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<http://www.mathassociation.net>

Email: MathThaiOrg@gmail.com

ผลเฉลยของสมการไดโอแฟนไทน์ $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$

Integral Solutions of the Diophantine Equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$

อภิสิทธิ์ ภาคพงศ์พันธุ์

Apisit Pakapongpun

Department of Mathematics, Faculty of Science,

Burapha University, Chon Buri 20131

Email: apisit.buu@gmail.com

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บทคัดย่อ

ในบทความนี้ เราได้หาผลเฉลยที่เป็นจำนวนเต็มบวก ของสมการไดโอแฟนไทน์ $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$

คำสำคัญ: สมการไดโอแฟนไทน์ ผลเฉลยที่เป็นจำนวนเต็มบวก

ABSTRACT

In this paper, we find positive integral solutions of the Diophantine equation

$$\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}.$$

Keywords: Diophantine equation, Positive integral solutions

1. Introduction

In 1948, Erdős and Straus [1], [3] have conjectured that for each $n \geq 2$, the Diophantine equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{n}$ has a positive integral solution (x, y, z) , where $x \neq y$, $y \neq z$ and $z \neq x$. Later, many researchers studied and solved this Diophantine equation, see [2], [4], [7] and [8].

In 2013, Rabago and Tagle [5] studied on the areas and volume of a certain regular solid and the Diophantine equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$. Sandor [6] offered an elementary approach to the solution of the Diophantine equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$.

In this paper, we try to solve for all positive integral solutions of the Diophantine equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$.

2. Method of Analysis

From the Diophantine equation $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{2}$, we get $x \geq 3, y \geq 5$ and $z \geq 7$.

We are going to split into three cases.

Case 1 If $x \leq y \leq z$ or $x \leq z \leq y$, then $\frac{1}{2} = \frac{1}{x} + \frac{2}{y} + \frac{3}{z} \leq \frac{6}{x}$. Hence, $3 \leq x \leq 12$.

$$\text{If } x=3, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{1}{6}, \text{ i.e., } (y-12)(z-18) = 216. \quad (1)$$

$$\text{If } x=4, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{1}{4}, \text{ i.e., } (y-8)(z-12) = 96. \quad (2)$$

$$\text{If } x=5, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{3}{10}, \text{ i.e., } (3y-20)(z-10) = 200. \quad (3)$$

$$\text{If } x=6, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{1}{3}, \text{ i.e., } (y-6)(z-9) = 54. \quad (4)$$

$$\text{If } x=8, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{3}{8}, \text{ i.e., } (3y-16)(z-8) = 128. \quad (5)$$

$$\text{If } x=10, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{2}{5}, \text{ i.e., } (y-5)(2z-15) = 75. \quad (6)$$

For (1), the only following cases are possible:

$$\begin{aligned}
 &y - 12 = 1, z - 18 = 216; & y - 12 = 2, z - 18 = 108; & y - 12 = 3, z - 18 = 72; \\
 &y - 12 = 4, z - 18 = 54; & y - 12 = 6, z - 18 = 36; & y - 12 = 8, z - 18 = 27; \\
 &y - 12 = 9, z - 18 = 24; & y - 12 = 12, z - 18 = 18; & y - 12 = 18, z - 18 = 12; \\
 &y - 12 = 24, z - 18 = 9; & y - 12 = 27, z - 18 = 8; & y - 12 = 36, z - 18 = 6; \\
 &y - 12 = 54, z - 18 = 4; & y - 12 = 72, z - 18 = 3; & y - 12 = 108, z - 18 = 2; \\
 &y - 12 = 216, z - 18 = 1.
 \end{aligned}$$

Leading to the solutions: $(x, y, z) = (3, 13, 234), (3, 14, 126), (3, 15, 90), (3, 16, 72), (3, 18, 54), (3, 20, 45), (3, 21, 42), (3, 24, 36), (3, 30, 30), (3, 36, 27), (3, 39, 26), (3, 48, 24), (3, 66, 22), (3, 84, 21), (3, 120, 20)$ and $(3, 228, 19)$.

In the same manner

(2) leads to $(x, y, z) = (4, 9, 108), (4, 10, 60), (4, 11, 44), (4, 12, 36), (4, 14, 28), (4, 16, 24), (4, 20, 20), (4, 24, 18), (4, 32, 16), (4, 40, 15), (4, 56, 14)$ and $(4, 104, 13)$.

(3) leads to $(x, y, z) = (5, 7, 210), (5, 8, 60), (5, 10, 30), (5, 15, 18), (5, 20, 15)$ and $(5, 40, 12)$.

(4) leads to $(x, y, z) = (6, 7, 63), (6, 8, 36), (6, 9, 27), (6, 12, 18), (6, 15, 15), (6, 24, 12), (6, 33, 11)$ and $(6, 60, 10)$.

(5) leads to $(x, y, z) = (8, 8, 24), (8, 16, 12)$ and $(8, 48, 9)$.

While, (6) leads to $(x, y, z) = (10, 10, 15)$ and $(10, 20, 10)$.

$$\text{If } x = 7, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{5}{14}. \quad (7)$$

$$\text{If } x = 9, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{7}{18}. \quad (8)$$

$$\text{If } x = 11, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{9}{22}. \quad (9)$$

$$\text{If } x = 12, \text{ then } \frac{2}{y} + \frac{3}{z} = \frac{5}{12}. \quad (10)$$

If $z \leq y$, then (7) gives $\frac{5}{14} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{z}$, that is, $7 \leq z \leq 14$. However, only $z \in \{9, 10, 14\}$ gives positive integral values of y . Hence, $(x, y, z) = (7, 84, 9), (7, 35, 10)$ and $(7, 14, 14)$.

If $y \leq z$, then (7) gives $\frac{5}{14} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{y}$, that is, $7 \leq y \leq 14$. However, only $y \in \{7, 8\}$ gives positive integral values of z . Hence, $(x, y, z) = (7, 7, 42)$ and $(7, 8, 28)$.

If $z \leq y$, then (8) gives $\frac{7}{18} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{z}$, that is, $9 \leq z \leq 12$. However, only $z \in \{9\}$ gives positive integral value of y . Hence, $(x, y, z) = (9, 36, 9)$.

If $y \leq z$, then (8) gives $\frac{7}{18} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{y}$, that is, $9 \leq y \leq 12$. However, only $y \in \{9\}$ gives positive integral value of z . Hence, $(x, y, z) = (9, 9, 18)$.

If $z \leq y$, then (9) gives $\frac{9}{22} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{z}$, that is, $11 \leq z \leq 12$. However, both of them do not give positive integral value of y . Hence, there is no solutions for (9).

If $y \leq z$, then (9) gives $\frac{9}{22} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{y}$, that is, $11 \leq y \leq 12$. However, both of them do not give positive integral value of z . Hence, there is no solutions for (9).

If $z \leq y$, then (10) gives $\frac{5}{12} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{z}$, that is, $z = 12$. Hence, $(x, y, z) = (12, 12, 12)$.

If $y \leq z$, then (10) gives $\frac{5}{12} = \frac{2}{y} + \frac{3}{z} \leq \frac{5}{y}$, that is, $y = 12$. Hence, $(x, y, z) = (12, 12, 12)$.

Case II If $y < x \leq z$ or $y \leq z < x$, then $\frac{1}{2} = \frac{1}{x} + \frac{2}{y} + \frac{3}{z} \leq \frac{6}{y}$. Hence, $5 \leq y \leq 12$.

$$\text{If } y = 5, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{1}{10}, \text{ i.e., } (x-10)(z-30) = 300. \quad (11)$$

$$\text{If } y = 6, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{1}{6}, \text{ i.e., } (x-6)(z-18) = 108. \quad (12)$$

$$\text{If } y = 7, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{3}{14}, \text{ i.e., } (3x-14)(z-14) = 196. \quad (13)$$

$$\text{If } y = 8, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{1}{4}, \text{ i.e., } (x-4)(z-12) = 48. \quad (14)$$

$$\text{If } y=10, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{3}{10}, \text{ i.e., } (3x-10)(z-10)=100. \quad (15)$$

$$\text{If } y=12, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{1}{3}, \text{ i.e., } (x-3)(z-9)=27. \quad (16)$$

For (11) the only following cases are possible :

$$\begin{array}{lll} x-10=1, z-30=300; & x-10=2, z-30=150; & x-10=3, z-30=100; \\ x-10=4, z-30=75; & x-10=5, z-30=60; & x-10=6, z-30=50; \\ x-10=10, z-30=30; & x-10=12, z-30=25; & x-10=15, z-30=20; \\ x-10=20, z-30=15; & x-10=25, z-30=12; & x-10=30, z-30=10; \\ x-10=50, z-30=6; & x-10=60, z-30=5; & x-10=75, z-30=4; \\ x-10=100, z-30=3; & x-10=150, z-30=2; & x-10=300, z-30=1. \end{array}$$

Leading to the solutions : $(x, y, z) = (11, 5, 330), (12, 5, 180), (13, 5, 130), (14, 5, 105), (15, 5, 90), (16, 5, 80), (20, 5, 60), (22, 5, 55), (25, 5, 50), (30, 5, 45), (35, 5, 42), (40, 5, 40), (60, 5, 36), (70, 5, 35), (85, 5, 34), (110, 5, 33), (160, 5, 32)$ and $(310, 5, 31)$.

In the same manner,

(12) leads to $(x, y, z) = (7, 6, 126), (8, 6, 72), (9, 6, 54), (10, 6, 45), (12, 6, 36), (15, 6, 30), (18, 6, 27), (24, 6, 24), (33, 6, 22), (42, 6, 21), (60, 6, 20)$ and $(114, 6, 19)$.

(13) leads to $(x, y, z) = (14, 7, 21), (21, 7, 18)$ and $(70, 7, 15)$.

(14) leads to $(x, y, z) = (10, 8, 20), (12, 8, 18), (16, 8, 16), (20, 8, 15), (628, 8, 14)$ and $(52, 8, 13)$.

(15) leads to $(x, y, z) = (20, 10, 12)$.

While, (16) has no solutions.

$$\text{If } y=9, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{5}{18}. \quad (17)$$

$$\text{If } y=11, \text{ then } \frac{1}{x} + \frac{3}{z} = \frac{7}{22}. \quad (18)$$

If $x \leq z$, then (17) gives $\frac{5}{18} = \frac{1}{x} + \frac{3}{z} \leq \frac{4}{x}$, that is, $9 < x \leq 14$. However, all values of x do not give positive integral value of z . Hence, there is no solutions for (17).

If $z < x$, then (17) gives $\frac{5}{18} = \frac{1}{x} + \frac{3}{z} < \frac{4}{z}$, that is, $9 \leq z \leq 14$. However, only $z \in \{11, 12\}$ gives positive integral values of x . Hence, $(x, y, z) = (198, 9, 11)$ and $(36, 9, 12)$.

If $x \leq z$, then (18) gives $\frac{7}{22} = \frac{1}{x} + \frac{3}{z} \leq \frac{4}{x}$, that is, $11 < x \leq 12$. However, both values of x do not give positive integral value of z . Hence, there is no solutions for (18).

If $z < x$, then (18) gives $\frac{7}{22} = \frac{1}{x} + \frac{3}{z} < \frac{4}{z}$, that is, $11 \leq z \leq 12$. However, only $z \in \{11\}$ gives positive integral value of x . Hence, $(x, y, z) = (22, 11, 11)$.

Case III If $z < x \leq y$ or $z < y < x$, then $\frac{1}{2} = \frac{1}{x} + \frac{2}{y} + \frac{3}{z} \leq \frac{6}{z}$. Hence, $7 \leq z \leq 12$.

$$\text{If } z = 7, \text{ then } \frac{1}{x} + \frac{2}{y} = \frac{1}{14}, \text{ i.e., } (x-14)(y-28) = 392. \quad (19)$$

$$\text{If } z = 8, \text{ then } \frac{1}{x} + \frac{2}{y} = \frac{1}{8}, \text{ i.e., } (x-8)(y-16) = 128. \quad (20)$$

$$\text{If } z = 9, \text{ then } \frac{1}{x} + \frac{2}{y} = \frac{1}{6}, \text{ i.e., } (x-6)(y-12) = 72. \quad (21)$$

$$\text{If } z = 10, \text{ then } \frac{1}{x} + \frac{2}{y} = \frac{1}{5}, \text{ i.e., } (x-5)(y-10) = 50. \quad (22)$$

$$\text{If } z = 12, \text{ then } \frac{1}{x} + \frac{2}{y} = \frac{1}{4}, \text{ i.e., } (x-4)(y-8) = 32. \quad (23)$$

(19) leads to $(x, y, z) = (15, 420, 7), (16, 224, 7), (18, 126, 7), (21, 84, 7), (22, 77, 7), (28, 56, 7), (42, 42, 7), (63, 36, 7), (70, 35, 7), (112, 32, 7), (210, 30, 7)$ and $(406, 29, 7)$.

(20) leads to $(x, y, z) = (9, 144, 8), (10, 80, 8), (12, 48, 8), (16, 32, 8), (24, 24, 8), (40, 20, 8), (72, 18, 8)$ and $(136, 17, 8)$.

(21) leads to $(x, y, z) = (10, 30, 9), (12, 24, 9), (14, 21, 9), (15, 20, 9), (18, 18, 9), (24, 16, 9), (30, 15, 9), (42, 14, 9)$ and $(78, 13, 9)$.

(22) leads to $(x, y, z) = (15, 15, 10), (30, 12, 10)$ and $(55, 11, 10)$.

While, (23) has no solutions.

$$\text{If } z = 11 \text{ then } \frac{1}{x} + \frac{2}{y} = \frac{5}{22}. \quad (24)$$

If $x \leq y$, then (24) gives $\frac{5}{22} = \frac{1}{x} + \frac{2}{y} \leq \frac{3}{x}$, that is, $12 \leq x \leq 13$. However, both values of x do not give positive integral value of y . Hence, there is no solution for (24).

If $y < x$ then (24) gives $\frac{5}{22} = \frac{1}{x} + \frac{2}{y} < \frac{3}{y}$ that is, $12 \leq y \leq 13$. However, both values of y do not give positive integral value of x . Hence, there is no solution for (24).

Suggested Problem

Let a, b, c, p and q be positive integers, we may ask for the positive integral solutions (x, y, z) for the Diophantine equation $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{p}{q}$ where x, y and z are positive integers.

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