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การกำกับบนเส้นเชื่อมอย่างมหัศจรรย์ยวดยิ่งผันกลับของกราฟ $P_2\square\, C_n$ และ $W_o(2,n)$

Reverse Super Edge-Magic Labelings of $P_2\square\, C_n$ and $W_o(2,n)$

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บทคัดย่อ

กราฟ $G=(V(G),E(G))$ ที่มี $|V(G)|=p$ และ $|E(G)|=q$ เป็นกราฟมหัศจรรย์ยวดยิ่งผันกลับเมื่อมีฟังก์ชันสมนัยหนึ่งต่อหนึ่ง f จาก $V(G)\cup E(G)$ ไปยัง $\{1,2,3,\dots,p+q\}$ และมีค่าคงตัว $c^{-1}(f)$ ที่ทำให้ $c^{-1}(f)=f(uv)-(f(u)+f(v))$ สำหรับทุก $uv\in E(G)$ และ $f\big(V(G)\big)=\{1,2,3,\dots,p\}$ เรียกฟังก์ชันสมนัยหนึ่งต่อหนึ่ง f นี้ว่าการกำกับอย่างมหัศจรรย์ยวดยิ่งผันกลับของ G และค่าต่ำสุดของ $c^{-1}(f)$ ทั้งหลายที่หาได้จากการกำกับอย่างมหัศจรรย์ยวดยิ่งผันกลับทุกแบบของ G ว่าความเข้มอย่างมหัศจรรย์ยวดยิ่งผันกลับของ G เขียนแทนด้วย $rsems(G)$ บทความวิจัยนี้สร้างการกำกับอย่างมหัศจรรย์ยวดยิ่งผันกลับให้ $P_2\square\, C_n$ และ $W_o(2,n)$ เมื่อ n เป็นจำนวนคี่ที่ $n\geq 3$ และพิสูจน์ว่า $rsems(P_2\square\, C_n)=\frac{3n-1}{2}$ และ $\frac{5n-2}{4}\leq rsems(W_o(2,n))\leq\frac{5n-1}{2}$ เมื่อ n เป็นจำนวนคี่ที่ $n\geq 3$

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ABSTRACT

A graph $G = (V(G), E(G))$ with $|V(G)| = p$ and $|E(G)| = q$, is called *reverse super edge-magic* if there exists a bijection f from $V(G) \cup E(G)$ onto $\{1,2,3, \dots, p+q\}$ and a constant $c^{-1}(f)$ such that $c^{-1}(f) = f(uv) - (f(u) + f(v))$ for all $uv \in E(G)$ and $f(V(G)) = \{1,2,3, \dots, p\}$. This bijection f is called a *reverse super edge-magic labeling* for G and the minimum of all constants $c^{-1}(f)$ taken over all reverse super edge-magic labelings of G is called the *reverse super edge-magic strength* of G and denoted by $rsems(G)$. This article constructs reverse super edge-magic labelings for $P_2 \square C_n$ and $W_o(2,n)$ for an odd integer n such that $n \geq 3$ and prove that $rsems(P_2 \square C_n) = \frac{3n-1}{2}$ and $\frac{5n-2}{4} \leq rsems(W_o(2,n)) \leq \frac{5n-1}{2}$ for an odd integer n such that $n \geq 3$.

Keywords: Reverse super edge-magic labeling, Reverse super edge-magic strength

1. Introduction

Graph labeling is an assignment of integers to the vertices or edges or both of the graph which satisfies certain conditions. There are many kinds of labeling such as graceful labeling, magic labeling. Most of the labelings and examples are collected in a dynamic survey [2]. Several situations can be modeled as labeled graphs. For examples, we can use graph labeling to design missile guidance codes, good radar type codes and convolution codes with optimal autocorrelation properties. We can also use the idea of graph labeling to assign each user terminal a node labeled subject to the constraint that all connecting edges receive distinct labels. Besides that, it can be applied widely in ambiguities in X-ray crystallography, communication network labeling, finite additive number theory and the Golomb's ruler problems, circuit layout, etc., see [1].

In 2013, the concept of a reverse super edge-magic labeling and reverse super edge-magic strength of a graph G was introduced by Hungund and Akka [3]. A Graph

$G=(V(G),E(G))$ is said to be *reverse super edge-magic* if there exists a bijection $f:V(G)\cup E(G)\to\{1,2,3,\dots,p+q\}$ and a constant $c^{-1}(f)$ such that

$$c^{-1}(f)=f(uv)-(f(u)+f(v))$$

for all $uv\in E(G)$ and $f(V(G))=\{1,2,3,\dots,p\}$. This bijection f is called *a reverse super edge-magic labeling* and the constant $c^{-1}(f)$ is called *a reverse super edge-magic* (*rsem*) *constant*. The minimum of all rsem constants $c^{-1}(f)$ of a graph G, where the minimum is taken over all rsem labelings of G, is called a *reverse super edge-magic strength* of the graph G, $rsems(G)$. They proved the following property for an rsem labeling of G.

Theorem 1.1 [3] *Let $G=(V(G),E(G))$ be a graph with $|E(G)|=q$ and f be a reverse super edge-magic labeling of G with its rsem constants $c^{-1}(f)$. Then,*

$$qc^{-1}(f)=\sum_{e\in E(G)} f(e)-\sum_{v\in V(G)} \deg(v)\,f(v).$$

In [3], they established reverse super edge-magic labelings and $rsems(G)$ of some well-known graphs such as the y-tree Y_n, the odd cycle C_{2n+1}, the generalized Petersen graph $P(m,k)$ and the disconnected graph $(2m+1)C_3$. The following results have been proved:

(1) $rsems(Y_{2n})=n-1$, for $n\geq 2$ and $rsems(Y_{2n-1})=n$, for $n\geq 2$;

(2) $rsems(C_{2n+1})=n$, for $n\geq 1$;

(3) $rsems((2m+1)C_3)=3m+1$, for $m\geq 1$.

In our article, we construct rsem labelings for the Cartesian product graph $P_2\,\square\,C_n$ and the generalized web graph without center $W_o(2,n)$ for an odd integer n such that $n\geq 3$. The article is organized in the following manner. In Section 2, definitions of $P_2\,\square\,C_n$, an rsem labeling for $P_2\,\square\,C_n$ and its $rsems$ are presented. In Section 3, definitions of $W_o(2,n)$ and the construction of an rsem labeling for $W_o(2,n)$ and its $rsems$ bounds for an odd integer n such that $n\geq 3$ are provided. Finally, conclusion and some discussion are given in Section 4.

2. RSEM labelings for $P_2 \square C_n$

In order to define $P_2 \square C_n$, let us recall the definitions of path graphs P_n and cycle graphs C_n as follows.

Definition 2.1 [4] A *path graph* P_n is a graph whose n vertices can be ordered in such a way that two vertices are adjacent if and only if they are consecutive in the list.

Figure 2.1 The path P_3

Definition 2.2 [4] A *cycle* C_n is a graph with an equal number of n vertices, $v_1, v_2, v_3, \ldots, v_n$, and n edges whose vertices can be placed around a circle in such a way that two vertices are adjacent if and only if they appear consecutively along the cycle.

In this article, we usually write a cycle C_n as $v_1 v_2 v_3 \ldots v_n$ and we name the vertices in the clockwise direction.

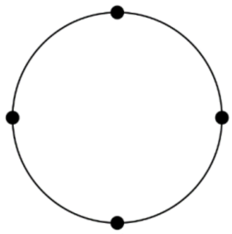

Figure 2.2 The cycle C_4

Definition 2.3 [4] The *Cartesian product* of $G = (V(G), E(G))$ and $H = (V(H), E(H))$ written by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

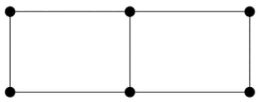

Figure 2.3 $P_2 \square P_3$

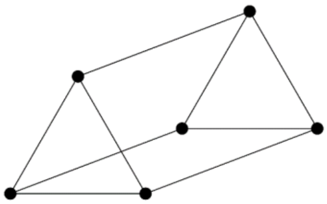

Figure 2.4 $P_2 \square C_3$

In some literature, $P_2 \square C_n$ is called the *prism of* C_n. Note that, for large n, to draw a diagram represented $P_2 \square C_n$, we can regard $P_2 \square C_n$ as two copies of "inner" cycle $C_n = v_{1,1}v_{1,2}v_{1,3} \dots v_{1,n}$ and "outer" cycle $C_n' = v_{2,1}v_{2,2}v_{2,3} \dots v_{2,n}$ and edges joining each corresponding vertices $v_{1,i}$ of C_n to $v_{2,i}$ of C_n' for all $i \in \{1,2,3, \dots, n\}$.

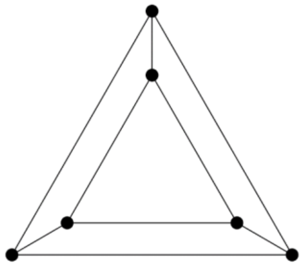

Figure 2.5 $P_2 \square C_3$

We are now ready to construct an rsem labeling and find the $\boldsymbol{rsems}$ of $P_2 \square C_n$. Note that $p = |V(P_2 \square C_n)| = 2n$ and $q = |E(P_2 \square C_n)| = 3n$.

Algorithm 2.1 Let n be an odd integer such that $n \geq 3$.

Define $f: V(P_2 \square C_n) \cup E(P_2 \square C_n) \to \{1,2,3, \dots, 5n\}$ by

i. $f(v_{1,2i-1}) = i$ for $i \in \left\{1,2,3, \dots, \frac{n+1}{2}\right\}$,

ii. $f(v_{1,2i}) = \frac{n+1}{2} + i$ for $i \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$,

iii. $f(v_{2,n}) = n + 1$,

iv. $f(v_{2,2i}) = n + 1 + i$ for $i \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$,

v. $f(v_{2,2i-1}) = \frac{3n+1}{2} + i$ for $i \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$,

vi. $f(rs) = \frac{3n-1}{2} + (f(r) + f(s))$ for $rs \in E(P_2 \square C_n)$.

Example 2.1 From Algorithm 2.1, we can label all vertices and edges of $P_2 \,\square\, C_5$ as shown in Figure 2.6.

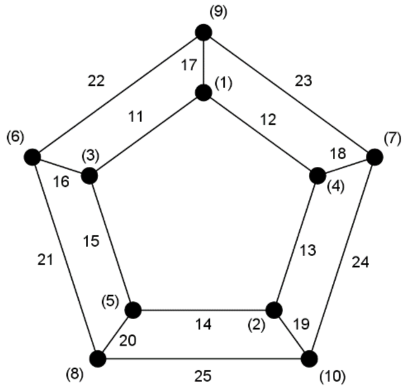

Figure 2.6 A labeling for $P_2 \,\square\, C_5$ using Algorithm 2.1

Theorem 2.1 *Let n be an odd integer such that $n \geq 3$. The edge labeling f of $P_2 \,\square\, C_n$ given by Algorithm 2.1 is an rsem labeling with $c^{-1}(f) = \frac{3n-1}{2}$.*

Proof. Let n be an odd integer such that $n \geq 3$. To show that f is a bijection, first, we illustrate each vertex with its corresponding labels by using Algorithm 2.1.

By Algorithm 2.1 (i),

$$\left\{v_{1,1}, v_{1,3}, v_{1,5}, \dots, v_{1,2i-1}, \dots, v_{1,2\left(\frac{n+1}{2}\right)-1} (= v_{1,n})\right\} \xrightarrow{f} \left\{1,2,3, \dots, i, \dots, \frac{n+1}{2}\right\}.$$

By Algorithm 2.1 (ii),

$$\left\{v_{1,2}, v_{1,4}, v_{1,6}, \dots, v_{1,2i}, \dots, v_{1,2\left(\frac{n-1}{2}\right)} (= v_{1,n-1})\right\} \xrightarrow{f} \left\{\frac{n+1}{2} + 1, \frac{n+1}{2} + 2, \frac{n+1}{2} + 3, \dots, \frac{n+1}{2} + i, \dots, \frac{n+1}{2} + \frac{n-1}{2} (= n)\right\}.$$

By Algorithm 2.1 (iii), $\{v_{2,n}\} \xrightarrow{f} \{n+1\}$.

By Algorithm 2.1 (iv),

$$\left\{v_{2,2}, v_{2,4}, v_{2,6}, \dots, v_{2,2i}, \dots, v_{2,2\left(\frac{n-1}{2}\right)} (= v_{2,n-1})\right\} \xrightarrow{f} \left\{(n+1) + 1, (n+1) + 2, (n+1) + 3, \dots, (n+1) + i, \dots, (n+1) + \frac{n-1}{2} \left(= \frac{3n+1}{2}\right)\right\}.$$

By Algorithm 2.1 (v),

$$\left\{v_{2,1}, v_{2,3}, v_{2,5}, \dots, v_{2,2i-1}, \dots, v_{2,2\left(\frac{n-1}{2}\right)-1} (= v_{2,n-2})\right\} \xrightarrow{f} \left\{\frac{3n+1}{2} + 1, \frac{3n+1}{2} + 2, \frac{3n+1}{2} + 3, \dots, \frac{3n+1}{2} + i, \dots, \frac{3n+1}{2} + \frac{n-1}{2} (= 2n)\right\}$$

Next, since $|E(P_2 \square C_n)| = |\{2n+1, 2n+2, 2n+3, \dots, 5n\}|$, it is enough to show that $f: E(P_2 \square C_n) \to \{2n+1, 2n+2, 2n+3, \dots, 5n\}$ is a surjection.

Let $a \in \{2n+1, 2n+2, 2n+3, \dots, 5n\}$.

Case 1 $a \in \{2n+1, 2n+2, 2n+3, \dots, 3n\}$.

Case 1.1 $a = 2n+1$.

Then, by Algorithm 2.1 (i), $f(v_{1,1}) + f(v_{1,n}) + \frac{3n-1}{2} = 1 + \frac{n+1}{2} + \frac{3n-1}{2} = 2n+1$.

Case 1.2 $a = 2n + 2\alpha$ for some $\alpha \in \left\{1,2,3,\dots,\frac{n-1}{2}\right\}$.

Then, by Algorithm 2.1 (i) and (ii),

$$f(v_{1,2\alpha-1}) + f(v_{1,2\alpha}) + \frac{3n-1}{2} = \alpha + \frac{n+1}{2} + \alpha + \frac{3n-1}{2} = 2n + 2\alpha.$$

Case 1.3 $a = 2n + 2\alpha + 1$ for some $\alpha \in \left\{1,2,3,\dots,\frac{n-1}{2}\right\}$.

Then, by Algorithm 2.1 (i) and (ii),

$$f(v_{1,2\alpha}) + f(v_{1,2\alpha+1}) + \frac{3n-1}{2} = \frac{n+1}{2} + \alpha + \alpha + 1 + \frac{3n-1}{2} = 2n + 2\alpha + 1.$$

Case 2 $a \in \{3n+1, 3n+2, 3n+3, \dots, 4n\}$.

Case 2.1 $a = 3n+1$.

Then, by Algorithm 2.1 (i) and (iii),

$$f(v_{1,n}) + f(v_{2,n}) + \frac{3n-1}{2} = \frac{n+1}{2} + n + 1 + \frac{3n-1}{2} = 3n+1.$$

Case 2.2 $a = 3n + 2\beta$ for some $\beta \in \left\{1,2,3,\dots,\frac{n-1}{2}\right\}$.

Then, by Algorithm 2.1 (i) and (v),

$$f(v_{1,2\beta-1}) + f(v_{2,2\beta-1}) + \frac{3n-1}{2} = \beta + \frac{3n+1}{2} + \beta + \frac{3n-1}{2} = 3n + 2\beta.$$

Case 2.3 $a = 3n + 2\beta + 1$ for some $\beta \in \left\{1,2,3,\dots,\frac{n-1}{2}\right\}$.

Then, by Algorithm 2.1 (ii) and (iv),

$$f(v_{1,2\beta}) + f(v_{2,2\beta}) + \frac{3n-1}{2} = \frac{n+1}{2} + \beta + n + 1 + \beta + \frac{3n-1}{2} = 3n + 2\beta + 1.$$

Case 3 $a \in \{4n+1, 4n+2, 4n+3, \dots, 5n\}$.

Case 3.1 $a = 4n+1$.

Then, by Algorithm 2.1 (iii) and (iv),

$$f(v_{2,n}) + f(v_{2,n-1}) + \frac{3n-1}{2} = n + 1 + n + 1 + \frac{n-1}{2} + \frac{3n-1}{2} = 4n+1.$$

Case 3.2 $a = 4n + 2$.

Then, by Algorithm 2.1 (iv) and (v),

$$f(v_{2,1}) + f(v_{2,n}) + \frac{3n-1}{2} = \frac{3n+1}{2} + 1 + n + 1 + \frac{3n-1}{2} = 4n + 2.$$

Case 3.3 $a = 4n + 2\gamma + 1$ for some $\gamma \in \left\{1,2,3,\dots,\frac{n-1}{2}\right\}$.

Then, by Algorithm 2.1 (iv) and (v),

$$f(v_{2,2\gamma}) + f(v_{2,2\gamma-1}) + \frac{3n-1}{2} = n + 1 + \gamma + \frac{3n+1}{2} + \gamma + \frac{3n-1}{2} = 4n + 2\gamma + 1.$$

Case 3.4 $a = 4n + 2\gamma + 2$ for some $\gamma \in \left\{1,2,3,\dots,\frac{n-3}{2}\right\}$.

Then, by Algorithm 2.1 (iv) and (v),

$$f(v_{2,2\gamma+1}) + f(v_{2,2\gamma}) + \frac{3n-1}{2} = \frac{3n+1}{2} + \gamma + 1 + n + 1 + \gamma + \frac{3n-1}{2} = 4n + 2\gamma + 2.$$

Thus, f defined in Algorithm 2.1 is an rsem labeling for $P_2 \,\square\, C_n$. Finally, from Algorithm 2.1 (vi), it is clear that $c^{-1}(f) = \frac{3n-1}{2}$. □

Theorem 2.2 *Let n be an odd integer such that $n \geq 3$. Then,*

$$rsems(P_2 \,\square\, C_n) = \frac{3n-1}{2}.$$

Proof. Let n be an odd integer such that $n \geq 3$ and $\bar{f}$ be any rsem labeling of $P_2 \,\square\, C_n$. Since $\deg(v) = 3$ for all $v \in V(P_2 \,\square\, C_n)$, by Theorem 1.1, we have

$$3nc^{-1}(\bar{f}) = \big((2n+1) + (2n+2) + (2n+3) + \cdots + 5n\big) - 3(1 + 2 + 3 + \cdots + 2n).$$

That is, $c^{-1}(\bar{f}) = \frac{3n-1}{2}$.

Hence, $rsems(P_2 \,\square\, C_n) = \frac{3n-1}{2}$, for odd integer n such that $n \geq 3$. □

Theorem 2.3. *Let n be an even integer such that $n \geq 4$. Then, $P_2 \,\square\, C_n$ is not rsem.*

Proof. Let n be an even integer such that $n \geq 4$. Assume that $P_2 \,\square\, C_n$ is rsem with a rsem labeling $\tilde{f}$. Then, by the same calculation as shown in Theorem 2.2, we have $c^{-1}(\tilde{f}) = \frac{3n-1}{2}$ which is not an integer, a contradiction. □

3. RSEM labelings for $W_o(2,n)$

Let $n \geq 3$. The graph $W_o(2,n)$ is constructed by attaching a single pendant edge at each vertex of an outer cycle of $P_2 \,\square\, C_n$. In some literature, $W_o(2,n)$ is called the *generalized web graph without center*. Note that $p=|V(W_o(2,n))|=3n$ and $q=|E(W_o(2,n))|=4n$.

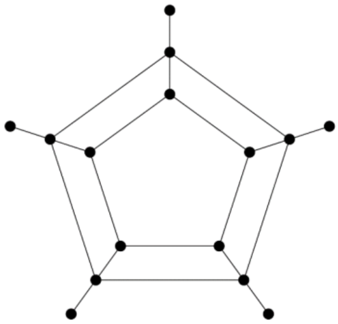

Figure 3.1 $W_o(2,5)$

In this section, we construct an rsem labeling f for $W_o(2,n)$ where n is an odd integer such that $n\geq 3$ for which $c^{-1}(f)$ is as small as we can. From that $c^{-1}(f)$ and Theorem 1.1, we can give upper and lower bounds for $rsems(W_o(2,n))$ where n is an odd integer such that $n\geq 3$.

Algorithm 3.1 Let n be an odd integer such that $n\geq 3$.

Define $f: V(W_o(2,n))\cup E(W_o(2,n))\to\{1,2,3,\dots,7n\}$ by

i. $f(v_{1,2i-1})=i$ for $i\in\left\{1,2,3,\dots,\frac{n+1}{2}\right\}$,

ii. $f(v_{1,2i})=\frac{n+1}{2}+i$ for $i\in\left\{1,2,3,\dots,\frac{n-1}{2}\right\}$,

iii. $f(v_{2,n})=n+1$,

iv. $f(v_{2,2i})=n+1+i$ for $i\in\left\{1,2,3,\dots,\frac{n-1}{2}\right\}$,

v. $f(v_{2,2i-1})=\frac{3n+1}{2}+i$ for $i\in\left\{1,2,3,\dots,\frac{n-1}{2}\right\}$,

vi. $f(v_{3,n-1})=2n+1$,

vii. $f(v_{3,2i-1})=2n+1+i$ for $i\in\left\{1,2,3,\dots,\frac{n+1}{2}\right\}$,

viii. $f(v_{3,2i})=\frac{5n+3}{2}+i$ for $i\in\left\{1,2,3,\dots,\frac{n-3}{2}\right\}$,

ix. $f(rs)=\frac{5n-1}{2}+(f(r)+f(s))$ for $rs\in E(W_o(2,n))$.

Example 3.1 From Algorithm 3.1, we can label all vertices and edges of $W_o(2,5)$ as shown in Figure 3.2.

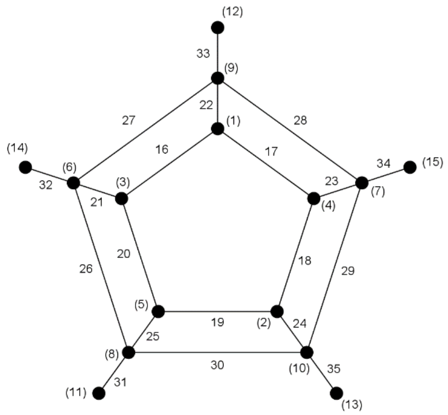

Figure 3.2 A labeling for $W_o(2,5)$ using Algorithm 3.1

Theorem 3.1 *Let n be an odd integer such that $n \geq 3$. The edge labeling of $W_o(2,n)$ given by Algorithm 3.1 is an rsem labeling with $c^{-1}(f) = \frac{5n-1}{2}$.*

Proof. Let n be an odd integer such that $n \geq 3$. Since (i) – (v) of Algorithm 3.1 are the same as (i) – (v) in Algorithm 2.1, we show the proof only on additional vertices $\{v_{3,1}, v_{3,2}, v_{3,3}, \dots, v_{3,n}\}$ of $W_o(2,n)$ that are different from those in $P_2 \,\square\, C_n$.

By Algorithm 3.1 (vi), $\{v_{3,n-1}\} \xrightarrow{f} \{2n+1\}$.

By Algorithm 3.1 (vii),

$$\left\{v_{3,1}, v_{3,3}, v_{3,5}, \dots, v_{3,2i-1}, \dots, v_{3,2\left(\frac{n+1}{2}\right)-1} (= v_{3,n})\right\} \xrightarrow{f} \left\{(2n+1)+1, (2n+1)+2, (2n+1)+3, \dots, (2n+1)+i, \dots, (2n+1)+\frac{n+1}{2}\left(=\frac{5n+3}{2}\right)\right\}.$$

By Algorithm 3.1 (viii),

$$\left\{v_{3,2}, v_{3,4}, v_{3,6}, \dots, v_{3,2i}, \dots, v_{3,2\left(\frac{n-3}{2}\right)} (= v_{3,n-3})\right\} \xrightarrow{f} \left\{\frac{5n+3}{2}+1, \frac{5n+3}{2}+2, \frac{5n+3}{2}+3, \dots, \frac{5n+3}{2}+i, \dots, \frac{5n+3}{2}+\frac{n-3}{2} (= 3n)\right\}.$$

Next, since $|E(W_o(2,n))| = |\{3n+1, 3n+2, 3n+3, \dots, 7n\}|$, it is enough to show that $f: E(W_o(2,n)) \to \{3n+1, 3n+2, 3n+3, \dots, 7n\}$ is a surjection.

Let $a \in \{3n+1, 3n+2, 3n+3, \dots, 7n\}$.

<u>Case 1</u> $a \in \{3n+1, 3n+2, 3n+3, \dots, 4n\}$.

<u>Case 1.1</u> $a = 3n+1$.

Then, by Algorithm 3.1 (i), $f(v_{1,1}) + f(v_{1,n}) + \frac{5n-1}{2} = 1 + \frac{n+1}{2} + \frac{5n-1}{2} = 3n+1$.

Case 1.2 $a = 3n + 2\alpha$ for some $\alpha \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$.

Then, by Algorithm 3.1 (i) and (ii),

$$f(v_{1,2\alpha-1}) + f(v_{1,2\alpha}) + \frac{5n-1}{2} = \alpha + \frac{n+1}{2} + \alpha + \frac{5n-1}{2} = 3n + 2\alpha.$$

Case 1.3 $a = 3n + 2\alpha + 1$ for some $\alpha \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$.

Then, by Algorithm 3.1 (i) and (ii),

$$f(v_{1,2\alpha}) + f(v_{1,2\alpha+1}) + \frac{5n-1}{2} = \frac{n+1}{2} + \alpha + \alpha + 1 + \frac{5n-1}{2} = 3n + 2\alpha + 1.$$

Case 2 $a \in \{4n + 1, 4n + 2, 4n + 3, \dots, 5n\}$.

Case 2.1 $a = 4n + 1$.

Then, by Algorithm 3.1 (i) and (iii),

$$f(v_{1,n}) + f(v_{2,n}) + \frac{5n-1}{2} = \frac{n+1}{2} + n + 1 + \frac{5n-1}{2} = 4n + 1.$$

Case 2.2 $a = 4n + 2\beta$ for some $\beta \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$.

Then, by Algorithm 3.1 (i) and (v),

$$f(v_{1,2\beta-1}) + f(v_{2,2\beta-1}) + \frac{5n-1}{2} = \beta + \frac{3n+1}{2} + \beta + \frac{5n-1}{2} = 4n + 2\beta.$$

Case 2.3 $a = 4n + 2\beta + 1$ for some $\beta \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$.

Then, by Algorithm 3.1 (ii) and (iv),

$$f(v_{1,2\beta}) + f(v_{2,2\beta}) + \frac{5n-1}{2} = \frac{n+1}{2} + \beta + n + 1 + \beta + \frac{5n-1}{2} = 4n + 2\beta + 1.$$

Case 3 $a \in \{5n + 1, 5n + 2, 5n + 3, \dots, 6n\}$.

Case 3.1 $a = 5n + 1$.

Then, by Algorithm 3.1 (iii) and (iv),

$$f(v_{2,n}) + f(v_{2,n-1}) + \frac{5n-1}{2} = n + 1 + n + 1 + \frac{n-1}{2} + \frac{5n-1}{2} = 5n + 1.$$

Case 3.2 $a = 5n + 2$.

Then, by Algorithm 3.1 (iii) and (v),

$$f(v_{2,1}) + f(v_{2,n}) + \frac{5n-1}{2} = \frac{3n+1}{2} + 1 + n + 1 + \frac{5n-1}{2} = 5n + 2.$$

Case 3.3 $a = 5n + 2\gamma + 1$ for some $\gamma \in \left\{1,2,3, \dots, \frac{n-1}{2}\right\}$.

Then, by Algorithm 3.1 (iv) and (v),

$$f(v_{2,2\gamma}) + f(v_{2,2\gamma-1}) + \frac{5n-1}{2} = n + 1 + \gamma + \frac{3n+1}{2} + \gamma + \frac{5n-1}{2} = 5n + 2\gamma + 1.$$

Case 3.4 $a = 5n + 2\gamma + 2$ for some $\gamma \in \left\{1,2,3, \dots, \frac{n-3}{2}\right\}$.

Then, by Algorithm 3.1 (iv) and (v),

$$f(v_{2,2\gamma+1}) + f(v_{2,2\gamma}) + \frac{5n-1}{2} = \frac{3n+1}{2} + \gamma + 1 + n + 1 + \gamma + \frac{5n-1}{2} = 5n + 2\gamma + 2.$$

Case 4 $a \in \{6n+1, 6n+2, 6n+3, \dots, 7n\}$.

If $n = 3$, then $a \in \{19,20,21\}$. We can see from the Algorithm 3.1 (v) and (viii); (iii) and (vi); and (iii) and (vi) that

$$f(v_{2,2}) + f(v_{3,2}) + \frac{5(3)-1}{2} = 5 + 7 + 7 = 19,$$
$$f(v_{2,3}) + f(v_{3,3}) + \frac{5(3)-1}{2} = 4 + 9 + 7 = 20 \text{ and}$$
$$f(v_{2,1}) + f(v_{3,1}) + \frac{5(3)-1}{2} = 6 + 8 + 7 = 21, \text{ respectively.}$$

For $n \geq 5$, we consider the following cases.

Case 4.1 $a = 6n + 1$.

Then, by Algorithm 3.1 (iv) and (vi),

$$f(v_{2,n-1}) + f(v_{3,n-1}) + \frac{5n-1}{2} = n + 1 + \frac{n-1}{2} + 2n + 1 + \frac{5n-1}{2} = 6n + 1.$$

Case 4.2 $a = 6n + 2$.

Then, by Algorithm 3.1 (iii) and (vii),

$$f(v_{2,n}) + f(v_{3,n}) + \frac{5n-1}{2} = n + 1 + 2n + 1 + \frac{n+1}{2} + \frac{5n-1}{2} = 6n + 2.$$

Case 4.3 $a = 6n + 2\delta + 1$ for some $\delta \in \left\{1,2,3,\dots,\frac{n-1}{2}\right\}$.

Then, by Algorithm 3.1 (v) and (vii),

$$f(v_{2,2\delta-1}) + f(v_{3,2\delta-1}) + \frac{5n-1}{2} = \frac{3n+1}{2} + \delta + 2n + 1 + \delta + \frac{5n-1}{2} = 6n + 2\delta + 1.$$

Case 4.4 $a = 6n + 2\delta + 2$ for some $\delta \in \left\{1,2,3,\dots,\frac{n-3}{2}\right\}$.

Then, by Algorithm 3.1 (iv) and (viii),

$$f(v_{2,2\delta}) + f(v_{3,2\delta}) + \frac{5n-1}{2} = n + 1 + \delta + \frac{5n+3}{2} + \delta + \frac{5n-1}{2} = 6n + 2\delta + 2.$$

Thus, f defined in Algorithm 3.1 is an rsem labeling for $W_o(2,n)$. Finally, from Algorithm 3.1 (ix), it is clear that $c^{-1}(f) = \frac{5n-1}{2}$. □

Theorem 3.2 *Let n be an odd integer such that $n \geq 3$. Then,*

$$\frac{5n-2}{4} \leq rsems(W_o(2,n)) \leq \frac{5n-1}{2}.$$

Proof. Let $\bar{f}$ be an rsem labeling of $W_o(2,n)$ with its rsem constant $c^{-1}(\bar{f})$. Then, by Theorem 1.1, we have

$$qc^{-1}(\bar{f}) = \sum_{e\in E(W_o(2,n))} \bar{f}(e) - \left(4\sum_{v\in V(W_o(2,n))} \bar{f}(v) - \left(\sum_{j=1}^{n} \bar{f}(v_{1,j}) + 3\sum_{j=1}^{n} \bar{f}(v_{3,j})\right)\right)$$

$$= \left((3n+1)+(3n+2)+(3n+3)+\cdots+(3n+4n)\right)$$
$$- \left(4(1+2+3+\cdots+3n) - \left(\sum_{j=1}^{n} \bar{f}(v_{1,j}) + 3\sum_{j=1}^{n} \bar{f}(v_{3,j})\right)\right)$$
$$= 2n^2 - 4n + \left(\sum_{j=1}^{n} \bar{f}(v_{1,j}) + 3\sum_{j=1}^{n} \bar{f}(v_{3,j})\right).$$

Consider

$$\sum_{j=1}^{n} \bar{f}(v_{1,j}) + 3\sum_{j=1}^{n} \bar{f}(v_{3,j}) \geq \left((n+1)+(n+2)+(n+3)+\cdots+2n\right)$$
$$+3(1+2+3+\cdots+n)$$
$$= 3n^2 + 2n.$$

That is,

$$4nc^{-1}(\bar{f}) \geq 2n^2 - 4n + (3n^2+2n) = 5n^2 - 2n.$$

Thus, $c^{-1}(\bar{f}) \geq \frac{5n-2}{4}$.

Therefore, $rsems(W_o(2,n)) \geq \frac{5n-2}{4}$.

Next, by Theorem 3.1, the labeling defined by Algorithm 3.1 is an rsem labeling, we can conclude that $rsems(W_o(2,n)) \leq \frac{5n-1}{4}$. Hence, $\frac{5n-2}{4} \leq rsems(W_o(2,n)) \leq \frac{5n-1}{2}$ for an odd integer n such that $n \geq 3$. ☐

4. Conclusion and Discussion

In this article, for an odd integer n such that $n \geq 3$, we can find that $rsems(P_2 \square C_n) = \frac{3n-1}{2}$ and $\frac{5n-2}{4} \leq rsems(W_o(2,n)) \leq \frac{5n-1}{2}$. There are several open problems that students may work as a mathematical project as follows.

(i) For an odd integer n such that $n \geq 3$, can we find the exact value of $rsems(W_o(2,n))$ or not ?

(ii) For an even integer n such that $n \geq 4$, can we find a rsem labeling for $W_o(2,n)$ or not ?

(iii) For integers n and k such that $n, k \geq 3$, can we find a rsem labeling for $P_k \,\square\, C_n$ and $W_o(k,n)$ or not ?

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