Distribution Solutions of Euler Equations

Alongkot Suvarnamani[1 *] and Gumpon Sritanratana[2]

[1]Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi

[2]Faculty of Science and Technology, Buriram Rajabhat University

[*]Email: kotmaster2@rmutt.ac.th

Abstract

Consider the n^{th} order non-homogeneous Euler equation of the form

$$m_0 t^n y^{(n)}(t) + m_1 t^{n-1} y^{(n-1)}(t) + \cdots + m_{n-1} t y'(t) + m_n y(t) = f(t) \qquad (1)$$

where m_0, m_1, m_2, …, m_n are real number $m_0 \neq 0$, $t \in R$ and $f(t)$ is a right-sided distribution. By using Laplace transform, we found that a complementary function of this equation in distribution sense is investigated under the conditions on the values of m_0, m_1, m_2, …, and m_n.

Keywords: Dirac-delta function, Tempered distribution, Laplace transform

Introduction

Generally, whenever we talk about the basic concept of differential equations, their solutions are called strong solutions and interpreted in the classical sense. However, this ordinary sense cannot be used to solve many problems of differential equations, so the distribution sense is applied. In 1999, Kananthai (1999) studied the solutions of the third order Euler equations. In 2000, Hongsit (2000) studied the weak solutions and the strong solutions of the fourth order Euler equations.

In this study, we will consider the n^{th} order Euler equations by using Laplace transform.

Definition 1.1

The space D consists of all complex-valued functions φ with continuous derivatives of all orders and with compact support; the support of φ is the closure of the set of all elements t in R such that $\varphi(t) \neq 0.$ We call each function $\varphi(t)$ in $D,$ the *testing function*.

Definition 1.2

A *functional* on the space D is a mapping $T : D \to C,$ where C is the set of all complex numbers. For all $\varphi \in D,$ the value of T acting on φ is denoted by $< T, \varphi >$.

Definition 1.3

A continuous linear functional T on the space D is called a *generalized function* or a *distribution*. The space of all such distributions is denoted by D'.

Example 1.1 The Dirac-delta function is a distribution defined by $< \delta, \varphi > = \varphi(0)$ for $\varphi \in D.$

Definition 1.4

Let f be a locally integrable function. We can define T through the convergent integral

$$< T, \varphi > = < f, \varphi > = < f(t), \varphi(t) > \equiv \int_{-\infty}^{\infty} f(t)\varphi(t)dt$$

for $\varphi \in D.$ Then a functional $T : D \to R$ defined is clearly a distribution. Such distribution is said to be generated by f.

Definition 1.5

A distribution T that is generated by a locally integrable function is called a *regular distribution*; otherwise, it is called a *singular distribution*.

Definition 1.6

The k^{th} order of derivative of a distribution T is denoted by $T^{(k)}$ and defined by

$$< T^{(k)}, \varphi > = < \mathrm{T}, (-1)^k \varphi^{(k)} > \text{ for all } \varphi \in D.$$

Example 1.2

$$< \delta', \varphi > \;=\; -\varphi'(0) \text{ for all } \varphi \in D.$$

$$< \delta^{(k)}, \varphi > \;=\; (-1)^k \varphi^{(k)}(0) \text{ for all } \varphi \in D.$$

A function H is called a *Heaviside function* defined by $H(t) = 1$ for $t > 0,$ and, $H(t) = 0$ for $t \leq 0.$ The first derivative of the Heaviside function is denoted by H' and defined by

$$< H', \varphi > \;=\; < \delta, \varphi > \qquad \text{for all } \varphi \in D.$$

Definition 1.7

Let α be an infinitely differentiable function and define the product of α with any distribution T in D' by $<\alpha T,\varphi>=<T,\alpha\varphi>$ for all $\varphi\in D.$

Example 1.3

$$<\alpha\delta,\varphi>=<\delta,\alpha\varphi>=\alpha(0)\varphi(0).$$

$$<t\delta,\varphi>\ =<\delta,t\varphi>\ =0\varphi(0)=0=<0,\varphi>.$$

It follows that $t\delta=0.$

$$<t\delta',\varphi>=<\delta',t\varphi>=<\delta,(-1)(t\varphi(t))'>=-\varphi(0)=<-\delta,\varphi>.$$

It follows that $t\delta'=-\delta.$

In general, $t\delta^{(m)}=-m\delta^{(m-1)},\ m=1,2,3,\dots$.

Definition 1.8

Let f be a locally integrable function that satisfies the following conditions: $f(t)=0$ for $-\infty<t<A,$ where A is a real constant;

There exists a real number c such that $e^{-ct}f(t)$ is absolutely integrable over $R.$ Then the Laplace transform of f is defined by

$$F(s)=L\{f(t)\}=\int_T^{\infty}f(t)e^{-st}dt \tag{2}$$

where s is a complex variable. The greatest lower bound σ_a on all positive values of $c,$ for which the condition (b) holds, is called the *abscissa of absolute convergence*.

It can be shown that F is an analytic function for the half-plane $\mathrm{Re}(s)>\sigma_a$, where σ_a is an abscissa of absolute convergence for $L\{f(t)\}.$

Now, equation (2) can be replaced by the notation

$$F(s)=L\{f(t)\}=<f(t),e^{-st}>. \tag{3}$$

Suppose f is a distribution, that is, $f\in D'.$ We try to define the Laplace transform of right-side distributions. Suppose f is a distribution whose support is bounded on the left and there exists a real number c for which $e^{-ct}f(t)$ is a tempered distribution.

Define

$$F(s)=L\{f(t)\}=<e^{-ct}f(t),X(t)e^{-(s-c)t}> \tag{4}$$

where $X(t)$ is any infinitely differentiable function with compact support on the left and equals to 1 over the neighborhood of support of $f(t)$.

For $\operatorname{Re}(s) > c,\ X(t)\,e^{-(s-c)t}$ is a testing function in the space S. Equation (4) can be deduced to the definition

$$F(s) = L\{f(t)\} = < f(t), e^{-st} >. \tag{5}$$

Then equation (5) possesses a distribution sense, which is similar to equation (3).

Now F is a function of s defined over the right side of the half-plane $\operatorname{Re}(s) > c$. By Zemanian [9], F is an analytic function in the region of convergence $\operatorname{Re}(s) > \sigma_1$ where σ_1 is the abscissa of convergence for which $e^{-ct} f(t) \in S'$.

Example 1.4 If $\operatorname{Re}(s) > 0$ and H is the Heaviside function, then

(i) $L\{H(t)\} = \dfrac{1}{s}$

(ii) $L\left\{\dfrac{t^k H(t)}{k!}\right\} = \dfrac{1}{s^{k+1}}$ for $k = 1, 2, 3, \ldots$.

(iii) $L\{\delta\} = 1$ for $\operatorname{Re}(s) \in R$ and δ is the Dirac-delta function whose support concentrated on a single point.

(iv) $L\{\delta^{(k)}\} = s^k$ for $\operatorname{Re}(s) \in R$, k is a positive integer.

(v) $L\{t^k f(t)\} = (-1)^k F^{(k)}(s)$ for $\operatorname{Re}(s) > \sigma_1$, where f is a distribution in the space D'_R of distributions whose supports are bounded on the left.

(vi) $L\{f^{(k)}(t)\} = s^k F(s)$ for $\operatorname{Re}(s) > \sigma_1$ where $f \in D'_R$.

Definition 1.9

The inverse Laplace transform of F is denoted by $L^{-1}\{F(s)\}$ and is defined by

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{y\to\infty} \int_{c-iy}^{c+iy} F(s) e^{st}\, ds \tag{6}$$

where $\operatorname{Re}(s) \geq c > \sigma_a$.

Example 1.5 For k is a positive integer,

(i) $L^{-1}\left\{\dfrac{1}{s^{k+1}}\right\} = \dfrac{t^k H(t)}{k!}$ for $\operatorname{Re}(s) > 0$.

(ii) $L^{-1}\{s^k\} = \delta^{(k)}$ for $\operatorname{Re}(s) \in R$.

Result

In this section, we find the solution of the n^{th} order non-homogeneous Euler equation by using Laplace transform.

Consider the n^{th} order non-homogeneous Euler equation of the form

$$b_0 t^n y^{(n)}(t) + b_1 t^{n-1} y^{(n-1)}(t) + \cdots + b_{n-1} t y'(t) + b_n y(t) = g(t) \qquad (7)$$

where b_0, b_1, b_2, ..., b_n are real numbers, $b_0 \neq 0$, $t \in R$ and $g(t)$ is a right-sided distribution.

From equation (7), we get

$$t^n y^{(n)}(t) + m_1 t^{n-1} y^{(n-1)}(t) + \cdots + m_{n-1} t y'(t) + m_n y(t) = f(t) \qquad (8)$$

where m_1, m_2, ..., m_n are real numbers $t \in R$ and $f(t)$ is a right-sided distribution. The solution of equation (8) is $y(t) = y_h(t) + y_p(t)$ where $y_h(t)$ is a complementary function and $y_p(t)$ is a particular solution.

From equation (8), if $f(t) = 0$ we get

$$t^n y^{(n)}(t) + m_1 t^{n-1} y^{(n-1)}(t) + \cdots + m_{n-1} t y'(t) + m_n y(t) = 0 \qquad (9)$$

where m_1, m_2, ..., m_n are real numbers and $t \in R$. The solution of equation (9) is $y_h(t)$.

Theorem 2.1

Consider the n^{th} order non-homogeneous Euler equation of the form

$$t^n y^{(n)}(t) + m_1 t^{n-1} y^{(n-1)}(t) + \cdots + m_{n-1} t y'(t) + m_n y(t) = f(t) \qquad (10)$$

where m_1, m_2, ..., m_n are real numbers, $t \in R$ and $f(t)$ is a right-sided distribution. Let r_1, r_2, ..., r_n are distinct complex numbers and X_i is defined by

$X_0 = 1$,

$X_1 = r_1 + r_2 + \cdots + r_n$,

$$X_2 = r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + r_2r_4 + \cdots + r_2r_n + \cdots + r_{n-1}r_n ,$$
$$X_3 = r_1r_2r_3 + r_1r_2r_4 + \cdots + r_1r_2r_n + r_1r_3r_4 + r_1r_3r_5 + \cdots + r_1r_3r_n + \cdots + r_{n-2}r_{n-1}r_n ,$$
$$\vdots$$
$$X_n = r_1r_2r_3 \cdots r_n .$$

If $m_k = \sum_{i=0}^{k} U_{n-k+1}\left(k+1-i\right)X_i$ for $k = 0,1,2,\ldots,n$ where $m_0 = 1$, $U_1\left(v\right) = 1,\ U_p\left(1\right) = 1$ and $U_p\left(v\right) = \sum_{i=1}^{p} iU_i\left(v-1\right)$, $p, v \in N$ then

(i) the complementary function of equation (10) is $y_h(t) = \sum_{i=1}^{n} c_i \delta^{(r_i)}(t)$ where $r_1, r_2, \ldots, r_n$ are distinct non-negative integers and $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) the complementary function of equation (10) is $y_h(t) = \sum_{i=1}^{n} c_i \dfrac{t^{-r_i-1}H(t)}{\Gamma(-r_i)}$ for $\mathrm{Re}(r_i) < 0$, $i = 1,2,3,\ldots,n$ and $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(iii) the complementary function of equation (10) is $y_h(t) = \sum_{i=1}^{n} c_i \dfrac{Pf\left(t^{-r_i-1}H(t)\right)}{\Gamma(-r_i)}$ for each i which there exists positive integer w_i such that $w_i - 1 < \mathrm{Re}(r_i) < w_i$ and $c_1, c_2, \ldots, c_n$ are arbitrary constants and $Pf\left(g(t)\right)$ is a pseudo-function which is a finite part of $\int_{-\infty}^{\infty} g(t)\phi(t)dt$.

Proof: Consider equation

$$t^n y^{(n)}(t) + m_1 t^{n-1} y^{(n-1)}(t) + \cdots + m_{n-1} t y'(t) + m_n y(t) = 0 \qquad (11)$$

where m_1, m_2, …, m_n are real numbers, $t \in R$ and $m_0 = 1$. We have

$$\sum_{i=0}^{n} m_{n-i}\, t^i y^{(i)}(t) = 0 \qquad (12)$$

Taking Laplace Transforms to equation (12), we get

$$\sum_{i=0}^{n} (-1)^i \frac{d^i}{ds^i}\left[m_{n-i} \cdot s^i Y(s)\right] = 0$$

$$\sum_{i=0}^{n}(-1)^i m_{n-i}\frac{d^i}{ds^i}\left[s^i Y(s)\right]=0$$

$$\sum_{i=0}^{n}(-1)^i m_{n-i}\sum_{k=0}^{i}\left[\binom{i}{k}\binom{i}{k}k!\,s^{i-k}Y^{(i-k)}(s)\right]=0. \qquad (13)$$

Let $Y(s)=s^r$ where r is an arbitrary constant.

From Example 1.4, we get $Y^{(i-k)}(s)=r(r-1)(r-2)\cdots(r-i+k-1)s^{r-i+k}$.

Now equation (13) becomes

$$\sum_{i=0}^{n}\left\{(-1)^i m_{n-i}\sum_{k=0}^{i}\left[\binom{i}{k}\binom{i}{k}k!\binom{r}{i-k}(i-k)!\,s^r\right]\right\}=0$$

$$s^r\sum_{i=0}^{n}\left\{(-1)^i m_{n-i}\sum_{k=0}^{i}\left[\binom{i}{k}\binom{i}{k}k!\binom{r}{i-k}(i-k)!\right]\right\}=0$$

Since $s^r\neq 0$, we get

$$\sum_{i=0}^{n}\left\{(-1)^i m_{n-i}\sum_{k=0}^{i}\left[\binom{i}{k}\binom{i}{k}k!\binom{r}{i-k}(i-k)!\right]\right\}=0$$

$$m_n-\sum_{i=1}^{n}\left\{(-1)^{i-1} m_{n-i}\sum_{k=0}^{i}\left[\binom{i}{k}\binom{i}{k}k!\binom{r}{i-k}(i-k)!\right]\right\}=0 \qquad (14)$$

Suppose $m_k=\sum_{i=0}^{k}U_{n-k+1}(k+1-i)X_i$ for $k=0,1,2,\ldots,n$ and $m_0=1$ where $U_p(v)=\sum_{i=1}^{p}iU_i(v-1)$ and $U_1(v)=1, U_p(1)=1$, $p,v\in N$ and X_i is defined by

$X_0=1,$

$X_1=r_1+r_2+\cdots+r_n,$

$X_2=r_1r_2+r_1r_3+\cdots+r_1r_n+r_2r_3+r_2r_4+\cdots+r_2r_n+\cdots+r_{n-1}r_n$

$X_3=r_1r_2r_3+r_1r_2r_4+\cdots+r_1r_2r_n+r_1r_3r_4+r_1r_3r_5+\cdots+r_1r_3r_n+\cdots+r_{n-2}r_{n-1}r_n,$

$\vdots$

$X_n=r_1r_2r_3\cdots r_n.$

From equation (12), we get

$$X_n-rX_{n-1}+r^2X_{n-2}-r^3X_{n-3}+\cdots+(-1)^{n-1}r^{n-1}X_1+(-1)^n r^n X_0=0$$

$$r^nX_0-r^{n-1}X_1+r^{n-2}X_2-r^{n-3}X_3+\cdots+(-1)^{n-1}rX_{n-1}+(-1)^n X_n=0$$

$$r^n-r^{n-1}(r_1+r_2+\cdots+r)+r^{n-2}(r_1r_2+r_1r_3+\cdots+r_{n-1}r)+\cdots+(-1)^n(r_1r_2r_3\cdots r)=0$$

$$(r-r_1)(r-r_2)(r-r_3)\cdots(r-r_n)=0$$

where $r = r_1, r_2, \ldots, r_n$.

Hence $Y(s) = s^{r_1}, s^{r_2}, \ldots, s^{r_n}$ are solutions of equation (13).

Taking inverse Laplace Transform, we get

(i) $y_h(t) = \sum_{i=1}^{n} c_i \delta^{(r_i)}(t)$ is the solution of equation (11) for $r_1, r_2, \ldots, r_n$ are non-negative integers and $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) $y_h(t) = \sum_{i=1}^{n} c_i \dfrac{t^{-r_i-1}H(t)}{\Gamma(-r_i)}$ is the solution of equation (11) for $\mathrm{Re}(r_i) < 0$, $i = 1,2,3,\ldots,n$ and $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(iii) $y_h(t) = \sum_{i=1}^{n} c_i \dfrac{Pf\left(t^{-r_i-1}H(t)\right)}{\Gamma(-r_i)}$ is the solution of equation (11) for each i which there exists positive integer w_i such that $w_i - 1 < \mathrm{Re}(r_i) < w_i$ and $c_1, c_2, \ldots, c_n$ are arbitrary constants.

□

Conclusion

By using Laplace transform, we found that a complementary function of the n^{th} order non-homogeneous Euler equation of the form

$$m_0 t^n y^{(n)}(t) + m_1 t^{n-1} y^{(n-1)}(t) + \cdots + m_{n-1} t y'(t) + m_n y(t) = f(t)$$

where m_0, m_1, m_2, …, m_n are real number $m_0 \neq 0$, $t \in R$ and $f(t)$ is a right-sided distribution. It is investigated under the conditions on the values of m_0, m_1, m_2, …, and m_n .

Acknowledgement

THIS RESEARCH WAS PARTLY SUPPORTED BY FACULTY OF SCIENCE AND TECHNOLOGY, RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI (RMUTT), PATHUM THANI, THAILAND.

References

Hongsit, N. (2000). On the Lattice Plane of the Fourth Order Euler Equation. Chiang Mai University.

Kananthai, A. (1999). Distribution Solutions of the Third Order Euler Equation. Southeast Asian Bulletin of Mathematics 23.