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สมบัติหลักมูลของฟีลด์อันดับ
Fundamental Properties of Ordered Fields

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บทคัดย่อ

เราอภิปรายสมบัติเชิงพีชคณิต สมบัติเชิงอันดับและสมบัติเชิงทอพอโลยีที่สำคัญของฟีลด์อันดับ ในข้อเท็จจริงนั้น ค่าสัมบูรณ์ในฟีลด์อันดับใด ๆ มีสมบัติคล้ายกับค่าสัมบูรณ์ของจำนวนจริง เราให้บทพิสูจน์อย่างง่ายของการสมมูลกันระหว่าง สมบัติอาร์คิมีดิสกับความหนาแน่น ของฟีลด์ย่อยตรรกยะ เรายังให้เงื่อนไขที่สมมูลกันสำหรับฟีลด์อันดับที่จะมีสมบัติอาร์คิมีดิส ซึ่งเกี่ยวกับการลู่เข้าของลำดับและการทดสอบอนุกรมเรขาคณิต

คำสำคัญ: ฟีลด์อันดับ สมบัติของอาร์คิมีดิส ฟีลด์ย่อยตรรกยะ อนุกรมเรขาคณิต

ABSTRACT

We discuss fundamental algebraic-order-topological properties of ordered fields. In fact, the absolute value in any ordered field has properties similar to those of real numbers. We give a simple proof of the equivalence between the Archimedean property and the density of the rational subfield. We also provide equivalent conditions for an ordered field to be Archimedean, involving convergence of certain sequences and the geometric series test.

Keywords: Ordered Field, Archimedean Property, Rational Subfield, Geometric Series

1. Introduction

In the course of undergraduate mathematical analysis (e.g. [1-3]), the first thing to learn is the real number system, i.e., the real numbers together with fundamental algebraic/order/topological properties. We begin with two algebraic operations, namely, the addition and the multiplications. From abstract-algebra point of view, the real numbers with these operations constitute a *field*. Then we discuss how to compare or order two real numbers, and how nice is the order relation related to algebraic operations. In fact, the real numbers is an example of an *ordered field*. The last axiom for the real numbers is the completeness axiom or the least-upper-bound property, making the real numbers a *complete ordered field*.

Consequently, the set of real numbers possesses the *Archimedean property* and the rational numbers is dense in the real line. From these properties, we can discuss convergence of sequences and series for real numbers, and limits, continuity, differentiation, integration, and other analytical concepts for real-valued functions. Therefore, it is significant to discuss properties of arbitrary ordered field. See more related discussions in [4-6].

In this paper, we focus on abstract properties of ordered fields. We provide the definition and examples of ordered fields, and discuss fundamental algebraic-order-topological properties. In fact, the absolute value in ordered field has properties similar to those of real numbers. In particular, every ordered field is a metric space. In any ordered field, there is a subset isomorphic to the rational numbers, called the rational subfield (see Section 2). We provide a simple proof for the fact that the Archi-medean property is equivalent to the density of the rational subfield (see Section 3). Moreover, this property is equivalent several properties involving convergence of certain sequences and the geometric series test (see Section 4).

2. Ordered Fields

In this section, we review fundamental properties of arbitrary ordered fields. The axiomatic properties of an ordered field are modeled from the real numbers.

Definition 1. A *order* on a set E is a binary relation $\prec$ on E with the following properties:

(i) *Trichotomy property*: For each $x, y \in E$, one and only one of the following statements hold: $x \prec y$, $x = y$, $y \prec x$.

(ii) *Transitivity*: For each $x, y, z \in E$, if $x \prec y$ and $y \prec z$ then $x \prec z$.

In this case, we say that E is an *ordered set* with respect to the order $\prec$.

Definition 2. An *ordered field* is a field $(F, +, \cdot)$ which is also an ordered set $(F, \prec)$ such that

(i) For each $x, y, z \in F$, if $y \prec z$ then $x + y \prec x + z$.

(ii) For each $x, y, z \in F$, if $0 \prec x$ and $0 \prec y$ then $0 \prec x \cdot y$.

Here, 0 denotes the additive identity in the field F.

The property (i) means that the order relation is compatible with addition. The property (ii) can be replaced by (ii)′. For each $x, y, z \in F$, if $x \prec y$ and $0 \prec z$ then $x \cdot z \prec y \cdot z$. The property (ii) or (ii)′ means that the order relation is compatible with multiplication. The collection of conditions (i) and (ii) is equivalent to the conditions (i) and (ii)′.

Example 3. The following examples are ordered fields:

1) The real numbers $\mathbb{R}$ with respect to the usual addition, the usual multiplication, and the usual order.
2) The field $\mathbb{R}(x)$ of real rational functions in the form $p(x)/q(x)$ where $p(x)$ and $q(x) \neq 0$ are polynomials with real coefficients, here for each $f, g \in \mathbb{R}(x)$ we define $f \prec g$ if and only if $f(k) < g(k)$ for all sufficiently large real numbers k.

Any subfield of an ordered field is an ordered field inheriting the algebraic and order structures. For example, the field of rational numbers and the field of real algebraic numbers are ordered subfield of the real numbers.

As a ring, every ordered field F always has characteristic zero since the elements

$$1, 1 + 1, 1 + 1 + 1, \ldots$$

are all different, so it contains copies of $\mathbb{Z}$. Thus, an ordered field necessarily must contain an infinite number of elements. It follows that every finite field cannot be ordered consistently with its algebraic structures. The universal mapping property of the quotient field implies that the ring monomorphism $\mathbb{Z} \to F$ can be extended to a monomorphism $\mathbb{Q} \to F$. Call the image of this map the rational subfield of F, denoted by $\mathbb{Q}_F$. We also denote

$$\mathbb{N}_F = \{1, 1+1, 1+1+1, \ldots\}$$

and $\mathbb{Z}_F = \mathbb{N}_F \cup \{0\} \cup (-\mathbb{N}_F)$

where $-\mathbb{N}_F = \{-x \mid x \in \mathbb{N}_F\}$. We can define nx and x^n for each natural number n and $x \in F$ by $x + x + \ldots + x$ (n times) and $xx \cdots x$ (n times), respectively. We can also define open and closed intervals in a similar manner to those in the real line. From the trichotomy property, we define

$$|a| = \max\{a, -a\} \text{ for each } a \in F.$$

Then the following properties hold for any $a, b, c \in F$ with $c > 0$.

- positivity: $|a| \geq 0$, and $|a| = 0$ if and only if $a = 0$.
- $-|a| \leq a \leq |a|$.
- $|a| \leq c$ if and only if $-c \leq a \leq c$.
- multiplicativity: $|ab| = |a||b|$; in particular $|a^n| = |a|^n$ for any $n \in \mathbb{N}$.
- triangle inequality: $|a+b| \leq |a| + |b|$.

In the context of ordered field, the binomial expansion theorem and Bernoulli's inequality also hold by mathematical induction.

We can equip an ordered field F with a topological structure as follows. For each $a \in F$ and $\varepsilon > 0$, the ε - neighborhood of a is given by

$$B_{\varepsilon}(a) = \{x \in F : |x - a| < \varepsilon\}.$$

From which one can define open sets, closed sets, continuity, convergence, and another topological/metric notions. In particular, every ordered field is a metric space.

3.Archimedean Property and Density of the Rationals Subfield in Ordered Fields

A fundamental fact in mathematical analysis, is the density of the rational numbers in the real line. We shall discuss this property in an ordered field (F, <):

(D): Given $x, y \in F$ with $x < y$, we can find an $r \in \mathbb{Q}_F$ such that $x < r < y$.

This fact is equivalent to the fact that the topological closure of $\mathbb{Q}_F$ in F is the whole space F. A usual proof of (D) in textbooks (see, *e.g*. [1, 2]) is given by expanding with sufficiently large spaces. Indeed, if x and y are elements of F with $x < y$, we will find $m, n \in \mathbb{Z}_F$ such that $x < m/n < y$ by scaling the interval [x, y] to [nx, ny] where n is large enough so that the interval [nx, ny] contains an element $m \in \mathbb{Z}_F$. This task is done by using a version of Archimedean property, namely:

(AP1): Given $x > 0$, there is an $n \in \mathbb{N}_F$ such that $1/n < x$.

Some authors (*e.g*. [3]) use (AP1) and the well-ordering principle to give a constructive proof of (D). There is a non-constructive proof using (AP1), the well-ordering principle and a contradiction, *e.g*., [7].

The next theorem provides a simple proof of the fact (D) by using (AP1) and the following property:

(AP2): $\mathbb{N}_F$ is not bounded above.

Theorem 4. In an ordered field (F, >), we have (AP1), (AP2), and (D) are mutually equivalent.

Proof. The equivalence between (AP1) and (AP2) in an ordered field is easy to see. Suppose (AP1) holds. The idea to prove (D) is “partitioning with sufficiently small spaces”. In order to get a fish, we first identify a “suitable place” it lives, then we use a fishnet with sufficiently small meshes. Indeed, by (AP2) there are $a, b \in \mathbb{Z}_F$ such that $a \le x < y \le b$. We shall partition the interval [a, b] into many intervals so that x and y belong to different intervals and the endpoints of each interval are in $\mathbb{Q}_F$. The proof is done if the length of each interval is less than y-x. Indeed, (AP1) allows the existence of an $n \in \mathbb{N}_F$ for which $1/n < y - x$. Now, we use $1/n$ as the length of each interval.

Conversely, suppose (D) holds. Let $x > 0$. The density of $\mathbb{Q}_F$ guarantees the existence of an $r \in \mathbb{Q}_F$ such that $0 < r < x$. With $r = m/n$ where $m, n \in \mathbb{N}_F$, we have $1/n \le m/n < x$. Thus, (AP1) holds.

4. Archimedean Property and Convergence of Certain Sequences and Series in Ordered Fields

Let us discuss the concept of Archimedean property from two abstract mathematical structures, namely, linearly ordered groups and normed fields.

In a linear ordered group $(G,*)$, an element a is said to be *infinitesimal* with respect to an element b if no positive integer multiple of a is greater than b. The linear order is called *Archimedean* if there are no infinitesimal elements.

Let F be a normed field, i.e., F is a field equipped with a function $|\bullet|: F \to [0,\infty)$ such that $|0| = 0$, $|x| > 0$ if $x \neq 0$, and satisfies $|xy| = |x|\,|y|$ and $|x+y| \leq |x| + |y|$ for all x, y in F. Then F is called *Archimedean* if, for each $x \in F$, there is a natural number n such that $|nx| > 1$.

Theorem 5. The following conditions are equivalent in an ordered field $(\mathbb{F},<)$:

(i) $\mathbb{F}$ is Archimedean;

(ii) the sequence $(r^n)_{n\in\mathbb{N}}$ converges to 0 whenever $|r|<1$;

(iii) the sequence $(1/k^n)_{n\in\mathbb{N}}$ converges to 0 for all $k \in \{2,3,4,...\}$;

(iv) there is a $k \in \{2,3,4,...\}$ such that the sequence . $(1/k^n)_{n\in\mathbb{N}}$. converges to 0;

(v) the geometric series $1+r+r^2+\cdots$ is convergent whenever $|r|<1$.

Proof. To prove (i) implies (ii), let $r \in \mathbb{F}-\{0\}$ be such that $|r|<1$. Then $c := (1/|r|)-1$ is a positive element of $\mathbb{F}$. Since Bernoulli's inequality also valid in $\mathbb{F}$, we have $1+nc \leq (1+c)^n$ for every natural number n. Let ε be a positive element in $\mathbb{F}$. The hypothesis (i) guarantees the existence of a natural number n such that $1/(c\varepsilon) < N$. Hence, for any natural number $n \geq N$, we have

$$0 < |r^n| = |r|^n = \frac{1}{(1+c)^n} \leq \frac{1}{1+nc} < \frac{1}{nc} < \varepsilon.$$

Thus, the sequence $(r^n)_{n\in\mathbb{N}}$ converges to 0.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear. We shall prove (iv) $\Rightarrow$ (i) via a contrapositive approach. Suppose that $\mathbb{F}$ is not Archimedean, i.e., there is a positive element a in that dominates 1, 2, 3,… Then for all natural numbers n, we have $a > k^n$ and hence $0 < 1/a < 1/k^n$. This shows that $(1/k^n)_{n\in\mathbb{N}}$ does not converge to 0.

The last job is to show the equivalences between (ii) and (v). Denote the nth partial sum $1+r+r^2+...+r^n$ of this series by s_n then $(1-r)s_n = 1 - r^{n+1}$ for all natural numbers n. If the sequence $(r^n)_{n\in\mathbb{N}}$ converges to 0, then the sequence $1 - r^{n+1}$ converges to 1 and the sequence s_n converges to $1/(1-r)$. On the other hand, if the sequence s_n converges, then the sequence $r^{n+1} = s_{n+1} - s_n$ converges to 0.

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