case 3: $s \geq 3$. Let $x \in dom\ \alpha$.

If $x \in A_1 = \{1\}$, then $x = 1$ and hence $x\alpha = a_1 = (\max A_1)\beta = 1\beta = x\beta$.

If $x \in A_i$ for some $i \in [2, s-1]$, then $x\alpha = a_i$. By Theorem 2(iv), we have that $x \in a_i\beta^{-1}$ which implies that $x\beta = a_i$. Thus, $x\beta = x\alpha$.

If $x \in A_s = \{n\}$, then $x = n$. By Theorem 2(iv), we have $x\beta = n\beta = (\min A_s)\beta = a_s = x\alpha$.

From above three cases we obtain $\alpha = \beta$, as required. ❑

Next, we give necessary and sufficient conditions for elements of COP_n to be maximal when $n \geq 3$.

Theorem 14. Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix} \in COP_n$. Then α is maximal if and only if $ran\ \alpha$ is a convex subset of $[n]$ and either one of the following conditions holds:

(i) if $n \in ran\ \alpha$, then $A_1 = \{1\}$ or

(ii) if $n \notin ran\ \alpha$, then $A_s = \{n\}$ and ($1 \in ran\ \alpha$ or $A_1 = \{1\}$).

Proof. *Necessity.* It follows from Lemma 6, 7, 8 and 9.

Sufficiency. We must prove that α is maximal of COP_n. Let $\beta \in COP_n$ be such that $\alpha \leq \beta$. By Theorem 2, $dom\ \alpha \subseteq dom\ \beta$, $ran\ \alpha \subseteq ran\ \beta$ and $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$. To prove that $\alpha = \beta$, let's divide it into two cases.

Case 1: $n \in ran\ \alpha$. By (i), we have $A_1 = \{1\}$. Suppose that $dom\ \beta \setminus dom\ \alpha \neq \emptyset$. Then there is an element $x \in dom\ \beta \setminus dom\ \alpha$. Then $x > 1$ and $x\beta = y$ for some $y \in ran\ \beta$. By Lemma 10, $y \notin ran\ \alpha$. Since $ran\ \alpha$ is convex, $A_1 = \{1\}$ and $n \in ran\ \alpha$, we obtain that $1\alpha = a_1$ and $ran\ \alpha = [a_1, n]$. This implies $y < a_1$. Since $a_1 \in ran\ \alpha \subseteq ran\ \beta$, there exists $a' \in [n]$ such that $a'\beta = a_1$. Hence, $(1, a') \in \alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and so $a'\alpha = 1\alpha = a_1$. Since $A_1 = \{1\}$, it follows that $a' = 1$ and hence $1\beta = a_1$. Since $\beta \in OP_n$ and $y < a_1$, $y\beta^{-1} < a_1\beta^{-1}$ by Proposition 1 and hence, $x < 1$ which is a contradiction. This proves that $dom\ \beta \setminus dom\ \alpha = \emptyset$. Hence, $dom\ \alpha = dom\ \beta$ comes directly from $dom\ \alpha \subseteq dom\ \beta$. By Lemma 11, $\alpha = \beta$.

Case 2: $n \notin ran\ \alpha$. By (ii), we have $A_s = \{n\}$ and $(1 \in ran\ \alpha \text{ or } A_1 = \{1\})$. Consider two subcases.

Subcase 2.1: $1 \in ran\ \alpha$. Suppose that $dom\ \beta \setminus dom\ \alpha \neq \emptyset$. Then there exists $x \in dom\ \beta \setminus dom\ \alpha$. Then $x\beta = y$ for some $y \in ran\ \beta$. By Lemma 10, $y \notin ran\ \alpha$.

Since $ran\ \alpha$ is convex, $1 \in ran\ \alpha$ and $A_s = \{n\}$, we obtain $ran\ \alpha = [1, a_s]$ and $n\alpha = a_s$. This implies $a_s < y$. Since $a_s \in ran\ \alpha \subseteq ran\ \beta$, there exists $a' \in [n]$ such that $a'\beta = a_s$. Hence, $(n, a') \in \alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and thus, $a'\alpha = n\alpha$. It follows that $a' = n$ and $n\beta = a_s$. Since $\beta \in OP_n$ and $a_s < y$ by Proposition 1, we obtain $a_s\beta^{-1} < y\beta^{-1}$, and hence, $n < x$. This is a contradiction which means that $dom\ \alpha = dom\ \beta$. By Lemma 12, $\alpha = \beta$.

Subcase 2.2: $A_1 = \{1\}$. Suppose that $dom\ \beta \setminus dom\ \alpha \neq \emptyset$. Then there exists $x \in dom\ \beta \setminus dom\ \alpha$. Then $x\beta = y$ for some $y \in ran\ \beta$. By Lemma 10, $y \notin ran\ \alpha$. Since $ran\ \alpha$ is convex, $ran\ \alpha = [a_1, a_s]$, $1\alpha = a_1$ and $n\alpha = a_s$. These imply that $y < a_1$ or $a_s < y$. Since $\beta \in OP_n$, $y\beta^{-1} < a_1\beta^{-1}$ or $a_s\beta^{-1} < y\beta^{-1}$ by Proposition 1. Hence, $x < 1$ or $n < y$. This is a contradiction. Therefore, $dom\ \alpha = dom\ \beta$. By Lemma 13, $\alpha = \beta$.

Considering the above two cases, we can conclude that α is a maximal element of COP_n. ❑

Let us, illustrate this theorem. For $n = 8$, let

$$\alpha_1 = \begin{pmatrix} \{1\} & \{3,5\} & \{6,7\} \\ 6 & 7 & 8 \end{pmatrix}, \alpha_2 = \begin{pmatrix} \{2\} & \{3,5,7\} & \{8\} \\ 1 & 2 & 3 \end{pmatrix}, \alpha_3 = \begin{pmatrix} \{1\} & \{3,5,7\} & \{8\} \\ 3 & 4 & 5 \end{pmatrix},$$

$$\text{and } \alpha_4 = \begin{pmatrix} \{1\} & \{2,3,4\} & \{5,6\} & \{7,8\} \\ 5 & 6 & 7 & 8 \end{pmatrix}.$$

It is easy to check that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in COP_n$. By Theorem 14, all such transformations are maximal.

It is clear that $\emptyset$ is the minimum element of COP_n. We say $\alpha \in COP_n \setminus \{\emptyset\}$ is a non-zero minimal element of COP_n if α is minimal among the non-zero elements of COP_n. Finally, we provide necessary and sufficient condition for non-zero elements in COP_n to be minimal.

Theorem 15. Let α be a non-zero element in COP_n. Then α is a non-zero minimal element of COP_n if and only if α is a constant mapping.

Proof. *Necessity.* We will prove the contrapositive. Suppose that α be nonconstant. Then $|ran\ \alpha| > 1$. We choose and fix an element $u \in ran\ \alpha$. Define $\beta \in COP_n$ by

$$x\beta = u \text{ for all } x \in u\alpha^{-1}.$$

It is clearly that $\emptyset \neq \beta \neq \alpha$. It suffices to verify that $\beta\alpha^{-1} \subseteq \beta\beta^{-1}$ and $\alpha\alpha^{-1} \cap (dom\,\alpha \times dom\,\beta) \subseteq \beta\beta^{-1}$. Let $(x,y) \in \beta\alpha^{-1}$. Then $x\beta = y\alpha$. Since $x\beta = u$, it follows that $y \in u\alpha^{-1}$. This implies $x\beta = y\beta$, and thus, $(x,y) \in \beta\beta^{-1}$. Hence, $\beta\alpha^{-1} \subseteq \beta\beta^{-1}$. Next, let $(x,y) \in \alpha\alpha^{-1} \cap (dom\,\alpha \times dom\,\beta)$. Then $x\alpha = y\alpha$ and $y \in dom\,\beta = u\alpha^{-1}$. Hence, $y\alpha = u$, so $x \in u\alpha^{-1}$. By the definition of β, $x\beta = u$. This shows that $(x,y) \in \beta\beta^{-1}$. By Theorem 3, $\beta \leq \alpha$. We proved that α is not a non-zero minimal element of COP_n.

Sufficiency. Assume that α is constant. Then there exists $y \in [n]$ such that $x\alpha = y$ for all $x \in dom\,\alpha$. Let $\gamma \in COP_n \setminus \{\emptyset\}$ be such that $\gamma \leq \alpha$. By Theorem 3, $dom\,\gamma \subseteq dom\,\alpha$, $ran\,\gamma \subseteq ran\,\alpha$ and $\gamma\alpha^{-1} \subseteq \gamma\gamma^{-1}$. It follows directly that $ran\,\gamma = \{y\}$. To show that $dom\,\alpha \subseteq dom\,\gamma$, let $a \in dom\,\alpha$. Then $a\alpha = y$. We choose an element $b \in dom\,\gamma$. Then $b\gamma = y$ and hence, $a\alpha = b\gamma$. This shows that $(b,a) \in \gamma\alpha^{-1} \subseteq \gamma\gamma^{-1}$, which means that $a \in dom\,\gamma$. Therefore, $dom\,\alpha \subseteq dom\,\gamma$, as required. Consequently, $\alpha = \gamma$. This completes the proof. ❑

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