On the Odd and Even Terms of (p,q) - Fibonacci Number and (p,q) - Lucas Number

Alongkot Suvarnamani *

Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathumthani, Thailand.

*Email: kotmaster2@rmutt.ac.th

Abstract

We consider the (p,q) - Fibonacci sequence and the (p,q) - Lucas sequence. By using the Binet's formulas, we get some properties of the odd and even terms of the (p,q) - Fibonacci number and the (p,q) - Lucas number.

Keywords: (p,q)- Fibonacci number, (p,q)- Lucas number, Binet's formula

INTRODUCTION

Falcon and Plaza (2007) considered the k - Fibonacci sequence $\{F_{k,n}\}$ which is defined as $F_{k,0}=0,\ \ F_{k,1}=1$ and $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$ for $n\geq 1,\ k\geq 1.$ We get the classical Fibonacci sequence $\{0,1,1,2,3,5,8,13,\ldots\}$ for $k=1$. The well-known Binet's formula for the k - Fibonacci number, see [1], is given by $F_{k,n}=\dfrac{r_1^n-r_2^n}{r_1-r_2}$ where $r_1=\dfrac{k+\sqrt{k^2+4}}{2}$ and $r_2=\dfrac{k-\sqrt{k^2+4}}{2}$ are roots of the characteristic equation $r^2-kr-1=0$.

In 2007, Falcon and Plaza (2007) studied the k - Fibonacci sequence and the Pascal 2- triangle. Next, they considered the 3 - dimensional k - Fibonacci spiral in (Falcon and Plaza, 2008)

Some combinatorial identities for the k - Fibonacci and k - Lucas numbers was proved such as $F_{k,2n}L_{k,2n}=F_{k,4n}$, $F_{k,2n+1}L_{k,2n}=F_{k,4n+1}+1$and $F_{k,2n}L_{k,2n+1}=F_{k,4n+1}-1$.

The (p,q) - Fibonacci number which be defined as $F_{p,q,0}=0,\ \ F_{p,q,1}=1$ and $F_{p,q,n}=pF_{p,q,n-1}+qF_{p,q,n-2}$ for $p\geq 1,\ q\geq 1$ and $n\geq 2.$ And the (p,q) - Lucas number which be defined as $L_{p,q,0}=2, L_{p,q,1}=p$ and $L_{p,q,n}=pL_{p,q,n-1}+qL_{p,q,n-2}$ for $p\geq 1,\ q\geq 1$ and $n\geq 2.$ Moreover, the Binet's formulas for the (p,q) - Fibonacci number and the (p,q) - Lucas number are given by $F_{p,q,n}=\frac{r_1^n-r_2^n}{r_1-r_2}$ and $L_{p,q,n}=r_1^n+r_2^n$ where $r_1=\frac{p+\sqrt{p^2+4q}}{2}$ and $r_2=\frac{p-\sqrt{p^2+4q}}{2}$ are roots of the characteristic equation $r^2-pr-q=0.$ We note that $r_1+r_2=p,\quad r_1r_2=-q$ and $r_1-r_2=\sqrt{p^2+4q}.$

In 2015, Suvarnamani and Tatong (2015) showed some properties of the (p,q) - Fibonacci number. For examples, $F_{p,q,n+1}F_{p,q,n-1}-F_{p,q,n}^2=(-1)^n q^{n-1}$ and $F_{p,q,m+n}=F_{p,q,m}F_{p,q,n+1}+qF_{p,q,m-1}F_{p,q,n}$. Then Suvarnamani (2016) showed some properties of the (p,q) - Lucas number in 2016 such as $L_{p,q,n+1}L_{p,q,n-1}-L_{p,q,n}^2=(-q)^{n-1}(p^2+4q)$ and $L_{p,q,n-2}L_{p,q,n+1}-L_{p,q,n-1}L_{p,q,n}=(-q)^{n-2}(p^3+4pq)$.

Now we use the Binet's formulas for finding some properties of the odd and even terms of the (p,q) - Fibonacci number and the (p,q) - Lucas number.

MAIN RESULT

We get 6 theorems which are some properties of the odd and even terms of the (p,q) - Fibonacci number and the (p,q) - Lucas number. Frist, we get the result of the even term of the (p,q) - Fibonacci number. We get the following result.

Theorem 2.1. Let $p,\ q$ and n be positive integers. Then $F_{p,q,2n}=F_{p,q,n}\cdot L_{p,q,n}$.

Proof. We have

$$F_{p,q,2n}=\frac{r_1^{2n}-r_2^{2n}}{r_1-r_2}$$

$$=\frac{\left(r_1^n\right)^2-\left(r_2^n\right)^2}{r_1-r_2}$$

$$=\frac{\left(r_1^n-r_2^n\right)\left(r_1^n+r_2^n\right)}{r_1-r_2}$$

$$= \frac{\left(r_1^n - r_2^n\right)}{r_1 - r_2}\left(r_1^n + r_2^n\right)$$

$$= F_{p,q,n} \cdot L_{p,q,n}.$$

□

Next, we get the result of the even term of the (p,q) - Lucas number. We get the following result.

Theorem 2.2. Let p, q and n be positive integers with $n \geq 2.$ Then $L_{p,q,2n} = \frac{\left(p^2 + 4q\right)F_{p,q,n}^2 + L_{p,q,n}^2}{2}.$

Proof. We have $L_{p,q,2n} = r_1^{2n} + r_2^{2n}$

$$= \frac{2r_1^{2n} + 2r_2^{2n}}{2}$$

$$= \frac{\left(r_1^{2n} - 2r_1^n r_2^n + r_2^{2n}\right) + \left(r_1^{2n} + 2r_1^n r_2^n + r_2^{2n}\right)}{2}$$

$$= \frac{\left(r_1^n - r_2^n\right)^2 + \left(r_1^n + r_2^n\right)^2}{2}$$

$$= \frac{\left(r_1^n - r_2^n\right)^2 \left(\frac{r_1 - r_2}{r_1 - r_2}\right)^2 + \left(r_1^n + r_2^n\right)^2}{2}$$

$$= \frac{\left(p^2 + 4q\right)F_{p,q,n}^2 + L_{p,q,n}^2}{2}.$$

□

Then we get the result of the product between the odd term of the (p,q) - Fibonacci number and the even term of the (p,q) - Fibonacci number. We get the following result.

Theorem 2.3. Let p, q and n be positive integers. Then $F_{p,q,2n} \cdot F_{p,q,2n+1} = \frac{L_{p,q,4n+1} - pq^{2n}}{p^2 + 4q}.$

Proof. We have $F_{p,q,2n} \cdot F_{p,q,2n+1} = \frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \cdot \frac{r_1^{2n+1} - r_2^{2n+1}}{r_1 - r_2}$

$$= \frac{r_1^{4n+1} - r_1^{2n+1} r_2^{2n} - r_1^{2n} r_2^{2n+1} + r_2^{4n+1}}{\left(r_1 - r_2\right)^2}$$

$$= \frac{r_1^{4n+1} + r_2^{4n+1} - \left(r_1^{2n+1} r_2^{2n} + r_1^{2n} r_2^{2n+1}\right)}{\left(r_1 - r_2\right)^2}$$

$$= \frac{r_1^{4n+1} + r_2^{4n+1} - r_1^{2n} r_2^{2n}\left(r_1 + r_2\right)}{\left(r_1 - r_2\right)^2}$$

$$= \frac{L_{p,q,4n+1} - (-q)^{2n}(p)}{p^2 + 4q}$$

$$= \frac{L_{p,q,4n+1} - pq^{2n}}{p^2 + 4q}.$$

□

Next, we get the result of the product between the odd term of the (p,q) - Lucas number and the even term of the (p,q) - Lucas number. We get the following result.

Theorem 2.4. Let p, q and n be positive integers. Then $L_{p,q,2n} \cdot L_{p,q,2n+1} = L_{p,q,4n+1} + pq^{2n}$.

Proof. We have

$$L_{p,q,2n} \cdot L_{p,q,2n+1} = \left(r_1^{2n} + r_2^{2n}\right) \cdot \left(r_1^{2n+1} + r_2^{2n+1}\right)$$

$$= r_1^{4n+1} + r_1^{2n+1} r_2^{2n} + r_1^{2n} r_2^{2n+1} + r_2^{4n+1}$$

$$= \left(r_1^{4n+1} + r_2^{4n+1}\right) + \left(r_1^{2n+1} r_2^{2n} + r_1^{2n} r_2^{2n+1}\right)$$

$$= \left(r_1^{4n+1} + r_2^{4n+1}\right) + r_1^{2n} r_2^{2n}\left(r_1 + r_2\right)$$

$$= L_{p,q,4n+1} + pq^{2n}.$$

□

Then we get the result of the product between the odd term of the (p,q) - Lucas number and the even term of the (p,q) - Fibonacci number. We get the following result.

Theorem 2.5. Let p, q and n be positive integers. Then $F_{p,q,2n} \cdot L_{p,q,2n+1} = F_{p,q,4n+1} - q^{2n}$.

Proof. We have

$$F_{p,q,2n} \cdot L_{p,q,2n+1} = \left(\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right) \cdot \left(r_1^{2n+1} + r_2^{2n+1}\right)$$

$$= \frac{r_1^{4n+1} - r_1^{2n+1} r_2^{2n} + r_1^{2n} r_2^{2n+1} - r_2^{4n+1}}{r_1 - r_2}$$

$$= \frac{\left(r_1^{4n+1} - r_2^{4n+1}\right) - \left(r_1^{2n+1} r_2^{2n} - r_1^{2n} r_2^{2n+1}\right)}{r_1 - r_2}$$

$$= \frac{\left(r_1^{4n+1} - r_2^{4n+1}\right) - r_1^{2n} r_2^{2n} \left(r_1 - r_2\right)}{r_1 - r_2}$$

$$= F_{p,q,4n+1} - q^{2n}.$$

□

Then we get the result of the product between the even term of the (p, q) - Lucas number and the odd term of the (p, q) - Fibonacci number. We get the following result.

Theorem 2.6. Let p, q and n be positive integers. Then $F_{p,q,2n+1} \cdot L_{p,q,2n} = F_{p,q,4n+1} + q^{2n}$.

Proof. We have

$$F_{p,q,2n+1} \cdot L_{p,q,2n} = \left(\frac{r_1^{2n+1} - r_2^{2n+1}}{r_1 - r_2}\right) \cdot \left(r_1^{2n} + r_2^{2n}\right)$$

$$= \frac{r_1^{4n+1} + r_1^{2n+1} r_2^{2n} - r_1^{2n} r_2^{2n+1} - r_2^{4n+1}}{r_1 - r_2}$$

$$= \frac{\left(r_1^{4n+1} - r_2^{4n+1}\right) + \left(r_1^{2n+1} r_2^{2n} - r_1^{2n} r_2^{2n+1}\right)}{r_1 - r_2}$$

$$= \frac{\left(r_1^{4n+1} - r_2^{4n+1}\right) + r_1^{2n} r_2^{2n} \left(r_1 - r_2\right)}{r_1 - r_2}$$

$$= F_{p,q,4n+1} + q^{2n}.$$

□

Remark 2.7. If $p = 1$ and $q = 1$ then we get the classical Fibonacci sequence $\{0,1,1,2,3,5,8,13,\ldots\}$ and the classical Lucas sequence $\{2,1,3,4,7,11,18,29,\ldots\}$. Then

1) From Theorem 2.1, we get $F_{1,1,2n} = F_{1,1,n} \cdot L_{1,1,n}$. It is similarly as $F_{2n} = F_n \cdot L_n$.

2) From Theorem 2.2, we get $L_{1,1,2n} = \dfrac{5F_{1,1,n}^2 + L_{1,1,n}^2}{2}$. It is similarly as $L_{2n} = \dfrac{5F_n^2 + L_n^2}{2}$.

3) From Theorem 2.3, we get $F_{1,1,2n} \cdot F_{1,1,2n+1} = \dfrac{L_{1,1,4n+1} - 1}{5}$. It is similarly as $F_{2n} \cdot F_{2n+1} = \dfrac{L_{4n+1} - 1}{5}$.

4) From Theorem 2.4, we get $L_{1,1,2n} \cdot L_{1,1,2n+1} = L_{1,1,4n+1} + 1$. It is similarly as $L_{2n} \cdot L_{2n+1} = L_{4n+1} + 1$.

5) From Theorem 2.5, we get $F_{1,1,2n} \cdot L_{1,1,2n+1} = F_{1,1,4n+1} - 1.$ It is similarly as $F_{2n} \cdot L_{2n+1} = F_{4n+1} - 1.$

6) From Theorem 2.6, we get $F_{1,1,2n+1} \cdot L_{1,1,2n} = F_{1,1,4n+1} + 1.$ It is similarly as $F_{2n+1} \cdot L_{2n} = F_{4n+1} + 1.$

Acknowledgement

This research was partly supported by Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathum Thani, THAILAND. (RMUTT Annual Government Statement of Expenditure in 2017)

REFERENCES

Sergio, F. & Ángel, P. (2007). On the Fibonacci k-numbers, Chaos, Solitons and Fractals, 32: 1615 - 1624.

Sergio, F. & Ángel, P. (2007). The k-Fibonacci Sequence and the Pascal 2- Triangle, Chaos, Solitons and Fractals, 33: 38 - 49.

Sergio F. & Ángel, P. (2008). On the 3-Dimensional k-Fibonacci Spiral, Chaos, Solitons and Fractals, 38: 993-1003.

Suvarnamani, A. & Tatong, M. (2015). Some Properties of (p,q)- Fibonacci Numbers, Science and Technology RMUTT Journal, 5(2): 17 - 21.

Suvarnamani, A. (2016). Some Properties of (p,q)- Lucas Number, Kyungpook Mathematical Journal, 56: 367-370.