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การเดินของม้าหมากรุกปิดแบบ (1, 2, 2) บนกระดานหมากรุกขนาด 3r × 4s × 4t เมื่อ r ≥ 1 และ s, t ≥ 2

A Closed (1, 2, 2)-Knight's Tour on the 3r × 4s × 4t Chessboard, where r ≥ 1 and s, t ≥ 2

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บทคัดย่อ

บทความนี้นำเสนอการเดินของม้าหมากรุกปิดแบบ (1, 2, 2) บนกระดานหมากรุกขนาด $3r \times 4s \times 4t$ เมื่อ $r \geq 1$ และ $s, t \geq 2$

คำสำคัญ: การเดินของม้าหมากรุกปิด กระดานหมากรุก

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ABSTRACT

In this paper, a closed (1, 2, 2)-knight's tour on the $3r \times 4s \times 4t$ chessboard, where $r \geq 1$ and $s, t \geq 2$, is obtained.

Keywords: Closed knight's tour, Chessboard

1. Introduction

A knight is an interesting chess piece on the chessboard because of its move. It moves like the L-shape, one square vertically or one square horizontally and then two squares at 90 degrees angle. Euler [4] was the first person to study the knight's moves on the chessboard. He found that, a knight can move from a square to every other squares exactly once and return to its starting square. This is called a *closed knight's tour*. Then, the chessboard is extended to an $m \times n$ chessboard, an array with m rows and n columns. The problem is which $m \times n$ chessboard contains a closed knight's tour. In 1991, Schwenk [4] answered this question.

Theorem 1.1 [4] The $m \times n$ chessboard with $m \leq n$ admits a closed knight's tour unless one or more of the following conditions holds:

(i) m and n are both odd; or

(ii) $m = 1$ or 2 or 4; or

(iii) $m = 3$ and $n = 4$ or 6 or 8.

Next, the knight's move was generalized by Chia [2] to move a squares vertically or a squares horizontally and then b squares at 90 degrees angle. It is called *an (a, b)-knight's move*. An (1, 2)-knight's move is an ordinary knight's move and a closed (1, 2)-knight's tour is a closed knight's tour. The authors in [2] obtained necessary conditions for existence of a closed (a, b)-knight's tour on the $m \times n$ chessboard.

Theorem 1.2 [2] Suppose that the $m \times n$ chessboard admits a closed (a, b)-knight's tour, where $a < b$ and $m \leq n$. Then,

(i) $a + b$ is odd;

(ii) m or n is even;

(iii) $m \geq a + b$; and

(iv) $n \geq 2b$.

Also, the chessboard is extended to the three-dimension chessboard called the *$k \times m \times n$ chessboard*. It is obtained from k copies of the $m \times n$ chessboard by putting each copy from the top to the bottom. We label each square by (l, i, j) where l is the level of the $k \times m \times n$ chessboard counting from the top to the bottom and (i, j) is the position of the square of the $m \times n$ chessboard. Note that it can be seen that a position of the three-dimension chessboard is a cube. Moreover, the (a, b)-knight's move is extended to an (a, b, c)-knight's move. If the knight moves with an (a, b, c)-knight's move from cube (i, j, k) to cube (r, s, t), then $\{|r - i|, |s - j|, |t - k|\} = \{a, b, c\}$. That is, if the knight stands at cube (l, i, j), then the knight can move to at most 24 cubes: $(l \pm a, i \pm b, j \pm c)$, $(l \pm b, i \pm a, j \pm c)$, and $(l \pm c, i \pm b, j \pm a)$. Also, a closed (a, b, c)-knight's tour is a series of (a, b, c)-knight's moves that visits every small cube of the $k \times m \times n$ chessboard. Bai et al. [1] obtained necessary conditions for existence of a closed (1, 2, 2)-knight's tour on the $k \times m \times n$ chessboard.

Theorem 1.3 [1] The $k \times m \times n$ chessboard with $k \leq m \leq n$ does not admit a closed (1, 2, 2)-knight's tour if one of the following conditions holds:

(i) k, m and n *are* odd;

(ii) $k \leq 2$;

(iii) $k = 3$ and $m \leq 7$;

(iv) $k = 4$.

Moreover, the authors in [1] showed closed (1, 2, 2)-knight's tours on the $3 \times 4s \times 4t$ chessboard, where $s, t \geq 2$.

Theorem 1.4 [1] Suppose $s \geq 2$ and $t \geq 2$. Then, the $3 \times 4s \times 4t$ chessboard admits a closed (1, 2, 2)-knight's tour.

The knight's tour problem is converted to question about a certain graph. A graph that represents all (a, b, c)-knight's moves on the $k \times m \times n$ chessboard is a graph with mnk vertices where each cube is replaced by a vertex and two vertices are joined by an edge if the knight can move by an (a, b, c)-knight's move between these cubes or vertices. Then, a closed (a, b, c)-knight's tour is a Hamiltonian cycle, a cycle that passed through each vertex exactly once, of the graph representing all (a, b, c)-knight's moves on the $k \times m \times n$ chessboard.

Motivated by these, this paper shows that the $3r \times 4s \times 4t$ chessboard, where $r \geq 1$ and $s, t \geq 2$, contains a closed (1, 2, 2)-knight's tour (in Theorem 2.1).

2. Main Results

First, we recall the construction of a closed (1, 2, 2)-knight's tour on the $3 \times 4s \times 4t$ chessboard, where $s, t \geq 2$, in the proof of Theorem 1.4 as follows.

Algorithm 2.1 A closed (1, 2, 2)-knight's tour on the $3 \times 4s \times 4t$ chessboard, where $s, t \geq 2$, is obtained as follows.

(i) A closed knight's tour C of the first level of the $3 \times 4s \times 4t$ chessboard is obtained by Theorem 1 of [3]. (We roughly describe in Section 2.1.)

(ii) C is extended to a closed (1, 2, 2)-knight's tour C^*: $v_1v_2 \ldots v_nv_1$ on the $3 \times 4s \times 4t$ chessboard where $v_1, v_4, v_7, \ldots, v_{n-2}$ (respectively $v_2, v_5, v_8, \ldots, v_{n-1}$ and $v_3, v_6, v_9, \ldots, v_n$) are vertices on the first (respectively second and third) level of the $3 \times 4s \times 4t$ chessboard.

(iii) If $C : (i_1, j_1)\,(i_2, j_2)\ldots(i_{4s}, j_{4t})\,(i_1, j_1)$ is a closed knight's tour of the $4s \times 4t$ chessboard, where r, s ≥ 2, then a path P: $(1, i_1, j_1)\ (2, \boldsymbol{\alpha}, \boldsymbol{\beta})\ (3, i_1, j_1)\ (1, i_2, j_2)$ of C^* is constructed where

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \begin{cases} (i_1 + 2, j_1 + 2) \text{ if } i_1 \pmod 4 \in \{1, 2\},\ j_1 \pmod 4 \in \{1, 2\} \\ (i_1 + 2, j_1 - 2) \text{ if } i_1 \pmod 4 \in \{1, 2\},\ j_1 \pmod 4 \in \{0, 3\} \\ (i_1 - 2, j_1 + 2) \text{ if } i_1 \pmod 4 \in \{0, 3\},\ j_1 \pmod 4 \in \{1, 2\} \\ (i_1 - 2, j_1 - 2) \text{ if } i_1 \pmod 4 \in \{0, 3\},\ j_1 \pmod 4 \in \{0, 3\}. \end{cases}$$

Repeat the above construction for the edge $(i_2, j_2)(i_3, j_3)$ by replacing i_1 and j_1 with i_2 and j_2, respectively, and so on. Note that in the first level and the third level, we obtain the same closed knight's tour C.

In this paper, we have two main results as follows.

(i) Introduce new algorithm to find a closed knight's tour of the $4s \times 4t$ chessboard, where $s, t \geq 2$.

(ii) Construct a closed (1, 2, 2)-knight's tour of the $3r \times 4s \times 4t$ chessboard, where $r \geq 1$ and $s, t \geq 2$.

2.1 The Construction of a Closed Knight's Tour of the 4*s* × 4*t* Chessboard where *s*, *t* ≥ 2

Parberry's algorithm is to find a closed knight's tour in all square-like chessboards, such as $n \times n$, $n \times (n+1)$ and $n \times (n+2)$. Lin and Wei [3] improved Parberry's algorithm to an arbitrary rectangular $m \times n$ chessboard. Lin and Wei's algorithm on the $m \times n$ chessboard where $m \leq n$ has three steps as follows.

First, partition the $m \times n$ chessboard. If $m = 3$, then partition into one $3 \times k$ for some $k < n$, and many 3×4 chessboards. If $4 \leq m \leq 10$, then divide into two rectangular chessboards. For $m > 10$, divide into four quadrants.

Second, find closed knight's tours on each smaller rectangular chessboard.

Finally, combine all closed knight's tours on smaller chessboards to a closed knight's tour on the $m \times n$ chessboard.

In this section, we introduce the new algorithm to find a closed knight's tour on the $4s \times 4t$ chessboard where s, $t \geq 2$. Our algorithm does not divide the $4s \times 4t$ chessboard into two or four small chessboards. We start with the 8×8 chessboard and its closed knight's tour. The extension of the 8×8 chessboard to the $4s \times 4t$ chessboard is the combination of the small chessboard with four columns chessboard on the right-side or with four rows chessboard on the bottom of the 8×8 chessboard. While Lin and Wei's algorithm finds closed knight's tours on each smaller chessboard, our construction obtains cycles on small chessboards, 4×4, 4×8 and 8×4 chessboards. Finally, connect each cycle with the closed knight's tour on the 8×8 chessboard to obtain a closed knight's tour on the combined chessboard.

For our construction, we give sequences of the knight's moves by labeling each square of the chessboard by numbers or letters. Thus, readers can consider cycle in appropriate direction by following each process of the construction from the smaller chessboards, 4×4, 4×8, 8×4 and 8×8 chessboards, and then obtain a Hamiltonian cycle or a closed knight's tour on $4s \times 4t$ chessboard where s, $t \geq 2$ as follows.

Process 1: Construct a closed knight's tour C : 1-2-3-⋯-64-1 on the 8×8 chessboard as shown in Figure 2.1(a).

Process 2: Construct four 8-cycles on the 8×4 chessboard, called C_1, C_2, C_3 and C_4 where C_1 : 1-2-3-4-5-6-7-8-1, C_2 : s-t-u-v-w-x-y-z-s, C_3 : A-B-C-D-E-F-G-H-A, and C_4 : a-b-c-d-e-f-g-h-a, as shown in Figure 2.1(b).

1	50	5	54	7	58	19	60
4	53	2	63	30	61	8	57
49	64	51	6	55	20	59	18
52	3	48	29	62	31	56	9
27	44	25	38	21	34	17	32
24	47	28	41	14	39	10	35
43	26	45	22	37	12	33	16
46	23	42	13	40	15	36	11

(a)

1	***s***	A	a
B	b	2	***t***
z	8	h	H
c	C	***u***	3
7	***y***	G	g
D	d	4	***v***
x	6	f	F
e	E	***w***	5

(b)

Figure 2.1 (a) A closed knight's tour on the 8 × 8 chessboard

and (b) Four 8-cycles on the 8 × 4 chessboard

Process 3: Construct a closed knight's tour on the 8 × 12 chessboard from a closed knight's tour of the 8 × 8 chessboard and four 8-cycles of the 8 × 4 chessboard by

- putting the 8 × 8 chessboard on the left of the 8 × 4 chessboard,
- deleting edges 8-9, 16-17, 32-33, and 56-57 of C, 7-8 of C_1, *y*-*z* of C_2, D-E of C_3, and d-e of C_4, and
- joining 8 of C with ***z*** of C_2, 9 of C with ***y*** of C_2 , 16 of C with E of C_3, 17 of C with D of C_3, 32 of C with d of C_4, 33 of C with e of C_4, 56 of C with 7 of C_1, and 57 of C with 8 of C_1, as shown in Figure 2.2.

1	50	5	54	7	58	19	60	1	***s***	A	a
4	53	2	63	30	61	8	57	B	b	2	***t***
49	64	51	6	55	20	59	18	***z***	8	h	H
52	3	48	29	62	31	56	9	c	C	***u***	3
27	44	25	38	21	34	17	32	7	***y***	G	g
24	47	28	41	14	39	10	35	D	d	4	***v***
43	26	45	22	37	12	33	16	***x***	6	f	F
46	23	42	13	40	15	36	11	e	E	***w***	5

Figure 2.2 A closed knight's tour on the 8 × 12 chessboard

Process 4: The 8 × 4t chessboard, where $t \geq 4$, is obtained from the 8 × 12 chessboard and t – 3 copies of the 8 × 4 chessboards by putting 8 × 12 chessboard on the left of t – 3 consecutive copies of the 8 × 4 chessboards. We do two steps to obtain the 8 × 4t chessboard.

4.1 Connect four 8-cycles of each of t – 3 consecutive of the 8 × 4 chessboard by

- for the ith copy, deleting edges 3-4 of C_1, t-u of C_2, G-H of C_3 and a-h of C_4,
- for the (i + 1)th copy, deleting edges 7-8 of C_1, s-z of C_2, C-D of C_3 and b-c of C_4, and
- joining 3 of the ith copy with 8 of the (i + 1)th copy, 4 of the ith copy with 7 of the (i + 1)th copy, t of the ith copy with s of the (i + 1)th copy, u of the ith copy with z of the (i + 1)th copy, H of the ith copy with C of the (i + 1)th copy, G of the ith copy with D of the (i + 1)th copy, a of the ith copy with b of the (i + 1)th copy, h of the ith copy with c of the (i + 1)th copy, as shown in Figure 2.3.

Then, the 8 × 4(t – 3) chessboard is obtained from t – 3 copies of the 8 × 4 chessboard by joining each of t – 3 consecutive copies and it contains four cycles.

4.2 Connect a closed knight's tour on the 8 × 12 chessboard with four 8-cycles of the first copy of the 8 × 4 chessboard. Since the last four columns of the 8 × 12 chessboard is obtained from the 8 × 4 chessboard, we can use the algorithm in step 4.1 to obtain a closed knight's tour on the combined chessboards.

1	50	5	54	7	58	19	60	1	*s*	A	a
4	53	2	63	30	61	8	57	B	b	2	*t*
49	64	51	6	55	20	59	18	*z*	8	h	H
52	3	48	29	62	31	56	9	c	C	*u*	3
27	44	25	38	21	34	17	32	7	*y*	G	g
24	47	28	41	14	39	10	35	D	d	4	*v*
43	26	45	22	37	12	33	16	*x*	6	f	F
46	23	42	13	40	15	36	11	e	E	*w*	5

. . .

i^{th} copy				$(i+1)^{th}$ copy			
1	*s*	A	a	1	*s*	A	a
B	b	2	*t*	B	b	2	*t*
z	8	h	H	*z*	8	h	H
c	C	*u*	3	c	C	*u*	3
7	*y*	G	g	7	*y*	G	g
D	d	4	*v*	D	d	4	*v*
x	6	f	F	*x*	6	f	F
e	E	*w*	5	e	E	*w*	5

. . .

Figure 2.3 A closed knight's tour on the 8 × 4t chessboard, where $t \geq 4$

Note that (i) when Process 4 ends, we obtain a closed knight's tour on the $8 \times 4t$ chessboard, and (ii) if we anti-clockwise rotate the 8×4 chessboard, we obtain the 4×8 chessboard, and also obtain four 8-cycles.

Process 5: Construct a closed knight's tour on the 12×8 chessboard from a closed knight's tour of the 8×8 and four 8-cycles of the 4×8 chessboards by

- deleting edges 11-12, 36-37, 22-23, and 42-43 of C, and 3-4 of C_1, *t*-*u* of C_2, A-H of C_3, and f-g of C_4

- joining 11 with f, 12 with g, 36 with 4, 37 with 3, 22 with H, 23 with A, 43 with *t* and 42 with *u*, as shown in Figure 2.4.

	1	50	5	54	7	58	19	60
	4	53	2	63	30	61	8	57
	49	64	51	6	55	20	59	18
	52	3	48	29	62	31	56	9
8×8	27	44	25	38	21	34	17	32
	24	47	28	41	14	39	10	35
	43	26	45	22	37	12	33	16
	46	23	42	13	40	15	36	11
	a	*t*	H	3	g	*v*	F	5
4×8	A	2	h	*u*	G	4	f	*w*
	s	b	8	C	*y*	d	6	E
	1	B	*z*	c	7	D	*x*	e

Figure 2.4 A closed knight's tour on the 12×8 chessboard

Process 6: Construct four 4–cycles on the 4×4 chessboard, called D_1, D_2, D_3 and D_4 where D_1 : 1-2-3-4-1, D_2 : *s*-*t*-*u*-*v*-*s*, D_3 : A-B-C-D-A, and D_4 : a-b-c-d-a, as shown in Figure 2.5.

1	*s*	A	a
B	b	2	*t*
v	4	d	D
c	C	*u*	3

Figure 2.5 Four 4-cycles on the 4 × 4 chessboard

Process 7: Construct four 12-cycles on the 4 × 12 chessboard from four 8–cycles of the 4 × 8 chessboard and four 4-cycles of the 4 × 4 chessboards by

- deleting edges 5-6 of C_1, *w-x* of C_2, E-F of C_3 and e-f of C_4, and 1 – 4 of D_1, *s-v* of D_2, B-C of D_3, and b-c of D_4

- joining F with B, E with C, e with 4, f with 1, 5 with b, 6 with c, *w* with *s*, *x* with *v*, as shown in Figure 2.6

a	*t*	H	3	g	*v*	F	5	1	*s*	A	a
A	2	h	*u*	G	4	f	*w*	B	b	2	*t*
s	b	8	C	*y*	d	6	E	*v*	4	d	D
1	B	*z*	c	7	D	*x*	e	c	C	*u*	3

4 × 8 4 × 4

Figure 2.6 Four 12-cycles on the 4 × 12 chessboard

Process 8: The 4 × 4*t* chessboard, where $t \geq 4$, is obtained from the 4 × 12 chessboard and *t* – 3 copies of the 4 × 4 chessboards by putting the 4 × 12 chessboard on the left of *t* – 3 consecutive copies of the 4 × 4 chessboards. We do two steps to obtain the 4 × 4*t* chessboard.

8.1 Connect four 8-cycles of each of *t* – 3 consecutive copies of the 4 × 4 chessboards by

- for the i^{th} copy, deleting edges 2-3 of D_1, *t-u* of D_2, A-D of D_3, a-d of D_4,

- for the $(i + 1)^{th}$ copy, deleting edges 1 – 4 of D_1, *s-v* of D_2, B-C of D_3, b-c of D_4, and

- joining 2 of the i^{th} copy with 1 of the $(i + 1)^{th}$ copy, 3 of the i^{th} copy with 4 of the $(i + 1)^{th}$ copy, *t* of the i^{th} copy with *s* of the $(i + 1)^{th}$ copy, *u* of the i^{th} copy with *v* of the (i + 1)th copy, A of the ith copy with B of the (i + 1)th copy, D of the ith copy with C of the (i + 1)th copy, a of the ith copy with b of the (i + 1)th copy, d of the ith copy with c of the (i + 1)th copy, as shown in Figure 2.7.

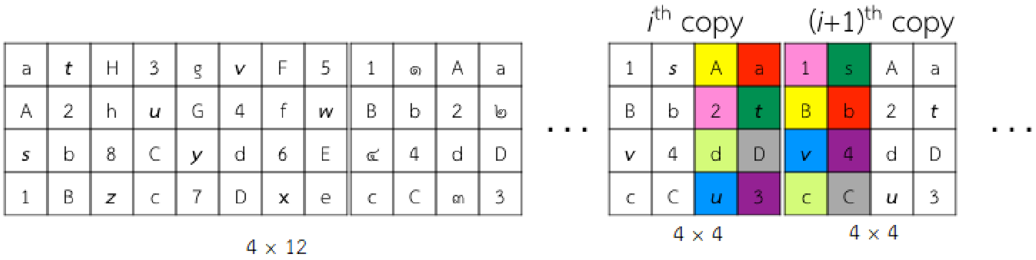

Figure 2.7 Four cycles on the 4 × 4t chessboard

Then, the 4 × 4(t – 3) chessboard is obtained from t – 3 copies of the 4 × 4 chessboard by joining each of t – 3 consecutive copies and it contains four cycles.

8.2 Connect four 12-cycles of the 4 × 12 chessboard with four 4-cycles of the first copy of the 4 × 4 chessboard. Since the last four columns of the 4 × 12 chessboard is obtained from the 4 × 4 chessboard, we can use the algorithm in step 8.1 to obtain four cycles on the combined chessboards.

Then, the 4 × 4t chessboard contains four cycles.

Process 9: Construct a closed knight's tour on the 12 × 4t chessboard, where $t \geq 2$. If $t = 2$, put the 8 × 8 chessboard on the top of the 4 × 8 chessboard. If $t = 3$, put the 8 × 12 chessboard on the top of the 4 × 12 chessboard. If $t \geq 4$, put the 8 × 4t chessboard (in Process 4) on the top of the 4 × 4t chessboard (in Process 8). Here, we note that the 8 × 12 chessboard and the 8 × 4t chessboard, where $t \geq 4$, are obtained from the 8 × 8 chessboard. In either case,

- delete edges 11-12, 36-37, 22-23 and 42-43 of the 8 × 8 chessboard and edges f-g, 3-4, A-H and t-u of the 4 × 8 chessboard, and

- join 11 with f, 12 with g, 36 with 4, 37 with 3, 22 with H, 23 with A, 43 with t and 42 with u, as shown in Figure 2.8.

Note that when Process 9 ends, we obtain a closed knight's tour on the 12 × 4*t* chessboard, where *t* ≥ 2.

	1	50	5	54	7	58	19	60	1	*s*	A	a		1	*s*	A	a	1	*s*	A	a	
	4	53	2	63	30	61	8	57	B	b	2	*t*		B	b	2	*t*	B	b	2	*t*	
	49	64	51	6	55	20	59	18	*z*	8	h	H		*z*	8	h	H	*z*	8	h	H	
	52	3	48	29	62	31	56	9	c	C	*u*	3	…	c	C	*u*	3	c	C	*u*	3	…
8 × 4*t*	27	44	25	38	21	34	17	32	7	*y*	G	g		7	*y*	G	g	7	*y*	G	g	
	24	47	28	41	14	39	10	35	D	d	4	*v*		D	d	4	*v*	D	d	4	*v*	
	43	26	45	22	37	12	33	16	*x*	6	f	F		*x*	6	f	F	*x*	6	f	F	
	46	23	42	13	40	15	36	11	e	E	*w*	5		e	E	*w*	5	e	E	*w*	5	
	a	*t*	H	3	g	*v*	F	5	1	s	A	a		1	*s*	A	a	1	*s*	A	a	
4 × 4*t*	A	2	h	*u*	G	4	f	*w*	B	b	2	t	…	B	b	2	*t*	B	b	2	*t*	…
	s	b	8	C	*y*	d	6	E	v	4	d	D		*v*	4	d	D	*v*	4	d	D	
	1	B	*z*	c	7	D	*x*	e	c	C	u	3		c	C	*u*	3	c	C	*u*	3	

Figure 2.8 A closed knight's tour on the 12 × 4*t* chessboard

Process 10: Construct a closed knight's tour on the 4*s* × 4*t* chessboard, when *s* ≥ 2. If *s* = 2, then the 8 × 4*t* chessboard is obtained from process 4. If *s* = 3, then the 12 × 4*t* chessboard is obtained from process 9. If *s* ≥ 4, then put the 12 × 4*t* chessboard on the top of *s*−3 consecutive copies of the 4 × 4*t* chessboards. We do two steps to obtain a closed knight's tour on the 4*s* × 4*t* chessboard.

10.1 Connect four cycles of each of *s*–3 consecutive copies of the 4 × 4*t* chessboards by

- for the ith copy, deleting edges 1 – 8 of C_1, *s*-*z* of C_2, B-C of C_3, b-c of C_4,
- for the (i + 1)th copy, deleting edges 2 – 3 of C_1, *t*-*u* of C_2, A-H of C_3, a-h of C_4, and
- joining 1 of the ith copy with 2 of the (i + 1)th copy, 8 of the ith copy with 3 of the (i + 1)th copy, *s* of the ith copy with *t* of the (i + 1)th copy, *z* of the ith copy with*u* of the (i + 1)th copy, B of the ith copy with A of the (i + 1)th copy, C of the ith copy with H

of the (i + 1)th copy, b of the ith copy with a of the (i + 1)th copy, c of the ith copy with h of the (i + 1)th copy, as shown in Figure 2.9.

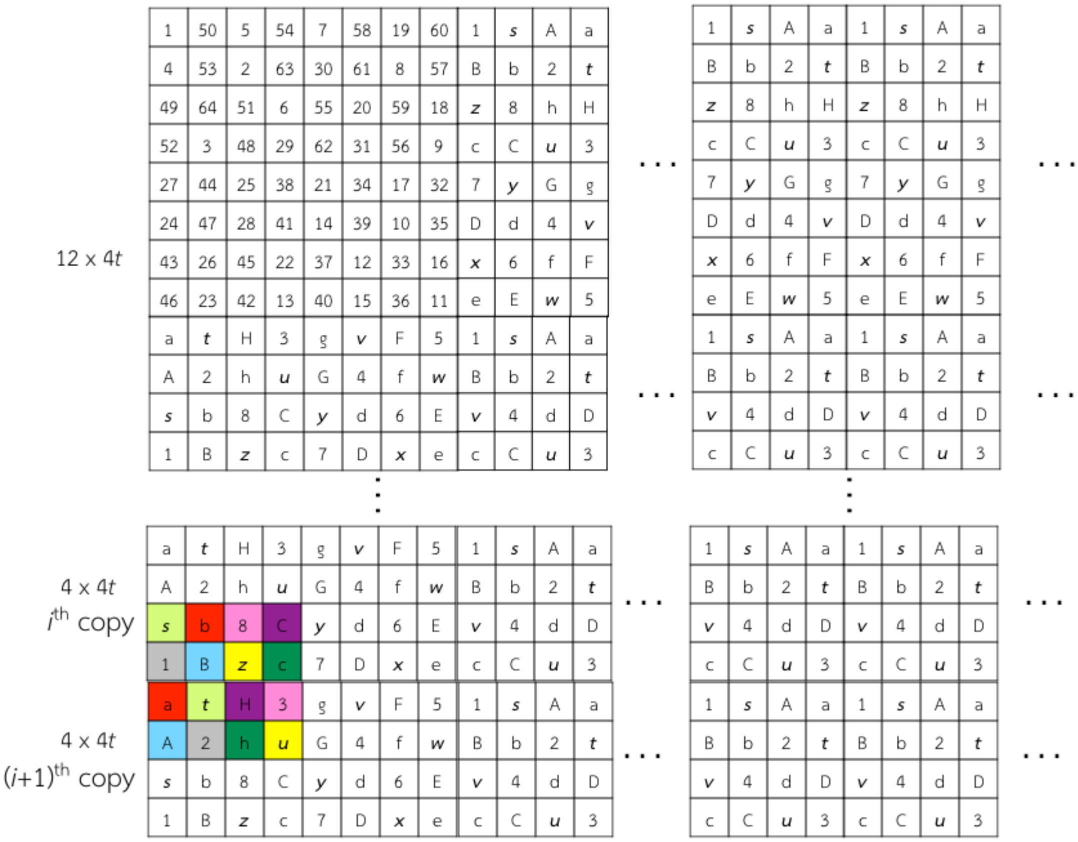

Figure 2.9 A closed knight's tour on the $4s \times 4t$ chessboard, when $s, t \geq 2$

10.2 Connect the closed knight's tour on the $12 \times 4t$ chessboard with four cycles of the first copy of the $4 \times 4t$ chessboard. Since the first eight columns of the $4 \times 4t$ chessboard is obtained from the 4×8 chessboard, we can use the algorithm in step 10.1 to obtain a closed knight's tour on the combined chessboards.

2.2 The Construction of a Closed (1, 2, 2)-Knight's Tour of the $3r \times 4s \times 4t$ Chessboard, where $r \geq 1$ and $s, t \geq 2$

First, we use Algorithm 2.1 to construct a closed (1, 2, 2)-knight's tour on the $3 \times 4s \times 4t$ chessboard by using a closed knight's tour in Section 2.1. In Figure 2.10, the closed knight's tours *C* from Section 2.1 are shown in the first and the third levels of the $3 \times 8 \times 8$ chessboard. Thus, we may say that small cubes of the $3 \times 8 \times 8$ chessboard are named by numbers of the closed knight's tour. In the second level, the number with a circle, x, is the position (obtained by Algorithm 2.1(iii)) that the knight can move to cube x in the first level or cube x in the third level by an (1, 2, 2)-knight's move. Also, by Algorithm 2.1 (iii), for each number x in the first level we obtain a path P_x : x-x-x-$(x+1)$, where the first x and $(x+1)$ in P_x are in the first level and x after x in P_x is in the third level. Connect each path P_x, we obtain a Hamiltonian cycle or a closed (1, 2, 2)-knight's tour C^*. In Figure 2.10, a closed (1, 2, 2)-knight's tour C^* is

1-1-1- 2-2-2-3-3-3- ... -64-64-64-1.

Next, we extend the $3 \times 4s \times 4t$ chessboard, where s, $t \geq 2$, to the $3r \times 4s \times 4t$ chessboard, where $r > 1$ and s, $t \geq 2$, we use r copies of the closed (1, 2, 2)-knight's tour C^* of the $3 \times 4s \times 4t$ chessboard by putting each copy from the top to the bottom. We do the following.

- Delete edge (2, 3, 3)-(3, 1, 1) of the i^{th} copy and edge (1, 4, 1)-(2, 2, 3) of the $(i+1)^{th}$ copy for all $i = 1, 2, 3, ..., r-1$.

- Join (2, 3, 3) of the i^{th} copy with (1, 4, 1) of the $(i + 1)^{th}$ copy and (3, 1, 1) of the i^{th} copy with (2, 2, 3) of the $(i+1)^{th}$ copy for all $i = 1, 2, 3, ..., r - 1$.

In Figure 2.10, edges 1-1 of the first copy and 52-52 of the second copy are deleted and then join 1 with 52 and 1 with 52.

Then, we obtain a closed (1, 2, 2)-knight's tour on the $3r \times 4s \times 4t$ chessboard where $r \geq 1$ and s, $t \geq 2$ in the following theorem.

Theorem 2.1 *Suppose* $r \geq 1$ *and* $s, t \geq 2$. *Then, the* $3r \times 4s \times 4t$ *chessboard admits a closed* (1, 2, 2)-*knight's tour.*

3. Conclusion

This paper shows a closed (1, 2, 2)-knight's tour on the $3r \times 4s \times 4t$ chessboard, where $r \geq 1$ and $s, t \geq 2$. This research extends the result of Bai et. al. [1] from $r = 1$ to $r > 1$. However, this is a special case of the $k \times m \times n$ chessboard where k, m and n are positive integers. The remaining cases are open problems. Moreover, we shall mention that the condition $k \leq m \leq n$ is quoted for the existence of a closed (1, 2, 2)-knight's tour on the $k \times m \times n$ chessboard in Theorem 1.3, while our result in Theorem 2.1 does not give the relation between r, s and t. There are no problems because of our construction.

	The 1st copy								The 2nd copy							
The 1st level	1	50	5	54	7	58	19	60	1	50	5	54	7	58	19	60
	4	53	2	63	30	61	8	57	4	53	2	63	30	61	8	57
	49	64	51	6	55	20	59	18	49	64	51	6	55	20	59	18
	52	3	48	29	62	31	56	9	52	3	48	29	62	31	56	9
	27	44	25	38	21	34	17	32	27	44	25	38	21	34	17	32
	24	47	28	41	14	39	10	35	24	47	28	41	14	39	10	35
	43	26	45	22	37	12	33	16	43	26	45	22	37	12	33	16
	46	23	42	13	40	15	36	11	46	23	42	13	40	15	36	11
The 2nd level	51	6	49	64	59	18	55	20	51	6	49	64	59	18	55	20
	48	29	52	3	56	9	62	31	48	29	52	3	56	9	62	31
	5	54	1	50	19	60	7	58	5	54	1	50	19	60	7	58
	2	63	4	53	8	57	30	61	2	63	4	53	8	57	30	61
	45	22	43	26	33	16	37	12	45	22	43	26	33	16	37	12
	42	13	46	23	36	11	40	15	42	13	46	23	36	11	40	15
	25	38	27	44	17	32	21	34	25	38	27	44	17	32	21	34
	28	41	24	47	10	35	14	39	28	41	24	47	10	35	14	39
The 3rd level	1	50	5	54	7	58	19	60	1	50	5	54	7	58	19	60
	4	53	2	63	30	61	8	57	4	53	2	63	30	61	8	57
	49	64	51	6	55	20	59	18	49	64	51	6	55	20	59	18
	52	3	48	29	62	31	56	9	52	3	48	29	62	31	56	9
	27	44	25	38	21	34	17	32	27	44	25	38	21	34	17	32
	24	47	28	41	14	39	10	35	24	47	28	41	14	39	10	35
	43	26	45	22	37	12	33	16	43	26	45	22	37	12	33	16
	46	23	42	13	40	15	36	11	46	23	42	13	40	15	36	11

Figure 2.10 A closed (1, 2, 2)-knight's tour on the $6 \times 8 \times 8$ chessboard

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