วารสารคณิตศาสตร์ MJ-MATh 65(700) มกราคม – เมษายน 2563
โดย สมาคมคณิตศาสตร์แห่งประเทศไทย ในพระบรมราชูปถัมภ์
http://www.mathassociation.net Email: MathThaiOrg@gmail.com

การส่งไดกราฟกำลังสามของริงสลับที่จำกัด

The Cubic Mapping Digraphs of Finite Commutative Rings

อิทธิวัฒน์ โตเจริญนิรัติศัย
สาขาวิชาคณิตศาสตร์ คณะครุศาสตร์ มหาวิทยาลัยราชภัฏธนบุรี กรุงเทพ 10600

Ittiwat Tocharoenirattisai
Division of Mathematics, Faculty of Education, Dhonburi Rajaphat Univercity, Bangkok 10600

Email: mathglass@hotmail.com

วันที่รับบทความ : 5 กันยายน 2562 วันที่แก้ไขบทความ : 21 ตุลาคม 2562 วันที่ตอบรับบทความ : 12 ธันวาคม 2562

บทคัดย่อ

ให้ R เป็นริงสลับที่จำกัดพร้อมเอกลักษณ์ เราศึกษาการส่งไดกราฟกำลังสาม $G(R)$ ซึ่งใช้จุดยอดใน R และมีทิศทางจาก a ไป b ก็ต่อเมื่อ $a^3 = b$ เราได้ศึกษาโครงสร้างของไดกราฟและได้ทฤษฎีที่เกี่ยวข้องกับจุดตรึง วัฏจักรความยาวที และความเป็นเซมิเรกูล่า นอกเหนือจากนั้นเราได้ศึกษาโครงสร้างของไดกราฟสำหรับสองและสามคอมโพเน็น

คำสำคัญ: ริงเฉพาะที่ วัฏจักรความยาวที เซมิเรกูล่า

ABSTRACT

Let R be a finite commutative ring with identity. We study the cubic mapping digraphs $G(R)$ whose vertex set is R and there is a directed edge from a to b if and only if $a^3 = b$. We investigate the structure of digraph and establish theorems about

fixed points, $t-$cycles and semiregularity. In addition, we work on the structure of digraphs for 2 and 3 components.

Keywords: Local rings, $t-$cycles, Semiregular

1. Introduction

A local ring is a commutative ring with identity which has a unique maximal ideal. The exponent of a finite group G, denoted by $\exp G$ is the least positive integer n such that $g^n = e$ for all $g \in G$. It is easy to see that $\exp G$ divides $|G|$. In particular, $\exp G = \text{lcm}\{o(a) : a \in G\}$, where $o(a)$ is the order of a in G . Moreover, if $G = G_1 \cup G_2$, then $\exp G = \text{lcm}\{\exp G_1, \exp G_2\}$. The exponent of a finite commutative ring R with identity is defined to be the exponent of the unit group $R^\times$ of R. We write $\lambda(R)$ for the exponent of R, that is, $\lambda(R) = \exp R^\times$. The order of k modulo d is the least positive integer t such that $d \mid k^t - 1$, denoted by $\text{ord}_d\, k$.

This research is motivated by Tocharoenirattisai and Meemark [4] who used $\lambda(R) = \exp R^\times$ to study the digraph $G^{(k)}(R)$, where R is a local extension of the Galois ring and Su, Tang and Wei [6] who studied the square mapping graphs of finite commutative rings.

Now, we study the cubic mapping digraphs over the finite commutative ring with identity and recall some definitions of structure of digraphs. Let R be a finite commutative ring with identity 1. For $k \geq 2$, the k^{th} power mapping digraph over R, denoted by $G^{(k)}(R)$, is the graph whose vertex set is R and there is a directed edge from a to b if and only if $a^k = b$.

A component of a digraph is a subdigraph which is a maximal connected subgraph of the associated nondirected graph. We consider two disjoint subdigraphs $G_1^{(k)}(R)$ and $G_2^{(k)}(R)$ of $G^{(k)}(R)$ induced on the set of vertices which are in the unit group $R^\times$ and induced on the remaining vertices which are not invertible, respectively. They are called the unit subdigraph and the zero divisor subdigraph, respectively. Observe that there are no edges between $G_1^{(k)}(R)$ and $G_2^{(k)}(R)$, that is, $G^{(k)}(R) = G_1^{(k)}(R) \dot{\cup} G_2^{(k)}(R)$.

A cycle of length $t \geq 1$ is said to be a $t-$cycle and we assume that all cycles are oriented counterclockwise. We call a cycle of length one a fixed point. The distance from a vertex $g \in R$ to a cycle is the length of the directed path from g to a vertex in the cycle.

The indegree (respectively, outdegree) of a vertex $a \in R$ of $G^{(k)}(R)$, is the number of directed edges entering (respectively, leaving) to a, and denoted by $\boldsymbol{indeg}^{(k)}\boldsymbol{a}$ (respectively, $\boldsymbol{outdeg}^{(k)}\boldsymbol{a}$). The definition of $G^{(k)}(R)$ implies that the outdegree of each vertex is equal to 1. For any fixed point such that its indegree is equal to 1, it is called an isolated fixed point. Moreover, the definition of $G^{(k)}(R)$ implies that each component of the digraph $G^{(k)}(R)$ has exactly one cycle, that is the number of components and the number of its cycles are equal.

For convenience, we use the notation $G(R)$ for $G^{(3)}(R)$. In this paper, we discover some theorems about fixed points in $G(R)$, existence of $t-$cycles in $G^{(k)}(R)$ which better [2] and the number of $t-$cycles in Section 2. The semiregularity is investigated in Section 3. Finally, we establish some theorems on the digraph $G(R)$ for 2 and 3 components in the last section.

2. The Structure of $G(R)$

In this section, we study the fixed points, $t-$cycles and the number of $t-$cycles which are the important structures of $G(R)$.

Let R be a finite commutative ring with identity. It is well-known that R is a finite direct product of finite local rings, that is,

$$R \cong R_1 \times R_2 \times \cdots \times R_s, \tag{1}$$

for some $s \in \mathbb{Z}^+$, where R_i is a finite local ring for any $1 \leq i \leq s$.

The following theorem on finite local rings is needed for this research.

Theorem 2.1. (Theorem 6.1.2 in [1]) Let R be a finite local ring with the unique maximal ideal M. Then $|R| = p^{nr}$, $|M| = p^{(n-1)r}$, $M^n = \{0\}$ and $char\, R = p^k$ for some prime p and some positive integers n, r, k $(1 \le k \le n)$.

Lemma 2.2. Let R be as in (1). Then the following statements hold.

(1) There are 3^s fixed points in $G(R)$.

(2) If $s = 1$ then $G_2(R)$ has the tree attached to 0.

(3) $G_2(R)$ has exactly one component if and only if $s = 1$.

(4) Let $a, b \in R^\times$. If a and b are in the same cycle in $G_1(R)$, then $o(a) = o(b)$.

Proof. (1) For any fixed point a in $G(R_i)$, $a^3 = a$, so $a = 0, 1, -1$. Then $G(R_i)$ has 3 fixed points. Hence $G(R)$ has 3^s fixed points.

(2) For $s = 1$, we have R is a finite local ring with unique maximal ideal, so we are done.

(3) Assume that $G_2(R)$ has exactly one component. Since $0^3 = 0$, so any vertex in $G_2(R)$ is attached to the 0. Consider the set $M = \left\{ a \in R \setminus R^\times \;\middle|\; \exists n \in \mathbb{N}, a^{3^n} = 0 \right\}$. It is not difficult to show that M is an ideal of R. This means that the set of all nonunit of R forms an ideal. This implies that R is a local ring, that is $s = 1$. The converse is clear by (2).

(4) Let $a, b \in R^\times$ and let $o(a) = l, o(b) = l'$. Without loss of generality, we let $l' \le l$. Since a and b are in the same cycle, so $b^{3^j} = a$ for some $j \ge 1$. Since $1 = (b^{3^j})^{l'} = a^{l'}$, so $l \mid l'$. This forces that $l = l'$. □

Theorem 2.3. Let R be as in (1). The element 0 is an isolated fixed point in $G_2(R)$ if and only if R_i is a field.

Proof. Suppose that $s = 1$. If R is a field, then it is clear that 0 is an isolated fixed point. Conversely, by Lemma 2.2(2), the unique maximal ideal of R is $\{0\}$. Thus R is a field. Now assume that $s > 1$. Suppose that 0 is an isolated fixed point. If R_i is not a field for some i, then there is $0 \ne \alpha_i \in R_i$ such that $\alpha_i^3 = 0$.

Let $\alpha=(0,\ldots,\alpha_i,\ldots,0)\in R$. Then $\alpha^3=0$ which is a contradiction since 0 is an isolated fixed point. Hence, each R_i is a field for all i. Conversely, if every R_i is a field, then it is clear that 0 is an isolated fixed point. □

Lemma 2.4. Let R be as in (1). If a nonidentity b is a vertex of a $t-$cycle $(t>1)$ in $G_1(R)$, then $\gcd\big(o(b),3\big)=1$.

Proof. Let a nonidentity b be a vertex of a $t-$cycle $(t>1)$ in $G_1(R)$. Then t is the least positive integer such that $b^{3^t}=b$. Since $b\in R^\times$, $b^{3^t-1}=1$. Thus $o(b)\mid 3^t-1$ and $\gcd\big(o(b),3\big)=1$. □

Lemma 2.5. Let R be as in (1). If a nonidentity $b\in R^\times$, $o(b)\neq 2$ and $\gcd\big(o(b),3\big)=1$, then b is a vertex of a $t-$cycle $(t>1)$ in $G_1(R)$.

Proof. Let $o(b)=l\neq 2$ and $\gcd\big(l,3\big)=1$. Then there is the least positive integer t such that $l\,|\,3^t-1$. Since $l\neq 2$, we have $t\neq 1$, that is, $t>1$. Says, $lk=3^t-1$. Then $b^{lk}=b^{3^t-1}=1$, that is, t is the least positive integer such that $b^{3^t}=b$. Thus b is a vertex of a $t-$cycle $(t>1)$ in $G_1(R)$. □

Next, we display the existence of a $t-$cycle with $t\geq 1$ in $G^{(k)}(R)$, such that these results are better [6] which displayed a $t-$cycle on $G_1^{(k)}(R)$. For a finite commutative ring R with identity, we set $\lambda(R)=uv$, where u is the largest divisor of $\lambda(R)$ relatively prime to k.

Theorem 2.6. Let R be as in (1). Let t be a positive integer, and $k\geq 2$.
The following statements are equivalent.

(1) There exists a $t-$cycle in $G^{(k)}(R)$.

(2) There exists $b\in R^\times$ where t is the least positive integer such that $o(b)\mid k^t-1$.

(3) $t=ord_d\,k$ for some divisor d of u.

Proof. (1) $\Rightarrow$ (2). Let a be a vertex of a $t-$cycle. Then t is the least positive integer such that $a^{k^t}=a$, so $a\left(a^{k^t-1}-1\right)=0$. If $a\in R^\times$, then $a^{k^t-1}-1=0$. Thus t is the least positive integer such that $a^{k^t-1}=1$, and in this case, we are done by setting $b=a$. Next,

suppose that $a \notin R^{\times}$. Let $A=(a),\ B=Ann(a)$, the annihilator of a in R. Then $AB=\{0\}$ and $a^{k^t-1}-1 \in B$. Since $a^{k^t-1}-\left(a^{k^t-1}-1\right)=1$, so $A+B=R$ and then $A \cap B = AB = \{0\}$. Define the ring isomorphism φ by $\varphi: R \to R/A \times R/B$ such that $\varphi(r)=(r+A,\ r+B)$ for any $r \in R$. Taking $b=1+a-a^{k^t-1}$. Then $\varphi(b)=(1+A,\ a+B)$
.

Thus

$$\varphi(b^{k^t-1})=(1+A,\ a^{k^t-1}+B)=(1+A,\ 1+B).$$

Since φ is a ring isomorphism, $b^{k^t-1}=1$, that is, t is the least positive integer such that $b^{k^t-1}=1$. Hence, we have (2) as required.

(2) $\Rightarrow$ (3). Suppose there exists $b \in R^{\times}$ such that $o(b) \mid k^t-1$, but $o(b) \nmid k^l-1$, for all $1 \le l < t$. Then t is the least positive integer such that $b^{k^t-1}=1$, and $\gcd(o(b),\ k)=1$, so $o(b) \mid u$. Set $d=o(b)$. Thus $t=ord_d\, k$ for some divisor d of u.

(3) $\Rightarrow$ (1). Suppose $t=ord_d\, k$ for some divisor d of u. Since $R^{\times}$ is abelian, there exists $a \in R^{\times}$ such that $o(a)=\lambda(R)$. Set $b=a^{\frac{\lambda(R)}{d}}$. Since $t=ord_d\, k$, t is the least positive integer such that $b^{k^t-1}=a^{\frac{(k^t-1)\lambda(R)}{d}}=1$. This means that $b^{k^t-1}=b$. Therefore, there exists a $t-$cycle in $G^{(k)}(R)$, where $t \ge 1$. □

For the number of $t-$cycles in $G^{(k)}(R)$ is denoted by $A_t\left(G^{(k)}(R)\right)$. Next, we investigate the number of $t-$cycles in $G(R)$.

Theorem 2.7. Let R be as in (1). Then the following statements hold.

(1) $A_1\left(G_1(R)\right)=2^s$ and $A_1\left(G_2(R)\right)=3^s-2^s$.

(2) For $t>1$, if $A_t\left(G_2(R)\right)\ge 1$, then $A_t\left(G_1(R)\right)\ge 1$.

Proof. Apply Lemma 2.2(1), we have that (1) is done.

Let $t>1$. Suppose that $A_t\left(G_2(R)\right)\ge 1$. Let a be a vertex of a $t-$cycle in $G_2(R)$. Then we set $b=1+a-a^{3^t-1}$. By the proof of Theorem 2.6, $b \in R^{\times}$ and t is the least positive integer such that $b^{3^t-1}=1$, that is, $b^{3^t}=b$, so b is a vertex of a $t-$cycle in $G_1(R)$. Hence $A_t\left(G_1(R)\right)\geq 1$. □

Lemma 2.8. Let R be as in (1) and q be an odd prime. Then $|R^\times|=q^m$ $(m\geq 0)$ if and only if R is a direct product of the rings where every direct product is isomorphic to F_{q^m+1}.

Proof. First, suppose that R is local. By Theorem 2.1, $|R^\times|=q^{(n-1)r}(p^r-1)=q^m$.
If $p>2$, then $p^r-1=1$, this forces that $p=2$ which is a contradiction. Thus $p=2$. Then $(n-1)r=0$, that is, $n=1$ and $p^r-1=q^m$. Hence $R\cong F_{q^m+1}$. The converse is clear. Secondary, we suppose that R is not local. Since $|R^\times|=q^m$, so $|R_i^\times|=q^{m_i}$ for all $1\leq i\leq s$. Then by above, the proof is completed. Again, the converse is clear. □

Theorem 2.9. Let R be as in (1).

(1) If $A_t\left(G_1(R)\right)=0$ for any $t>1$, then $|R^\times|=2^l3^m$ $(l,m\geq 0)$.

(2) If $|R^\times|=3^m$ $(m\geq 0)$, then $A_t\left(G_1(R)\right)=0$ for any $t>1$.

(3) If $A_t\left(G_2(R)\right)=0$ for any $t>1$, then $|R^\times|=2^l3^m$ $(l,m\geq 0)$ or R is local.

(4) If $|R^\times|=3^m$ $(m\geq 0)$ or R is local, then $A_t\left(G_2(R)\right)=0$ for any $t>1$.

Proof. (1) Suppose that $A_t\left(G_1(R)\right)=0$ for any $t>1$. Assume that there is a prime $p>3$ such that $p\,\big|\,|R^\times|$. Since $R^\times$ is abelian, there is an element $b\in R^\times$ such that $o(b)=p$. And since $p\neq 2$ and $\gcd(p,3)=1$, by Lemma 2.5, $A_t\left(G_1(R)\right)\neq 0$ $(t>1)$. This is a contradiction. Hence $|R^\times|=2^l3^m$ $(l,m\geq 0)$.

(2) Suppose that $|R^\times|=3^m$ $(m\geq 0)$. Then any nonidentity element $g\in R^\times$, $o(g)=3^i$ for some $1\leq i\leq m$. Since $\gcd(3^i,3^t-1)=1$ for any $t>1$, by Theorem 2.6, $A_t\left(G_1(R)\right)=0$ for any $t>1$.

(3) Suppose that $A_t\left(G_2(R)\right)=0$ for any $t>1$. Assume that R is not local. If there is a prime $p>3$ such that $p\,\big|\,|R^\times|$, then since $R^\times$ is abelian, there is an element $g\in R^\times$ such that $o(g)=p$. Since $\gcd(p,3)=1$, there is the least positive integer t

such that $p \mid 3^t - 1$, says $pk = 3^t - 1$. Let $\alpha = (g, 0, 0, \ldots, 0)$. Then $0 \neq \alpha \notin R^\times$, that is, α is a zero divisor of R and $\alpha^{3^t - 1} = \alpha^{pk} = (g^{pk}, 0, 0, \ldots, 0) = (1, 0, 0, \ldots, 0)$. Thus $\alpha^{3^t} = \alpha$, that is, α is a vertex of a $t-$cycle in $G_2(R)$ which is a contradiction. Hence $|R^\times| = 2^l 3^m$ $(l, m \geq 0)$. For another way, if we assume that $|R^\times| \neq 2^l 3^m$ for any $l, m \geq 0$, then there is a prime $p > 3$ such that $p \mid |R^\times|$. Above discussion forces that R is local.

(4) Suppose that $|R^\times| = 3^m$ $(m \geq 0)$. By (2), $A_t\left(G_1(R)\right) = 0$ for any $t > 1$. If $t > 1$, $A_t\left(G_2(R)\right) \geq 1$, then by Theorem 2.7(2), $A_t\left(G_1(R)\right) \geq 1$ which is a contradiction. Thus $A_t\left(G_2(R)\right) = 0$ for any $t > 1$. Another one for local ring R is obvious. □

3. Semiregularity of $G^{(k)}(R)$

In this section, we study the semiregularity of $G^{(k)}(R)$, where R is a finite commutative ring with identity as in (1). Now, we recall the definition of semiregular. The digraph $G^{(k)}(R)$ is called semiregular if there is a positive integer d such that for each vertex $g \in R$, $\operatorname{indeg}^{(k)} g = 0$ or $\operatorname{indeg}^{(k)} g = d$.

Theorem 3.1. Let R be as in (1). Then $G_1^{(k)}(R)$ is semiregular.

Proof. We want to show that for any $g \in R^\times$, if $\operatorname{indeg}^{(k)} g \neq 0$, then $\operatorname{indeg}^{(k)} g = \operatorname{indeg}^{(k)} 1$. Suppose that $\operatorname{indeg}^{(k)} g \neq 0$. Then there is $h \in R^\times$ such that $h^k = g$. Since, for any $s \in R^\times$, $s^k = g$ if and only if $(h^{-1}s)^k = 1$. Thus $\operatorname{indeg}^{(k)} g = \operatorname{indeg}^{(k)} 1$. □

It is clear to see that the digraph $G_1(R)$ is semiregular by taking $k = 3$ in Theorem 3.1 which is shown below.

Corollary 3.2. Let R be as in (1). Then $G_1(R)$ is semiregular.

Theorem 3.3. Let R be a finite local ring with unique maximal ideal M. If $M^k = 0$, then $G_2^{(k)}(R)$ is semiregular.

Proof. Suppose that $M^k = 0$. Then it is clear that any element in M is a vertex in $G_2^{(k)}(R)$ attaching to a fixed point 0. Thus $G_2^{(k)}(R)$ is semiregular. □

Again, it is clear that the digraph $G_2(R)$ is semiregular by taking $k = 3$ in Theorem 3.3 which is shown below.

Corollary 3.4. Let R be a finite local ring with the unique maximal ideal M. If $M^3 = 0$, then $G_2(R)$ is semiregular.

4. The Digraphs $G(R)$ with Two and Three Components

In last section, we work on the structure of digraphs $G(R)$ with two and three components. First of all, we will talk about some theorems on the digraph over a finite cyclic group in [3].

Lemma 4.1. Let C_n be a finite cyclic group of order n and $k \geq 2$.

(1) If $\gcd(n, k) = 1$, then $G^{(k)}(C_n) = \bigcup_{d|n} \underbrace{\left(\sigma(ord_d\, k) \cup \cdots \cup \sigma(ord_d\, k)\right)}_{\frac{\varphi(d)}{ord_d\, k}\text{terms}}$,

where $\sigma(l)$ is an $l-$cycle and φ is the Euler $\varphi-$function.

(2) If $\gcd(n, 3) = 1$, then $A_t\left(G(C_n)\right) = \sum_{d|n} \frac{\varphi(d)}{ord_d\, 3}$ for $t \geq 1$ and

$A_t\left(G(C_n)\right) = \sum_{\substack{d|n \\ d\neq 1,2}} \frac{\varphi(d)}{ord_d\, 3}$ for $t > 1$.

Proof. The proof of (1) can be seen in Proposition 4.2 (1) in [3].

For (2), we apply (1) by taking $k = 3$. Then we have that $A_t\left(G(C_n)\right) = \sum_{d|n} \frac{\varphi(d)}{ord_d\, 3}$ for $t \geq 1$. Since $ord_d\, 3 = 1$ if and only if $d = 1, 2$, so $A_t\left(G(C_n)\right) = \sum_{\substack{d|n \\ d\neq 1,2}} \frac{\varphi(d)}{ord_d\, 3}$ for $t > 1$. □

Theorem 4.2. Let R be as in (1). If $G(R)$ has exactly two components, then R is local, $char\, R = 2$ and $|R^\times| = 2^l 3^m$ $(l, m \geq 0)$.

Proof. Suppose that $G(R)$ has exactly two components. Since $0, 1, -1$ are fixed points of $G(R)$, if follows that $A_t\left(G(R)\right) = 0$ for any $t > 1$ and $l = -1$, that is,

$char\,R = 2$. Since $A_t\left(G_2(R)\right) = 0$ for any $t > 1$, by Lemma 2.2(3) and by Theorem 2.9(3), R is local and $|R^\times| = 2^l 3^m$ $(l, m \ge 0)$. □

Theorem 4.3. Let R be as in (1). If R is local, $char\,R = 2$ and $|R^\times| = 3^m$ $(m \ge 0)$, then $G(R)$ has exactly two components.

Proof. It is obvious by Theorem 2.9. □

Theorem 4.4. Let R be as in (1). If $G(R)$ has exactly three components, then R satisfies one of the following statements.

(1) R is local, $char\,R \ne 2$ and $|R^\times| = 2^l 3^m$ $(l, m \ge 0)$.

(2) R is local, $char\,R = 2$ and $|R| = 2^n = 2|M|$.

(3) $R \cong F_{q+1}$, where q is a prime that $q > 3$, $char\,R = 2$ and 3 is a primitive root modulo q.

(4) R is local, $char\,R = 2$ with $|R^\times| = 2^i 3^m q^j$ $(m \ne 0,\, i, j \ge 1)$ or $(i = 0,\, m, j \ge 1)$ or $(j = 0,\ i, m \ge 1)$.

Proof. Suppose that $G(R)$ has exactly three components. Since $0, 1, -1$ are fixed points of $G(R)$, if $char\,R \ne 2$, then $1 \ne -1$, we have $A_t\left(G(R)\right) = 0$ for any $t > 1$. Thus R is local and by Theorem 2.9, we have $|R^\times| = 2^l 3^m$ $(l, m \ge 0)$. Now, we consider if $char\,R = 2$, then $1 = -1$. For $s > 1$, we have that $G(R)$ contains 2^s fixed points, that is, $G(R)$ contains 2^s components such that $2^s > 3$ which is a contradiction. Thus $s = 1$, that is, R is local and $G(R)$ contains only two fixed points and only one $t-$cycle for some $t > 1$ in $G_1(R)$. By Lemma 2.5, there is a nonidentity $b \in R^\times$, $o(b) \ne 2$, and $\gcd(o(b), 3) = 1$. Now we will investigate the order of $|R^\times|$. Suppose that there are distinct primes $p, q > 3$ such that $p, q \,\big|\, |R^\times|$. Since $|R^\times|$ is abelian, there are elements $b, c \in R^\times$ such that $o(b) = p$ and $o(c) = q$. By Lemma 2.2(4), b, c are in different $t-$ cycles for some $t > 1$ in $G_1(R)$ which is a contradiction. Thus no distinct primes $p, q > 3$ such that $p, q \,\big|\, |R^\times|$. By Lemma 2.4, it is clear that $|R^\times| \ne 3^m$ for any $m \ge 0$. Now, we consider the other cases of $|R^\times|$.

Case 1 If $|R^{\times}| = 2^{i+1}$ $(i \geq 1)$, then since $|R^{\times}| = 2^{(n-1)r}(2^r - 1) = 2^{i+1}$, it follows that $r = 1$ and $|R^{\times}| = 2^{(n-1)} = |M|$. Hence $|R| = 2^n = 2|M|$.

Case 2 If $|R^{\times}| = q^j$ $(j \geq 1)$, where q is a prime that $q > 3$, then since $|R^{\times}| = 2^{(n-1)r}(2^r - 1) = q^j$, we have $(n-1)r = 0$, that is, $n = 1$ and $2^r - 1 = q^j$. Since $A_t\left(G_1(R)\right) = 0$ $(t > 1)$ and $R^{\times} \cong F^{\times}_{q^j+1} = C_{q^j}$, by Lemma 4.1 (2), $1 = A_t\left(G(C_{q^j})\right)$ $= \sum_{\substack{d|q^j \\ d \neq 1,2}} \frac{\varphi(d)}{ord_d 3}$. This forces that $j = 1,\ d = q$, and $\varphi(d) = ord_d 3$, that is, 3 is a primitive root modulo q. Thus $R \cong F_{q+1}$.

Case 3 $|R^{\times}| = 2^i q^j$ $(i, j \geq 1)$. Since $|R^{\times}| = 2^{(n-1)r}(2^r - 1) = 2^i q^j$, says $R^{\times} \cong Q_1 \times Q_2$, where Q_1 is an abelian of order 2^i and Q_2 is a cyclic of order q^j. Since $q \,\big|\, |R^{\times}|$, there is $b \in R^{\times}$ such that $o(b) = q$. By Lemma 2.5, b is a vertex of a $t-$cycle $(t > 1)$ in $G_1(R)$. If there is $c \in R^{\times}$ such that $o(c) = 2^l$ $(l \geq 2)$, then by Lemma 2.5, c is a vertex of a $t-$cycle $(t > 1)$ in $G_1(R)$. Since $o(b) \neq o(c)$, by Lemma 2.2(4), b, c are in different $t-$cycles $(t > 1)$ in $G_1(R)$ which is a contradiction. If there is $c \in R^{\times}$ such that $o(c) = q^l$ $(l \geq 2)$ or $o(c) = 2^i q^j$ $(i, j \geq 1)$, then we have a contradiction again by above discussion. Thus $|R^{\times}| \neq 2^i q^j$ for any $i, j \geq 1$. Hence, we have $|R^{\times}| = 2^i 3^m q^j$ $(m \neq 0,\ i, j \geq 1)$ or $(i = 0,\ m, j \geq 1)$ or $(j = 0,\ i, m \geq 1)$. □

Theorem 4.5. Let R be as in (1). If $R \cong F_{q+1}$, where q is a prime that $q > 3$, $char\, R = 2$, and 3 is a primitive root modulo q, then $G(R)$ has exactly three components.

Proof. It is immediate from Lemma 2.5 and Lemma 4.1(2) □

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References

[1] Bini, G., and Flamini, F. (2002). *Finite Commutative Rings and Their Applications*. New York: Kluwer Accademic Publishers.

[2] Su, HD. Tang, GH., and Wei, YJ. (2012). The Square Mapping Graphs of Finite Commutative rings. *Algebra Colloquium*, 19, p. 569 - 580.

[3] Tang, GH., and Wei, YJ. (2015). The Iteration Digraphs of Finite Commutative Rings. *Turkish Journal of Mathematics*, 39, p. 872 - 883.

[4] Tocharoenirattisai, I., & Meemark, Y. (2017). Exponent of Local Ring Extensions of Galois Rings and Digraphs of the k^{th} Power Mapping. *Turkish Journal of Mathematics*, 41, p. 223 - 234.