A Comparison of Methods for the Estimation of two-parameter Weibull Distribution with Outlier

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Abstract

In this study, we compare the three different methods for the joint estimation of both scale and shape parameters for two-parameter in Weibull distribution when data are contaminated with outliers: the method of moment (MOM), the maximum likelihood method (MLE) and the weighted likelihood method (WLE). The performance of these methods is compared using the Monte Carlo s imulation and the efficiency of these methods is compared based on RMSE. From simulation results, we found that the WLE outperforms the other methods for the scale parameter in term of RMSE of the estimator for scale parameter for $\beta = 0.5$. For the RMSE of the estimator for shape parameter of the MLE approach provides better results for the scale parameter when outliers in the data set are small. In term of RMSE for two parameter estimators, the WLE performs better than the other methods when $\beta = 0.5$. For $\beta = 1$ and 1.5, the MOM also performs well, especially and outliers in the data set are small.

Keywords: Maximum Likelihood Estimator, Weighted Likelihood Estimator, Method of Moment, Contamination

Introduction

The Weibull distribution is widely used in reliability and life data analysis its usefulness in many fields. When samples are collected, the two–parameter Weibull is often used as the first step of the Weibull analysis. Applications of the two-parameter Weibull distribution, we mention wind speed amount, prediction of water levels, and analysis of lifetime of materials. The distribution function for the two-parameter Weibull distribution is

$$F(x;\delta,\beta) = 1 - e^{-\left(x/\delta\right)^{\beta}} \qquad ;x \geq 0, \delta > 0, \beta > 0, \tag{1}$$

and the probability density function is

$$f(x;\delta,\beta) = \frac{\beta}{\delta^{\beta}} x^{\beta-1} e^{-\left(x/\delta\right)^{\beta}} \qquad ;x \geq 0, \delta > 0, \beta > 0, \tag{2}$$

where δ is the scale parameter and β is the shape parameter of the distribution (Murthy et al., 2004). Let $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ be sample values from a distribution with a density function $f(x;\delta,\beta)$ the maximum likelihood estimator (MLE) of $\{\delta,\beta\}$ are obtained by maximizing the maximum likelihood function

$$L(\delta,\beta|x^{(n)})=\sum_{i=1}^{n} ln\, f(x_i;\delta,\beta). \quad (3)$$

The Weibull distribution can be used to model many of life behaviors. For the two-parameter Weibull distribution plays a central role in lifetime models. If a dataset is contaminated with outliers, the maximum likelihood estimator (MLE) can be very unreliable (Boudt et al., 2011). Ahmed et al. (2005) propose the method of weighted likelihood estimator (WLE). They consider an estimation method based on the maximizing of weighted likelihood function of exponential distribution parameters. They establish that the suggested weighted likelihood method provides α-trimmed mean type estimators and suggest a special choice of weights which connected with the theory of robust estimation, and based on the maximum likelihood method with rejection of spurious observations. The WLE basically provides the MLE if we assume all weights are equal one. The statistical asymptotic properties of the WLE are developed and a simulation study is conducted to appraise the behavior of the proposed estimators for moderate samples.

Boonlha (2013) proposed for robust of the weighted likelihood estimator (WLE) for the Weibull distribution parameters were considered only an estimator of the scale parameter of the Weibull distribution and assume that the shape parameter is known. The WLE method can be extended to estimate the two-parameter Weibull distribution when we assume both of the scale and shape parameter are unknown. Hence, in this study the WLE is applied to the Weibull distribution for two parameters, when the data set are contaminated with outliers. However, the WLE method can be extended to estimate the two-parameter Weibull distribution, we assume both scale and shape parameters are unknown. So, in this simulation studies are extended to compare the MLE, WLE, and MOM methods based on root mean square error.

The rest of this paper is organized as follows. In section two we define the WLE for the two-parameter Weibull distribution. In next section, we compare the MLE, WLE and MOM in terms of root mean square error. Some conclusions remarks are finally made in last section.

Estimation methods

The distribution function for the two-parameter Weibull distribution is

$$F(x;\delta,\beta) = 1-e^{-\left(x/\delta\right)^{\beta}} \qquad ; x \ge 0, \delta > 0, \beta > 0, \quad (4)$$

and the probability density function is

$$f(x;\delta,\beta)=\frac{\beta}{\delta^{\beta}}x^{\beta-1}e^{-(x/\delta)^{\beta}} \quad ;x\geq 0,\delta>0,\beta>0, \tag{5}$$

where δ is the scale parameter and β is the shape parameter of the distribution (Murthy et al., 2004). Assume that the sample $x_1,x_2,\dots,x_n$ is taken from a population that follows a distribution with the distribution function $G_{\varepsilon}(x)$ to be defined now. We define the ε – contamination model as

$$G_{\varepsilon}(x)=(1-\varepsilon)F(x;\delta,\beta)+\varepsilon F_1(x;\delta_1,\beta_1) \tag{6}$$

where $F(x;\delta,\beta)$ is the Weibull distribution with parameters (δ,β), and contamination $F_1(x;\delta_1,\beta_1)$ is the Weibull distribution with parameters (δ_1,β_1) where $\delta_1=\Delta_1\delta,\ \beta_1=\Delta_2\beta,\quad \Delta_1,\Delta_2\geq 0$, and ε denotes the contamination proportion, $0\leq\varepsilon\leq 1$. For instance, one may use of 90 % the Weibull distribution, and 1 % the Weibull distribution with the same shape but difference scale which representing outliers. Figure 1 shows the probability density function plot for the Weibull distribution with the Weibull distribution contamination when shape parameter is 1.5, scale parameter is 1, $\Delta_1=2$ and $\Delta_2=1$. The goal is still to estimate the parameters from the data, as if the contamination did not exist.

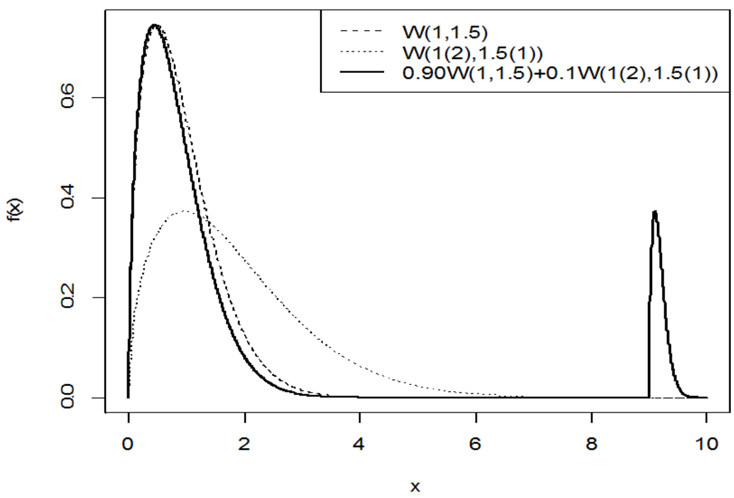

Figure 1. The probability density function plot for the Weibull distribution with the Weibull distribution contamination when shape parameter is 1.5, scale parameter is 1, $(\Delta_1,\Delta_2)=(2,1)$

The method of moments (MOM)

The k^{th} moment of the Weibull distribution (Murthy et al., 2004) about the mean, μ_k, is given by

$$\mu_k = \delta^k \sum_{j=0}^{k} (-1)^k \binom{k}{j} \Gamma\left(\frac{k-j}{\beta-1}\right)\left(\Gamma\left(\frac{1}{\beta+1}\right)\right)^j . \tag{7}$$

The mean μ is given by

$$\mu = \delta\Gamma\left(1+\frac{1}{\beta}\right). \tag{8}$$

The variance σ^2 is given by

$$\sigma^2 = \delta^2\left(\Gamma\left(\frac{2}{\beta+1}\right)-\left(\Gamma\left(\frac{1}{\beta+1}\right)\right)^2\right)+6\delta^4\left(\Gamma\left(\frac{2}{\beta+1}\right)\left(\Gamma\left(\frac{1}{\beta+1}\right)\right)^3-3\left(\Gamma\left(\frac{1}{\beta+1}\right)\right)^3\right). \tag{9}$$

The method of moments is based on expressing *k* moments in terms of the parameters of the distribution. Using sample moments in place of the moments in these relationships yields k equations containing the *k* unknown parameters. Solving these yields the estimates. Note that in most cases, the estimates need to be obtained using numerical techniques. One can use the moments $M_k=E[X^k]$, the central moments $\mu_k=E[(X-\mu)^k]$ or a combination of the two. In the case of complete data $x_1, x_2, ..., x_n$, the sample moments $\hat{M}_k(k=1,2,3)$ and sample central moments are obtained as follows: $\hat{M}_k=\frac{1}{n}\sum_{i=1}^{n} x_i^k$ and $\hat{\mu}_k=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^k$ where $\bar{x}=\hat{M}_1$. For the most, momemt estimators are asymptotically consistent and are normally distribution.

The maximum likelihood estimator (MLE)

Let $x^{(n)}=\{x_1,x_2,...,x_n\}$ be sample values from a distribution, the maximum likelihood estimator $\{\delta,\beta\}$ of the parameter $\{\delta,\beta\}$ be defined as the solution of the equation

$$\frac{\sum_{i=1}^{n}\partial f(x_i;\delta,\beta)}{\partial\delta} = 0 \quad \text{and} \quad \frac{\sum_{i=1}^{n}\partial f(x_i;\delta,\beta)}{\partial\beta} = 0, \tag{10}$$

The log-likelihood function is

$$ln\, f(x_i;\delta,\beta) = ln(\beta) - \beta ln(\delta) + (\beta-1)ln(x_i) - \left(\frac{x_i}{\delta}\right)^\beta . \tag{11}$$

Differentiating (11) with respect to δ and β we have

$$\frac{\partial ln\, f(x_i;\delta,\beta)}{\partial\delta} = -\frac{\beta}{\delta}+\frac{\beta}{\delta}\left(\frac{x_i}{\delta}\right)^\beta , \tag{12}$$

and

$$\frac{\partial lnf(x_i;\delta,\beta)}{\partial\beta} = \frac{1}{\beta} - ln(\delta) + ln(x_i) - \left(\frac{x_i}{\delta}\right)^{\beta} ln\left(\frac{x_i}{\delta}\right) = \frac{1}{\beta} + ln\left(\frac{x_i}{\delta}\right) - \left(\frac{x_i}{\delta}\right)^{\beta} ln\left(\frac{x_i}{\delta}\right). \tag{13}$$

Then

$$\sum_{i=1}^{n} \frac{\partial lnf(x_i;\delta,\beta)}{\partial\delta} = \sum_{i=1}^{n}\left(-\frac{\beta}{\delta}+\frac{\beta}{\delta}\left(\frac{x_i}{\delta}\right)^{\beta}\right) = -\frac{n\beta}{\delta} + \frac{\beta}{\delta}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\beta}, \tag{14}$$

and

$$\sum_{i=1}^{n} \frac{\partial ln\, f(x_i;\delta,\beta)}{\partial\beta} = \sum_{i=1}^{n}\left(\frac{1}{\beta}+ln\left(\frac{x_i}{\delta}\right)-\left(\frac{x_i}{\delta}\right)^{\beta} ln\left(\frac{x_i}{\delta}\right)\right) = \frac{n}{\beta}+\sum_{i=1}^{n} ln\left(\frac{x_i}{\delta}\right)-\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\beta} ln\left(\frac{x_i}{\delta}\right). \tag{15}$$

Set the (12) and (13) equal to zero, from (14), we get

$$\delta=\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{1}{\beta}}. \tag{16}$$

If β is obtained, the δ can be determined. To solve β by using Newton–Raphson method.

The weighted likelihood estimator (WLE)

The method of weighted likelihood estimator (WLE) basically provides the MLE if we assume all weights are equal one. The statistical asymptotic properties of the WLE are developed and a simulation study is conducted to appraise the behavior of the proposed estimators for moderate samples. Estimators of $\{\delta,\beta\}$ obtained by maximizing the weighted log likelihood function

$$L(\delta,\beta \mid x^{(n)}) = \sum_{i=1}^{n} w_i(x^{(n)}) ln\left(f(x_i;\delta,\beta)\right), \tag{17}$$

where $w_i(x^{(n)}),\ 1 \le i \le n$ depends on the sample are called weighted likelihood estimators (WLE) of $\{\delta,\beta\}$. Following the idea presented in Ahmed et al. (2005), let the weight w_i that corresponds to the i^{th} observation to be 1, if its estimated likelihood is sufficiently large, and 0 elsewhere. To be more precise, let

$$w_i = \begin{cases} 1 & \text{if} \quad f(x_i;\delta,\beta)>C \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

where $\{\delta,\beta\}$ be the MLE of the parameter $\{\delta,\beta\}$. This means that we delete all improbable observations from the sample, we reject only extreme order statistics. Following the ideas of Ahmed, Volodin, and Hussein (2005), we suggest this not be considered as a constant. Rather assume that *C* is chosen from the condition of a small probability of rejection of an observation when we sample from the non contamination Weibull distribution with cumulative distribution function, $F(x;\delta,\beta)$. Hence, we define *C* by the given pre-assigned small probability α as $P\left[f\left(\max_{1\le i\le n} X_i;\delta,\beta\right)<C\right]=\alpha.$ So we get, $C\approx\frac{\alpha\beta}{n\delta}$. The WLE estimator $\{\delta,\beta\}$ of the parameter $\{\delta,\beta\}$ is defined as the solution of the equation

$$\frac{\sum_{k=1}^{m}\partial f(x_{i_k};\delta,\beta)}{\partial\delta}=0 \quad\text{and}\quad \frac{\sum_{k=1}^{m}\partial f(x_{i_k};\delta,\beta)}{\partial\beta}=0, \tag{19}$$

where $x_{i_1}, x_{i_2}, ..., x_{i_m}$ are the remaining observations in the sample after the rejection method. The log-likelihood function is

$$ln\, f(x_{i_k};\delta,\beta)=ln(\beta)-\beta ln(\delta)+(\beta-1)ln(x_{i_k})-\left(\frac{x_{i_k}}{\delta}\right)^{\beta}. \tag{20}$$

Differentiating (20) with respect to δ and β we have

$$\frac{\partial\, ln\, f(x_{i_k};\delta,\beta)}{\partial\delta}=-\frac{\beta}{\delta}+\frac{\beta}{\delta}\left(\frac{x_{i_k}}{\delta}\right)^{\beta}, \tag{21}$$

and

$$\frac{\partial\, ln\, f(x_{i_k};\delta,\beta)}{\partial\beta}=\frac{1}{\beta}-ln(\delta)+ln(x_{i_k})-\left(\frac{x_{i_k}}{\delta}\right)^{\beta}ln\left(\frac{x_{i_k}}{\delta}\right)=\frac{1}{\beta}+ln\left(\frac{x_{i_k}}{\delta}\right)-\left(\frac{x_{i_k}}{\delta}\right)^{\beta}ln\left(\frac{x_{i_k}}{\delta}\right). \tag{22}$$

Then

$$\sum_{k=1}^{m}\frac{\partial\, ln\, f(x_{i_k};\delta,\beta)}{\partial\delta}=\sum_{k=1}^{m}\left(-\frac{\beta}{\delta}+\frac{\beta}{\delta}\left(\frac{x_{i_k}}{\delta}\right)^{\beta}\right)=-\frac{m\beta}{\delta}+\frac{\beta}{\delta}\sum_{k=1}^{m}\left(\frac{x_{i_k}}{\delta}\right)^{\beta}, \tag{23}$$

and

$$\sum_{k=1}^{m}\frac{\partial\, ln\, f(x_{i_k};\delta,\beta)}{\partial\beta}=\sum_{k=1}^{m}\left(\frac{1}{\beta}+ln\left(\frac{x_{i_k}}{\delta}\right)-\left(\frac{x_{ik}}{\delta}\right)^{\beta}ln\left(\frac{x_{i_k}}{\delta}\right)\right)=\frac{m}{\beta}+\sum_{k=1}^{m}ln\left(\frac{x_{i_k}}{\delta}\right)-\sum_{k=1}^{m}\left(\frac{x_{ik}}{\delta}\right)^{\beta}ln\left(\frac{x_{i_k}}{\delta}\right). \tag{24}$$

Set the (21) and (22) equal to zero, from (23), we get

$$\delta=\left(\frac{1}{m}\sum_{k=1}^{m}x_{i_k}^{\beta}\right)^{\frac{1}{\beta}}. \tag{25}$$

If β is obtained,the δ can be determined. To solve β by using Newton–Raphson method.

Simulation Study

The main objective of this study is to compare the performance of three different methods to estimate the shape and scale parameters of two-parameter Weibull distribution with outlier. A simulation study we generate the random sample data from the ε –contamination model with two-parameter Weibull distribution performance of three different methods to estimate the scale and shape parameters. The central distribution to be the Weibull with scale parameter $\delta = 1$ and shape parameter β = 0.5, 1, and 1.5. The contamination has the Weibull distribution with scale parameter $\delta_2 = \Delta_1\delta$ where $\Delta_1 = 1, 2$ and shape parameter $\beta_2 = \Delta_2\beta$ where $\Delta_2 = 1, 2$ and the contamination proportion ε = 0.01 and 0.10 and the values of α =0.01, 0.05 and 0.10. In this simulation study, we have chosen sample n =30, 50 and 100 to represent small, moderate and large sample size. All the following simulations results are based on 5,000 replicates and the simulation was done using statistical software R version 3.3.1. Then we compute the root mean square error (RMSE) of the estimator for scale and shape parameter $RMSE_\delta = \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\delta}_i - \delta\right)^2}$, and $RMSE_\beta = \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\beta}_i - \beta\right)^2}$. To compare the performance of the methods we compute the sample root mean square error (RMSE) of two parameter given by $RMSE = \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}\left[\left(\hat{\delta}_i - \delta\right)^2 + \left(\hat{\beta}_i - \beta\right)^2\right]}$. The estimator with smaller RMSE are preferred.

Results and discussion

The results of the simulation study are summarized and tabulates in Table 1 to 3 for the RMSE of the three estimators for all sample, β and ε respectively. It is evident that as the sample size increase the values of the RMSE all methods decrease and hence the estimation precision of the parameters increases. When shape parameter increase, the RMSE decrease when all parameter fixed. And the RMSE is increase when ε increase. Consider the RMSE of the estimator scale parameter, for outlier through shape parameter the results in Table 1 obtained from $\Delta_1 = 1$ and $\Delta_2 = 2$. The comparison shows that the WLE outperforms the other methods for the scale parameter in term of RMSE for $\beta = 0.5$. Consider the RMSE of the estimator for scale parameter, of the MLE approach provides better results for the scale parameter if outliers in the data set is small that is $\varepsilon = 0.01$, and large values of ε, say $\varepsilon = 0.10$ the MOM approach performs better than the other methods in term of $\beta = 1$. For $\beta = 1.5$, we found that MOM, MLE approach performs slightly better than the WLE approach in estimating scale parameter for Weibull distribution. In case the RMSE of the estimator for shape parameter, in general the RMSE of the shape parameter of the MOM outperforms the other methods for the shape parameter. In term of RMSE for two parameter estimators, the MLE also performs well, especially and outliers in the data set are small, say $\varepsilon = 0.01$. To generate outliers through only shape parameter, keeping the scale parameter fixed, the results in Table 2 obtained from $\Delta_1 = 2$ and $\Delta_2 = 1$. The comparison shows that the WLE outperforms the other methods for the scale parameter in term of RMSE for $\beta = 0.5$ if outlier in the data set is small value at $\varepsilon = 0.10$, the

MOM approach performs better than the other methods for all β. Consider the RMSE of the estimator for shape parameter, of the MLE approach provides better results for the scale parameter if outlier in the data set is small. In term of RMSE for two parameter estimators when $\beta = 0.5$ and 1 the WLE perform better than the other methods if $\varepsilon = 0.01$ and the MOM approach performs better than the other methods. To generate outliers through shape and scale parameter the results in Table 3 obtained from $\Delta_1 = 2$ and $\Delta_2 = 2$. The comparison shows that the WLE outperforms the other methods for the scale parameter in term of RMSE for $\beta = 0.5$ if outliers in the data set are small. For $\beta = 1$ and 1.5 the MLE approach provides better results for the scale parameter if $\varepsilon = 0.01$. When large values the MOM approach performs better than the other methods for all β. Some situations, MOM performs well, especially for shape estimates. In addition, it can be observed that the sample RMSE of two parameters in almost all case WLE performed better than the MLE and MOM when $\varepsilon = 0.01$.

Table 1. RMSE for scale and shape parameters of Weibull distribution obtained from a contamination $W(1, \beta) + \varepsilon W(1, 0.5(2))$.

β	n	ε	α	RMSE$_\delta$			RMSE$_\beta$			RMSE		
				MOM	MLE	WLE	MOM	MLE	WLE	MOM	MLE	WLE
0.5	30	0.01	0.01	0.558	0.430	**0.429**	0.153	**0.104**	0.109	0.579	**0.442**	0.443
			0.05	0.558	0.430	**0.420**	0.153	**0.104**	0.124	0.579	**0.442**	0.438
			0.1	0.558	0.430	**0.410**	0.153	**0.104**	0.133	0.579	**0.442**	0.431
		0.10	0.01	0.525	0.515	**0.508**	**0.183**	0.205	0.219	0.557	**0.554**	**0.554**
			0.05	0.525	0.515	**0.478**	**0.183**	0.205	0.245	0.557	0.554	**0.538**
			0.1	0.525	0.515	**0.453**	**0.183**	0.205	0.261	0.557	0.554	**0.523**
	50	0.01	0.01	0.425	0.329	**0.329**	0.120	**0.082**	0.087	0.442	**0.339**	0.340
			0.05	0.425	0.329	**0.320**	0.120	**0.082**	0.097	0.442	0.339	**0.334**
			0.1	0.425	0.329	**0.312**	0.120	**0.082**	0.104	0.442	0.339	**0.329**
		0.10	0.01	**0.404**	0.430	0.419	**0.144**	0.180	0.194	**0.429**	0.467	0.462
			0.05	0.404	0.430	**0.392**	**0.144**	0.180	0.212	**0.429**	0.467	0.445
			0.1	0.404	0.430	**0.370**	**0.144**	0.180	0.224	**0.429**	0.467	0.433
	100	0.01	0.01	0.309	0.240	**0.238**	0.088	**0.065**	0.069	0.321	**0.248**	**0.248**
			0.05	0.309	0.240	**0.230**	0.088	**0.065**	0.076	0.321	0.248	**0.242**
			0.1	0.309	0.240	**0.224**	0.088	**0.065**	0.081	0.321	0.248	**0.238**
		0.10	0.01	**0.298**	0.365	0.352	**0.107**	0.162	0.174	**0.316**	0.399	0.392
			0.05	**0.298**	0.365	0.328	**0.107**	0.162	0.186	**0.316**	0.399	0.377
			0.1	**0.298**	0.365	0.312	**0.107**	0.162	0.195	**0.316**	0.399	0.368

Table 1. (Cont.)

β	n	ε	α	$RMSE_{\delta}$			$RMSE_{\beta}$			RMSE		
				MOM	MLE	WLE	MOM	MLE	WLE	MOM	MLE	WLE
1	30	0.01	0.01	0.194	**0.189**	**0.189**	0.187	**0.173**	0.174	0.269	**0.256**	0.257
			0.05	0.194	**0.189**	0.190	0.187	**0.173**	0.178	0.269	**0.256**	0.261
			0.1	0.194	**0.189**	0.192	0.187	**0.173**	0.184	0.269	**0.256**	0.266
		0.10	0.01	**0.177**	0.181	0.181	**0.256**	0.296	0.298	**0.312**	0.347	0.348
			0.05	**0.177**	0.181	0.182	**0.256**	0.296	0.308	**0.312**	0.347	0.357
			0.1	**0.177**	0.181	0.182	**0.256**	0.296	0.321	**0.312**	0.347	0.369
	50	0.01	0.01	0.150	**0.147**	**0.147**	0.142	**0.127**	0.128	**0.207**	0.194	0.194
			0.05	0.150	**0.147**	**0.147**	0.142	**0.127**	0.132	**0.207**	0.194	0.197
			0.1	0.150	**0.147**	0.148	0.142	**0.127**	0.135	**0.207**	0.194	0.200
		0.10	0.01	**0.138**	0.143	0.143	**0.207**	0.251	0.253	**0.249**	0.289	0.291
			0.05	**0.138**	0.143	0.143	**0.207**	0.251	0.262	**0.249**	0.289	0.299
			0.1	**0.138**	0.143	0.143	**0.207**	0.251	0.270	**0.249**	0.289	0.306
	100	0.01	0.01	0.107	**0.104**	**0.104**	0.103	**0.090**	0.090	0.148	**0.137**	0.138
			0.05	0.107	**0.104**	**0.104**	0.103	**0.090**	0.093	0.148	**0.137**	0.140
			0.1	0.107	**0.104**	0.105	0.103	**0.090**	0.095	0.148	**0.137**	0.141
		0.10	0.01	**0.174**	0.200	0.198	**0.253**	0.339	0.344	**0.307**	0.393	0.397
			0.05	**0.174**	0.200	0.196	**0.253**	0.339	0.354	**0.307**	0.393	0.404
			0.1	**0.174**	0.200	0.193	**0.253**	0.339	0.361	**0.307**	0.393	0.409
1.5	30	0.01	0.01	0.126	**0.125**	**0.125**	**0.248**	0.255	0.255	**0.278**	0.284	0.284
			0.05	0.126	**0.125**	**0.125**	**0.248**	0.255	0.258	**0.278**	0.284	0.287
			0.1	0.126	**0.125**	0.126	**0.248**	0.255	0.262	**0.278**	0.284	0.290
		0.10	0.01	**0.114**	**0.114**	**0.114**	**0.354**	0.396	0.396	**0.372**	0.412	0.412
			0.05	**0.114**	**0.114**	**0.114**	**0.354**	0.396	0.400	**0.372**	0.412	0.416
			0.1	**0.114**	**0.114**	0.115	**0.354**	0.396	0.406	**0.372**	0.412	0.422
	50	0.01	0.01	0.098	**0.097**	**0.097**	**0.185**	0.186	0.186	**0.210**	**0.210**	**0.210**
			0.05	0.098	**0.097**	**0.097**	**0.185**	0.186	0.187	**0.210**	**0.210**	0.211
			0.1	0.098	**0.097**	0.098	**0.185**	0.186	0.190	**0.210**	**0.210**	0.214
		0.10	0.01	**0.089**	**0.089**	**0.089**	**0.291**	0.328	0.328	**0.304**	**0.340**	**0.340**
			0.05	**0.089**	**0.089**	**0.089**	**0.291**	0.328	0.332	**0.304**	**0.340**	0.344
			0.1	**0.089**	**0.089**	**0.089**	**0.291**	0.328	0.337	**0.304**	**0.340**	0.349
	100	0.01	0.01	**0.069**	**0.069**	**0.069**	0.132	**0.129**	**0.129**	0.149	**0.146**	**0.146**
			0.05	**0.069**	**0.069**	**0.069**	0.132	**0.129**	0.130	0.149	**0.146**	0.147
			0.1	**0.069**	**0.069**	**0.069**	0.132	**0.129**	0.132	0.149	**0.146**	0.149
		0.10	0.01	**0.063**	0.064	0.064	**0.241**	0.277	0.278	**0.250**	0.284	0.285
			0.05	**0.063**	0.064	0.064	**0.241**	0.277	0.281	**0.250**	0.284	0.289
			0.1	**0.063**	0.064	0.064	**0.241**	0.277	0.285	**0.250**	0.284	0.292

Table 2. RMSE for scale and shape parameters of Weibull distribution obtained from a contamination $W(1,\ \beta)+\varepsilon W(1(2),\ 0.5(1))$.

β	n	ε	α	$RMSE_\delta$			$RMSE_\beta$			RMSE		
				MOM	MLE	WLE	MOM	MLE	WLE	MOM	MLE	WLE
0.05	30	0.01	0.01	0.578	0.473	**0.470**	0.162	**0.126**	0.132	0.601	**0.489**	**0.489**
			0.05	0.578	0.473	**0.452**	0.162	**0.126**	0.150	0.601	0.489	**0.477**
			0.1	0.578	0.473	**0.436**	0.162	**0.126**	0.160	0.601	0.489	**0.464**
		0.10	0.01	**0.798**	0.814	0.804	0.243	**0.239**	0.248	0.834	0.849	**0.842**
			0.05	**0.798**	0.814	0.765	0.243	**0.239**	0.269	0.834	0.849	**0.811**
			0.1	**0.798**	0.814	0.728	0.243	**0.239**	0.283	0.834	0.849	**0.781**
	50	0.01	0.01	0.445	0.373	**0.370**	0.127	**0.103**	0.109	0.463	0.387	**0.386**
			0.05	0.445	0.373	**0.354**	0.127	**0.103**	0.121	0.463	0.387	**0.374**
			0.1	0.445	0.373	**0.339**	0.127	**0.103**	0.129	0.463	0.387	**0.363**
		0.10	0.01	**0.678**	0.736	0.722	**0.199**	0.214	0.223	**0.707**	0.767	0.756
			0.05	**0.678**	0.736	0.689	**0.199**	0.214	0.239	**0.707**	0.767	0.729
			0.1	**0.678**	0.736	0.661	**0.199**	0.214	0.249	**0.707**	0.767	0.706
	100	0.01	0.01	0.327	0.290	**0.285**	0.093	**0.086**	0.091	0.340	0.303	**0.299**
			0.05	0.327	0.290	**0.271**	0.093	**0.086**	0.099	0.340	0.303	**0.289**
			0.1	0.327	0.290	**0.261**	0.093	**0.086**	0.104	0.340	0.303	**0.281**
		0.10	0.01	**0.570**	0.681	0.666	**0.156**	0.194	0.203	**0.591**	0.708	0.697
			0.05	**0.570**	0.681	0.641	**0.156**	0.194	0.214	**0.591**	0.708	0.676
			0.1	**0.570**	0.681	0.622	**0.156**	0.194	0.221	**0.591**	0.708	0.660
1	30	0.01	0.01	0.195	**0.192**	**0.192**	0.194	**0.186**	0.187	0.275	**0.267**	0.268
			0.05	0.195	0.192	**0.193**	0.194	**0.186**	0.192	0.275	**0.267**	0.272
			0.1	0.195	0.192	**0.194**	0.194	**0.186**	0.199	0.275	**0.267**	0.277
		0.10	0.01	**0.249**	0.265	0.264	**0.347**	0.395	0.396	**0.427**	0.475	0.476
			0.05	**0.249**	0.265	0.263	**0.347**	0.395	0.407	**0.427**	0.475	0.485
			0.1	**0.249**	0.265	0.259	**0.347**	0.395	0.421	**0.427**	0.475	0.494
	50	0.01	0.01	0.152	**0.150**	**0.150**	0.148	**0.140**	0.141	0.212	**0.205**	0.205
			0.05	0.152	**0.150**	**0.150**	0.148	**0.140**	0.145	0.212	**0.205**	0.209
			0.1	0.152	**0.150**	**0.150**	0.148	**0.140**	0.149	0.212	**0.205**	0.212
		0.10	0.01	**0.219**	0.237	0.237	**0.297**	0.347	0.349	**0.369**	0.420	0.422
			0.05	**0.219**	0.237	0.234	**0.297**	0.347	0.360	**0.369**	0.420	0.429
			0.1	**0.219**	0.237	0.231	**0.297**	0.347	0.369	**0.369**	0.420	0.435
	100	0.01	0.01	0.109	**0.108**	**0.108**	0.108	**0.103**	0.104	0.153	**0.149**	0.150
			0.05	0.109	**0.108**	**0.108**	0.108	**0.103**	0.107	0.153	**0.149**	0.152
			0.1	0.109	**0.108**	**0.108**	0.108	**0.103**	0.110	0.153	**0.149**	0.154

Table 2. (Cont.)

β	n	ε	α	$RMSE_{\delta}$			$RMSE_{\beta}$			RMSE		
				MOM	MLE	WLE	MOM	MLE	WLE	MOM	MLE	WLE
		0.10	0.01	**0.196**	0.216	0.215	**0.253**	0.311	0.315	**0.320**	0.378	0.381
			0.05	**0.196**	0.216	0.213	**0.253**	0.311	0.322	**0.320**	0.378	0.386
			0.1	**0.196**	0.216	0.210	**0.253**	0.311	0.328	**0.320**	0.378	0.390
1.5	30	0.01	0.01	**0.126**	**0.126**	**0.126**	**0.255**	0.266	0.266	**0.285**	0.294	0.294
			0.05	**0.126**	**0.126**	**0.126**	**0.255**	0.266	0.269	**0.285**	0.294	0.297
			0.1	**0.126**	**0.126**	**0.126**	**0.255**	0.266	0.273	**0.285**	0.294	0.301
		0.10	0.01	**0.163**	0.167	0.167	**0.479**	0.530	0.530	**0.506**	0.556	0.556
			0.05	**0.163**	0.167	0.167	**0.479**	0.530	0.534	**0.506**	0.556	0.560
			0.1	**0.163**	0.167	0.167	**0.479**	0.530	0.542	**0.506**	0.556	0.567
	50	0.01	0.01	**0.098**	**0.098**	**0.098**	**0.192**	0.196	0.196	**0.216**	0.219	0.219
			0.05	**0.098**	**0.098**	**0.098**	**0.192**	0.196	0.198	**0.216**	0.219	0.221
			0.1	**0.098**	**0.098**	**0.098**	**0.192**	0.196	0.201	**0.216**	0.219	0.224
		0.10	0.01	**0.146**	0.150	0.150	**0.420**	0.461	0.462	**0.444**	0.485	0.485
			0.05	**0.146**	0.150	0.150	**0.420**	0.461	0.467	**0.444**	0.485	0.490
			0.1	**0.146**	0.150	0.149	**0.420**	0.461	0.473	**0.444**	0.485	0.496
	100	0.01	0.01	**0.070**	**0.070**	**0.070**	**0.138**	0.139	0.139	**0.155**	0.156	0.156
			0.05	**0.070**	**0.070**	**0.070**	**0.138**	0.139	0.141	**0.155**	0.156	0.158
			0.1	**0.070**	**0.070**	0.071	**0.138**	0.139	0.143	**0.155**	0.156	0.159
		0.10	0.01	**0.133**	0.137	0.137	**0.373**	0.409	0.410	**0.396**	0.432	0.432
			0.05	**0.133**	0.137	0.136	**0.373**	0.409	0.415	**0.396**	0.432	0.437
			0.1	**0.133**	0.137	0.136	**0.373**	0.409	0.420	**0.396**	0.432	0.441

Table 3. RMSE for scale and shape parameters of Weibull distribution obtained from a contamination $W(1,\ \beta)+\varepsilon W(1(2),\ 0.5(2))$.

β	n	ε	α	$RMSE_{\delta}$			$RMSE_{\beta}$			RMSE		
				MOM	MLE	WLE	MOM	MLE	WLE	MOM	MLE	WLE
0.05	30	0.01	0.01	0.565	0.453	**0.451**	0.156	**0.120**	0.126	0.586	0.468	**0.468**
			0.05	0.565	0.453	**0.437**	0.156	**0.120**	0.144	0.586	0.468	**0.460**
			0.1	0.565	0.453	**0.423**	0.156	**0.120**	0.154	0.586	0.468	**0.450**
		0.10	0.01	**0.606**	0.678	0.662	**0.217**	0.269	0.291	**0.644**	0.730	0.723
			0.05	**0.606**	0.678	0.616	**0.217**	0.269	0.321	**0.644**	0.730	0.695
			0.1	**0.606**	0.678	0.580	**0.217**	0.269	0.340	**0.644**	0.730	0.672
	50	0.01	0.01	0.431	0.353	**0.351**	0.122	**0.098**	0.104	0.448	0.366	**0.366**
			0.05	0.431	0.353	**0.337**	0.122	**0.098**	0.116	0.448	0.366	**0.357**
			0.1	0.431	0.353	**0.325**	0.122	**0.098**	0.124	0.448	0.366	**0.348**
		0.10	0.01	**0.487**	0.604	0.583	**0.172**	0.241	0.260	**0.516**	0.651	0.639
			0.05	**0.487**	0.604	0.544	**0.172**	0.241	0.283	**0.516**	0.651	0.613
			0.1	**0.487**	0.604	0.514	**0.172**	0.241	0.297	**0.516**	0.651	0.593
	100	0.01	0.01	0.314	**0.268**	0.264	0.090	**0.082**	0.087	0.327	0.280	**0.278**
			0.05	0.314	**0.268**	0.252	0.090	**0.082**	0.094	0.327	0.280	**0.269**
			0.1	0.314	**0.268**	0.243	0.090	**0.082**	0.100	0.327	0.280	**0.263**
		0.10	0.01	**0.379**	0.552	0.529	**0.129**	0.220	0.237	**0.400**	0.594	0.580
			0.05	**0.379**	0.552	0.498	**0.129**	0.220	0.253	**0.400**	0.594	0.559
			0.1	**0.379**	0.552	0.476	**0.129**	0.220	0.263	**0.400**	0.594	0.544
1	30	0.01	0.01	0.194	**0.191**	**0.191**	0.192	**0.184**	0.185	0.273	0.265	**0.266**
			0.05	0.194	**0.191**	0.192	0.192	**0.184**	0.190	0.273	0.265	**0.270**
			0.1	0.194	**0.191**	0.193	0.192	**0.184**	0.197	0.273	0.265	**0.276**
		0.10	0.01	**0.228**	0.248	0.248	**0.350**	0.421	0.424	**0.418**	0.489	0.491
			0.05	**0.228**	0.248	0.247	**0.350**	0.421	0.438	**0.418**	0.489	0.502
			0.1	**0.228**	0.248	0.243	**0.350**	0.421	0.456	**0.418**	0.489	0.517
	50	0.01	0.01	0.151	**0.149**	**0.149**	0.147	**0.138**	0.139	0.211	**0.203**	0.204
			0.05	0.151	**0.149**	**0.149**	0.147	**0.138**	0.143	0.211	**0.203**	0.207
			0.1	0.151	**0.149**	0.150	0.147	**0.138**	0.148	0.211	**0.203**	0.210
		0.10	0.01	**0.198**	0.221	0.220	**0.297**	0.374	0.378	**0.357**	0.434	0.437
			0.05	**0.198**	0.221	0.218	**0.297**	0.374	0.391	**0.357**	0.434	0.447
			0.1	**0.198**	0.221	0.214	**0.297**	0.374	0.402	**0.357**	0.434	0.455

Table 3. (Cont.)

β	n	ε	α	$RMSE_\delta$			$RMSE_\beta$			RMSE		
				MOM	MLE	WLE	MOM	MLE	WLE	MOM	MLE	WLE
	100	0.01	0.01	0.108	**0.107**	**0.107**	0.107	**0.102**	**0.102**	0.152	**0.148**	0.148
			0.05	0.108	**0.107**	**0.107**	0.107	**0.102**	0.105	0.152	**0.148**	0.151
			0.1	0.108	**0.107**	**0.107**	0.107	**0.102**	0.108	0.152	**0.148**	0.152
		0.10	0.01	**0.174**	0.200	0.198	**0.253**	0.339	0.344	**0.307**	0.393	0.397
			0.05	**0.174**	0.200	0.196	**0.253**	0.339	0.354	**0.307**	0.393	0.404
			0.1	**0.174**	0.200	0.193	**0.253**	0.339	0.361	**0.307**	0.393	0.409
1.5	30	0.01	0.01	**0.126**	**0.126**	**0.126**	**0.255**	0.266	0.266	**0.285**	0.294	0.294
			0.05	**0.126**	**0.126**	**0.126**	**0.255**	0.266	0.269	**0.285**	0.294	0.297
			0.1	**0.126**	**0.126**	**0.126**	**0.255**	0.266	0.273	**0.285**	0.294	0.300
		0.10	0.01	**0.160**	0.165	0.165	**0.510**	0.568	0.568	**0.534**	0.592	0.592
			0.05	**0.160**	0.165	0.165	**0.510**	0.568	0.573	**0.534**	0.592	0.596
			0.1	**0.160**	0.165	0.164	**0.510**	0.568	0.582	**0.534**	0.592	0.605
	50	0.01	0.01	**0.098**	**0.098**	**0.098**	**0.192**	0.196	0.196	**0.216**	0.219	0.219
			0.05	**0.098**	**0.098**	**0.098**	**0.192**	0.196	0.198	**0.216**	0.219	0.221
			0.1	**0.098**	**0.098**	**0.098**	**0.192**	0.196	0.201	**0.216**	0.219	0.224
		0.10	0.01	**0.144**	0.149	0.149	**0.449**	**0.499**	0.501	**0.472**	0.521	0.522
			0.05	**0.144**	0.149	0.148	**0.449**	**0.499**	0.507	**0.472**	0.521	0.528
			0.1	**0.144**	0.149	0.148	**0.449**	**0.499**	0.513	**0.472**	0.521	0.534
	100	0.01	0.01	**0.070**	**0.070**	**0.070**	**0.138**	0.139	0.140	**0.155**	0.156	0.156
			0.05	**0.070**	**0.070**	**0.070**	**0.138**	0.139	0.141	**0.155**	0.156	0.158
			0.1	**0.070**	**0.070**	**0.070**	**0.138**	0.139	0.143	**0.155**	0.156	0.159
		0.10	0.01	**0.131**	0.136	0.136	**0.403**	0.449	0.450	**0.424**	0.469	0.470
			0.05	**0.131**	0.136	0.135	**0.403**	0.449	0.457	**0.424**	0.469	0.476
			0.1	**0.131**	0.136	0.135	**0.403**	0.449	0.461	**0.424**	0.469	0.481

Conclusion

The performance of the method of moment (MOM), the maximum likelihood method (MLE) and the weighted likelihood method (WLE) for the joint estimation of both scale and shape parameters for two-parameter Weibull distribution are compared using the Monte Carlo simulation and the efficiency of these methods is compared based on RMSE. From simulation results, it is evident that as the sample size increase the values of the RMSE all methods decrease and hence the estimation precision of the parameters increases. This is expected because most estimator in statistical theory perform better when sample size increases. To examine the performance of the WLE method in comparison with the MLE and MOM methods, we found that the WLE outperforms the other methods for the scale parameter in term of RMSE of the estimator for scale parameter for $\beta = 0.5$ with outliers in the data set. Consider the RMSE of the estimator for shape parameter the MLE approach provides better results for the scale parameter if outliers in the data set is small and the MOM approach performs better than the other methods if outliers in the data set is large values. In almost all cases WLE perform better than the other methods when $\beta = 0.5$ in term of RMSE for two parameter estimators. For $\beta = 1$ and 1.5, the MOM also performs well, especially and outliers in the data set is small. As future work we may mention that the WLE method can be extended to some further modifications of the Weibull distribution. One extened this approach can also be extened to case when data contain the censored observations. In addition, in this study we were considered only the two-parameter Weibull distribution, the WLE method can be extended to estimate the three-parameter Weibull distribution when data are contaminated with outliers.

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