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http://www.mathassociation.net Email: MathThaiOrg@gmail.com

ผลบวกของอนุกรมแฟกทอเรียล

Sum of Factorial Series

สมพงษ์ ฉุยสุริฉาย และ มนทกานติ เพชรอภิรักษ์

Sompong Chuysurichay[1] and Montakarn Petapirak[2,*]

[1,2]Algebra and Applications Research Unit, Department of Mathematics and Statistics,
Faculty of Science, Prince of Songkla University, Songkhla, 90110

Email: [1]sompong.c@psu.ac.th [2]montakarn.p@psu.ac.th

[*]Corresponding author

บทคัดย่อ

ในงานวิจัยเรื่องนี้ผู้วิจัยศึกษาการหาผลบวกของอนุกรมในรูป $\sum_{n=0}^{\infty}\frac{1}{(a_n)!}$ เมื่อ $(a_n)_{n=0}^{\infty}$ เป็นลำดับเลขคณิตของจำนวนเต็มที่ไม่เป็นลบ

คำสำคัญ: ลำดับเลขคณิต แฟกทอเรียล อนุกรม

ABSTRACT

In this research, we find the sum of the series $\sum_{n=0}^{\infty}\frac{1}{(a_n)!}$ where $(a_n)_{n=0}^{\infty}$ is an arithmetic progression of nonnegative integers.

Keywords: Arithmetic Progression, Factorial, Series

1. Introduction

This work is motivated by the sum of the series $\sum_{n=0}^{\infty}\frac{1}{(2n)!}$ and $\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$ which can be evaluated by the Maclaurin series of hyperbolic cosine and hyperbolic sine functions, i.e., $\sum_{n=0}^{\infty}\frac{1}{(2n)!}=\cosh(1)$ and $\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}=\sinh(1)$ (see [4] for more details). In this paper, we give a closed form for the sum of the series $\sum_{n=0}^{\infty}\frac{1}{(a_n)!}$ where $(a_n)_{n=0}^{\infty}$ is any arithmetic progression of nonnegative integers.

2. Preliminaries

Let d be a positive integer greater than 1. We define $f_d(x)=\sum_{n=0}^{\infty}\frac{x^{dn}}{(dn)!}$. The Absolute Convergence Test shows that the series $\sum_{n=0}^{\infty}\frac{x^{dn}}{(dn)!}$ is absolutely convergent for all $x\in\mathbb{R}$. Thus $f_d(x)$ is well-defined for all $x\in\mathbb{R}$. Let $\omega_d=\cos\left(\frac{2\pi}{d}\right)+i\sin\left(\frac{2\pi}{d}\right)$. The following results are known in [1].

Theorem 2.1 $1,\omega_d,\omega_d^2,\ldots,\omega_d^{d-1}$ are roots of the polynomial x^d-1.

Proof. It is clear by De Moivre formula that $\omega_d^d=1$. Thus

$$\left(\omega_d^i\right)^d-1=\left(\omega_d^d\right)^i-1=1-1=0$$

for all $i\in\{0,1,\ldots,d-1\}$. This proves that $1,\omega_d,\omega_d^2,\ldots,\omega_d^{d-1}$ are roots of the polynomial x^d-1. □

Theorem 2.2 $\omega_d,\omega_d^2,\ldots,\omega_d^{d-1}$ are roots of the polynomial $\sum_{i=0}^{d-1}x^i$.

Proof. Since $x^d-1=(x-1)\left(\sum_{i=0}^{d-1}x^i\right)$ and $\omega_d^l\neq 1$ for all $l\in\{1,2,\ldots,d-1\}$, by Theorem 2.1, we obtain that $\omega_d,\omega_d^2,\ldots,\omega_d^{d-1}$ are roots of $\sum_{i=0}^{d-1}x^i$. □

Let

$$A_d=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \omega_d & \omega_d^2 & \omega_d^3 & \cdots & \omega_d^d \\ \omega_d^2 & \omega_d^4 & \omega_d^6 & \cdots & \omega_d^{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_d^{d-1} & \omega_d^{2(d-1)} & \omega_d^{3(d-1)} & \cdots & \omega_d^{d(d-1)} \end{pmatrix},$$

$u=(1,1,\ldots,1)$ and $v=(1,0,\ldots,0)$. It is worth noting that $\det(A_d)=\prod_{1\le i<j\le n-1}\left(\omega_d^j-\omega_d^i\right)\neq 0$ and hence A_d is invertible.

Lemma 2.3 Let A_d, u and v be defined as above. Then $A_d u^T=dv^T$.

Proof. Let $p(x)=\sum_{i=0}^{d-1}x^i$.

By Theorem 2.2, we have that $p\left(\omega_d^k\right)=0$ for all $k\in\{1,2,\ldots,d-1\}$. Then

$$\begin{aligned} A_d u^T &= \left(p(1), p(\omega_d), p\left(\omega_d^2\right),\ldots, p\left(\omega_d^{d-1}\right)\right)^T \\ &= (d,0,0,\ldots,0)^T \\ &= dv^T \end{aligned}$$

□

Theorem 2.4 The particular solution of the initial value problem

$$y^{(d)}-y=0,\ y(0)=1,\ y^{(i)}(0)=0 \text{ for all } i\in\{1,2,\ldots,d-1\} \tag{1}$$

is $y(x)=\frac{1}{d}\left(\sum_{i=1}^{d}e^{\omega_d^i x}\right)$.

Proof. The auxiliary equation is $x^d-1=0$. The roots of x^d-1 are $\omega_d,\omega_d^2,\ldots,\omega_d^d$ and they are all distinct. Thus, the general solution of $y^{(d)}-y=0$ is

$$y(x)=\sum_{i=1}^{d}c_i e^{\omega_d^i x}$$

where $c_1,c_2,\ldots,c_d\in\mathbb{C}$. Then

$$y^{(i)}(x)=\sum_{j=1}^{d}c_j\omega_d^{ij}e^{\omega_d^j x}$$

for all $i\in\{1,2,\ldots,d-1\}$. Hence the initial conditions give the system of equations

$$\sum_{i=1}^{d} c_i = 1$$

$$\sum_{i=1}^{d} \omega_d^i c_i = 0$$

$$\vdots$$

$$\sum_{i=1}^{d} \omega_d^{i(d-1)} c_i = 0$$

which can be written as $A_d c^T = v^T$ where $c = (c_1, c_2, \ldots, c_d)$. By Lemma 2.3, we get

$$c^T = A_d^{-1} v^T = \frac{1}{d} u^T .$$

Thus $c_i = \frac{1}{d}$ for all $i \in \{1, 2, \ldots, d\}$. Therefore, the particular solution of the initial value problem is

$$y(x) = \frac{1}{d}\left(\sum_{i=1}^{d} e^{\omega_d^i x}\right). \qquad \square$$

Theorem 2.5 $f_d(x) = \sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!}$ satisfies problem (1).

Proof. Let $y = \sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!}$. Then

$$y^{(i)} = \sum_{n=1}^{\infty} \frac{x^{dn-i}}{(dn-i)!}$$

for all $i \in \{1, 2, \ldots, d\}$. In particular,

$$\begin{aligned} y^{(d)} &= \sum_{n=1}^{\infty} \frac{x^{dn-d}}{(dn-d)!} \\ &= \sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!} \\ &= y. \end{aligned}$$

It is clear that $y(0) = 1$ and $y^{(i)}(0) = 0$ for all $i \in \{1, 2, \ldots, d-1\}$.

Thus $f_d(x)$ satisfies (1). $\square$

3. Sum of Factorial Series

The following theorems are needed for our main results.

Theorem 3.1 For any positive integer d ,

$$\sum_{n=0}^{\infty}\frac{x^{dn}}{(dn)!}=\frac{1}{d}\left(\sum_{i=1}^{d}e^{\omega_d^i x}\right)$$

Proof. For $d=1$, $\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$.

For $d>1$, we get by Theorem 2.4, Theorem 2.5 and the uniqueness of the solution of the initial value problem (1) [5], that

$$\sum_{n=0}^{\infty}\frac{x^{dn}}{(dn)!}=\frac{1}{d}\left(\sum_{i=1}^{d}e^{\omega_d^i x}\right). \qquad \square$$

Theorem 3.2 For any positive integer $d>1$, and $1\le i\le d$,

$$\sum_{n=1}^{\infty}\frac{x^{dn-i}}{(dn-i)!}=\frac{1}{d}\left(\sum_{j=1}^{d}\omega_d^{ij}e^{\omega_d^i x}\right).$$

Proof. The desired result follows from

$$\sum_{n=1}^{\infty}\frac{x^{dn-i}}{(dn-i)!}=y^{(i)}(x)=\frac{1}{d}\left(\sum_{j=1}^{d}\omega_d^{ij}e^{\omega_d^i x}\right). \qquad \square$$

Let $(a_n)_{n=0}^{\infty}$ be an arithmetic progression of nonnegative integers with common difference $d>0$. Then $a_n=dn+r$ where $r\ge 0$. We consider two cases:

1. $0\le r<d$, and
2. $r\ge d$. Write $r=kd+s$, where $k\ge 1$ and $0\le s\le d-1$. Then $a_n=d(n+k)+s$.

Theorem 3.3 The following results hold:

1. If $a_n=dn+r$, where $0\le r<d$, then

$$\sum_{n=0}^{\infty}\frac{1}{(a_n)!}=\frac{1}{d}\left(\sum_{i=1}^{d}\omega_d^{i(d-r)}e^{\omega_d^i}\right).$$

2. If $a_n=d(n+k)+s$, where $k\ge 1$ and $0\le s\le d-1$, then

$$\sum_{n=0}^{\infty}\frac{1}{(a_n)!}=\frac{1}{d}\left(\sum_{i=1}^{d}\omega_d^{i(d-s)}e^{\omega_d^i}\right)-\sum_{i=0}^{k-1}\frac{1}{(di+s)!}.$$

Proof.

1. The statement follows directly from Theorem 3.2.

2. We have

$$\sum_{n=0}^{\infty}\frac{1}{(a_n)!}=\sum_{n=0}^{\infty}\frac{1}{(dn+s)!}-\sum_{i=0}^{k-1}\frac{1}{(di+s)!}$$
$$=\frac{1}{d}\left(\sum_{i=1}^{d}\omega_d^{i(d-s)}e^{\omega_d^i}\right)-\sum_{i=0}^{k-1}\frac{1}{(di+s)!}.$$

□

4. Examples

We give examples when $d=3$. By Theorem 3.3, we have

1. $\sum_{n=0}^{\infty}\frac{1}{(3n)!}=\frac{1}{3}\left[e+2e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)\right]$
2. $\sum_{n=0}^{\infty}\frac{1}{(3n+1)!}=\frac{1}{3}\left[e-e^{-\frac{1}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\right)\right)\right]$
3. $\sum_{n=0}^{\infty}\frac{1}{(3n+2)!}=\frac{1}{3}\left[e-e^{-\frac{1}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\right)\right)\right]$
4. $\sum_{n=0}^{\infty}\frac{1}{(3n+3)!}=\frac{1}{3}\left[e+2e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)\right]-\frac{1}{0!}$
5. $\sum_{n=0}^{\infty}\frac{1}{(3n+4)!}=\frac{1}{3}\left[e-e^{-\frac{1}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\right)\right)\right]-\frac{1}{1!}$
6. $\sum_{n=0}^{\infty}\frac{1}{(3n+5)!}=\frac{1}{3}\left[e-e^{-\frac{1}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\right)\right)\right]-\frac{1}{2!}$

Note that Examples 1 - 3 are guaranteed in [2] using a different method as well as in [3].

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