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http://www.mathassociation.net Email: MathThaiOrg@gmail.com

ลำดับโมเบียส

Mobius Sequences

สมพงษ์ ฉุยสุริฉาย[1,*] และ ลลิตา อภิสรพาณิชย์[2]

[1]หน่วยวิจัยพีชคณิตและการประยุต์ สาขาวิทยาศาสตร์การคำนวณ คณะวิทยาศาสตร์

มหาวิทยาลัยสงขลานครินทร์ สงขลา 90110

[2]สาขาวิทยาศาสตร์การคำนวณ คณะวิทยาศาสตร์ มหาวิทยาลัยสงขลานครินทร์ สงขลา 90110

Sompong Chuysurichay[1,*] and Lalita Apisornpanich[2]

[1]Algebra and Applications Research Unit, Division of Computational Science,

Faculty of Science, Prince of Songkla University, Songkhla 90110

[2]Division of Computational Science, Faculty of Science,

Prince of Songkla University, Songkhla 90110

Email: [1]sompong.c@psu.ac.th [2]5910210268@psu.ac.th

วันที่รับบทความ : 10 มกราคม 2563 วันที่แก้ไขบทความ : 6 พฤษภาคม 2563 วันที่ตอบรับบทความ : 21 สิงหาคม 2563

บทคัดย่อ

ในงานวิจัยเรื่องนี้ผู้วิจัยหารูปแบบปิดของลำดับโมเบียส ซึ่งนิยามโดย

$$z_{n+1} = \frac{az_n + b}{cz_n + d}$$

เมื่อ a, b, c และ d เป็นจำนวนจริง โดยที่ $ad - bc \neq 0$

คำสำคัญ: โมเบียส ลำดับ

[*] ผู้เขียนหลัก

ABSTRACT

In this research, we obtain a new derivation of the closed-form solution of a Mobius sequence defined by

$$z_{n+1} = \frac{az_n + b}{cz_n + d}$$

where a, b, c and d are real numbers with $ad - bc \neq 0$.

Keywords: Mobius, Sequence

1. Introduction

A Mobius sequence is a sequence defined by using iterated applications of a single Mobius transformation to an initial point on the real line (or more generally, on the extended complex plane). It is a common sequence appeared in many literatures yet a complete set of solutions are still rare to find. Many authors studied Mobius sequences in terms of their dynamical properties [1, 2, 5]. It is also applied in the study of continued fractions [3]. The complete set of solutions, in nontrivial cases, can be found in [4]. Liebeck [4] obtained the solution to the Mobius sequence by assuming that the sequence must converge to some complex number and constructing an *auxiliary equation* from this recurrence relation. He also explained the behavior of the solution in each case. See Theorem 2R in [4] for more details.

The aim of this paper is to collect, with proof different from [4], the closed-form solution of Mobius sequences defined on the real line. The novelty of our method is to define another *auxiliary sequence* which later becomes a homogeneous linear recurrence relation of degree 2 and can be easily solved. Our main result is summarized in Theorem 1 below.

2. Mobius Sequences and Their Closed-Form

A Mobius sequence is a sequence of real numbers $\{z_n\}$, with initial value z_1, defined for any $n \in \mathbb{N}$ by the recurrence relation

$$z_{n+1} = \frac{az_n + b}{cz_n + d} \tag{1}$$

where a, b, c and d are real numbers with $ad - bc \neq 0.$ The closed-form solution of a Mobius sequence is summarized in the following theorem.

Theorem 1 Let $\{z_n\}$ be a Mobius sequence defined recursively by (1).

(1) If $c = 0$ and $a = d,$ then $z_n = z_1 + (n-1)\dfrac{b}{d}$ for all $n \in \mathbb{N}.$

(2) If $c = 0$ and $a \neq d,$ then

$$z_n = \left(\frac{a}{d}\right)^{n-1} z_1 + \frac{b}{d^{n-1}}\left(\frac{a^{n-1} - d^{n-1}}{a - d}\right)$$

for all $n \in \mathbb{N}.$

(3) If $c \neq 0$ and $(a-d)^2 + 4bc > 0,$ then

$$z_n = \frac{\left(\dfrac{\alpha - d}{c}\right)(\beta - d - cz_1)\alpha^{n-1} + \left(\dfrac{\beta - d}{c}\right)(cz_1 + d - \alpha)\beta^{n-1}}{(\beta - d - cz_1)\alpha^{n-1} + (cz_1 + d - \alpha)\beta^{n-1}}$$

for all $n \in \mathbb{N},$ where

$$\alpha = \frac{a + d - \sqrt{(a-d)^2 + 4bc}}{2} \quad \text{and} \quad \beta = \frac{a + d + \sqrt{(a-d)^2 + 4bc}}{2}.$$

(4) If $c \neq 0$ and $(a-d)^2 + 4bc = 0,$ then

$$z_n = \frac{1}{2c}\left[\frac{(a-d)(2cz_1 + d - a)n + (a-d)^2 + 4cdz_1}{(2cz_1 + d - a)n + 2(a - cz_1)}\right]$$

for all $n \in \mathbb{N}.$

(5) If $c \neq 0$ and $(a-d)^2 + 4bc < 0,$ then

$$z_n = \frac{r}{c}\left[\frac{(cz_1 + d)\sin n\theta - r\sin(n-1)\theta}{(cz_1 + d)\sin(n-1)\theta - r\sin(n-2)\theta}\right] - \frac{d}{c}$$

for all $n \in \mathbb{N},$ where $r = \sqrt{ad - bc}$ and $\theta = \cos^{-1}\left(\dfrac{a + d}{2\sqrt{ad - bc}}\right).$

Proof. We consider the following cases:

Case 1 $c=0.$ Then, $ad\neq 0$ and (1) reduces to

$$z_{n+1}=\frac{a}{d}z_n+\frac{b}{d}. \tag{2}$$

From $(2),$ we get

$$z_n=\frac{a}{d}z_{n-1}+\frac{b}{d}. \tag{3}$$

Substituting (3) in (2) and continuing in the same manner, we have

$$\begin{aligned}
z_{n+1}&=\frac{a}{d}\left(\frac{a}{d}z_{n-1}+\frac{b}{d}\right)+\frac{b}{d}\\
&=\left(\frac{a}{d}\right)^2 z_{n-1}+\frac{b}{d}\left(\frac{a}{d}+1\right)\\
&=\left(\frac{a}{d}\right)^2\left(\frac{a}{d}z_{n-2}+\frac{b}{d}\right)+\frac{b}{d}\left(\frac{a}{d}+1\right)\\
&=\left(\frac{a}{d}\right)^3 z_{n-2}+\frac{b}{d}\left(\left(\frac{a}{d}\right)^2+\left(\frac{a}{d}\right)+1\right)\\
&\;\;\vdots\\
&=\left(\frac{a}{d}\right)^n z_1+\frac{b}{d}\left(\left(\frac{a}{d}\right)^{n-1}+\left(\frac{a}{d}\right)^{n-2}+\cdots+\left(\frac{a}{d}\right)+1\right).
\end{aligned}$$

Thus, we have the following results.

(1) If $c=0$ and $a=d$, then $z_{n+1}=z_1+\frac{bn}{d}.$ Hence, $z_n=z_1+\frac{b}{d}(n-1)$ for all $n\in\mathbb{N}.$

(2) If $c=0$ and $a\neq d$, then $z_{n+1}=\left(\frac{a}{d}\right)^n z_1+\frac{b}{d}\left(\frac{\left(\frac{a}{d}\right)^n-1}{\frac{a}{d}-1}\right).$ Hence,

$$z_n=\left(\frac{a}{d}\right)^{n-1}z_1+\frac{b}{d^{n-1}}\left(\frac{a^{n-1}-d^{n-1}}{a-d}\right) \text{ for all } n\in\mathbb{N}.$$

Case 2 $c \neq 0.$ Define a sequence $\{x_n\}$ by

$$x_{n+1} = (cz_n + d)x_n. \tag{4}$$

Since $cz_n + d \neq 0,$ $x_n \neq 0$ for all $n \in \mathbb{N}.$ Then, we have

$$z_n = \frac{x_{n+1}}{cx_n} - \frac{d}{c}. \tag{5}$$

Substituting (5) in $(1),$ we get

$$\frac{x_{n+2}}{cx_{n+1}} - \frac{d}{c} = \frac{a\left(\frac{x_{n+1}}{cx_n} - \frac{d}{c}\right) + b}{c\left(\frac{x_{n+1}}{cx_n} - \frac{d}{c}\right) + d}$$

$$\frac{x_{n+2} - dx_{n+1}}{cx_{n+1}} = \frac{ax_{n+1} - adx_n + bcx_n}{cx_{n+1}}$$

$$x_{n+2} - (a+d)x_{n+1} + (ad - bc)x_n = 0. \tag{6}$$

The auxiliary equation of (6) is

$$x^2 - (a+d)x + (ad - bc) = 0.$$

Solving for $x,$ we get

$$x = \frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2}.$$

Let $\alpha = \dfrac{a + d - \sqrt{(a-d)^2 + 4bc}}{2}$ and $\beta = \dfrac{a + d + \sqrt{(a-d)^2 + 4bc}}{2}.$

We consider 3 subcases.

Case 2.1 $(a-d)^2 + 4bc > 0.$ Since $ad - bc \neq 0,$ we have $\alpha, \beta \neq 0.$ Also, $\alpha \neq \beta$ because $(a-d)^2 + 4bc > 0.$ Thus, the solution of (6) is

$$x_n = k_1\alpha^n + k_2\beta^n \tag{7}$$

where k_1 and k_2 are constants.

Putting $n = 1, 2$ in $(7),$ we get the system of equations

$$\left.\begin{aligned}\alpha k_1+\beta k_2&=x_1,\\ \alpha^2k_1+\beta^2k_2&=x_2.\end{aligned}\right\}\tag{8}$$

Using Cramer's rule, the solutions for (8) are

$$k_1=\frac{\beta x_1-x_2}{\alpha(\beta-\alpha)}\text{ and }k_2=\frac{x_2-\alpha x_1}{\beta(\beta-\alpha)}.$$

Thus,

$$x_n=\left(\frac{\beta x_1-x_2}{\beta-\alpha}\right)\alpha^{n-1}+\left(\frac{x_2-\alpha x_1}{\beta-\alpha}\right)\beta^{n-1}.\tag{9}$$

Putting (9) in (5) and simplifying, we get

$$z_n=\frac{\left(\frac{\alpha-d}{c}\right)(\beta-d-cz_1)\alpha^{n-1}+\left(\frac{\beta-d}{c}\right)(cz_1+d-\alpha)\beta^{n-1}}{(\beta-d-cz_1)\alpha^{n-1}+(cz_1+d-\alpha)\beta^{n-1}}$$

for all $n\in\mathbb{N}$.

Case 2.2 $(a-d)^2+4bc=0.$ Then $\alpha=\beta=\frac{a+d}{2}.$ Thus, the solution of (6) is

$$x_n=(k_1+k_2n)\alpha^n,\tag{10}$$

where k_1 and k_2 are constants.

Putting $n=1,2$ in $(10),$ we get the system of equations

$$\left.\begin{aligned}\alpha k_1+\ \ \alpha k_2&=x_1,\\ \alpha^2k_1+2\alpha^2k_2&=x_2.\end{aligned}\right\}\tag{11}$$

Using Cramer's rule, the solutions for (11) are

$$k_1=\frac{x_1}{\alpha^2}(2\alpha-cz_1-d)\text{ and }k_2=\frac{x_1}{\alpha^2}(cz_1+d-\alpha).$$

Thus,

$$\begin{aligned}x_n&=\left[\frac{x_1}{\alpha^2}(2\alpha-cz_1-d)+\frac{x_1}{\alpha^2}(ncz_1+nd-n\alpha)\right]\alpha^n\\ &=x_1\alpha^{n-2}\left[a-cz_1+ncz_1+\frac{n}{2}(d-a)\right]\end{aligned}$$

$$= x_1\alpha^{n-2}\left[(n-1)cz_1 + \frac{(d-a)}{2}n + a\right]. \tag{12}$$

Putting (12) in (5) and simplifying, we have

$$z_n = \frac{x_1\alpha^{n-1}\left[ncz_1 + \left(\frac{d-a}{2}\right)(n+1) + a\right]}{cx_1\alpha^{n-2}\left[(n-1)cz_1 + \left(\frac{d-a}{2}\right)n + a\right]} - \frac{d}{c}$$

$$= \left(\frac{a+d}{2c}\right)\left[\frac{ncz_1 + \left(\frac{d-a}{2}\right)(n+1) + a}{(n-1)cz_1 + \left(\frac{d-a}{2}\right)n + a}\right] - \frac{d}{c}$$

$$= \frac{1}{2c}\left[\frac{(a-d)(2cz_1 + d - a)n + (a-d)^2 + 4cdz_1}{(2cz_1 + d - a)n + 2(a - cz_1)}\right]$$

for all $n \in \mathbb{N}$.

Case 2.3 $(a-d)^2 + 4bc < 0$. Then, α, β are nonreal complex numbers. Note that

$$ad - bc = \frac{1}{4}(a+d)^2 - \frac{1}{4}\left[(a-d)^2 + 4bc\right] \ > \ 0.$$

We also have

$$\alpha = \frac{a + d - i\sqrt{-(a-d)^2 - 4bc}}{2} \text{ and } \beta = \frac{a + d + i\sqrt{-(a-d)^2 - 4bc}}{2}.$$

Let θ be the argument of β and $r = |\beta|$.

Then, $r = \sqrt{ad - bc}$ and $\sin\theta = \sqrt{\frac{-(a-d)^2 - 4bc}{4(ad-bc)}} > 0$. Thus, $\theta \in (0,\pi)$ and hence, $\theta = \cos^{-1}\left(\frac{a+d}{2\sqrt{ad-bc}}\right)$. Then, $\alpha = r(\cos\theta - i\sin\theta)$ and $\beta = r(\cos\theta + i\sin\theta)$.

Thus, the solution of (6) is

$$x_n = r^n\left(k_1\cos n\theta + k_2\sin n\theta\right), \tag{13}$$

where k_1 and k_2 are constants.

Putting $n = 1, 2$ in (13), we get the system of equations

$$\left.\begin{aligned}(r\cos\theta)k_1 \;+\; (r\sin\theta)k_2 &= x_1,\\ (r^2\cos 2\theta)k_1+(r^2\sin 2\theta)k_2 &= x_2.\end{aligned}\right\} \tag{14}$$

Using Cramer's rule and $x_2=(cz_1+d)x_1$, the solutions for (14) are

$$\begin{aligned}k_1 &= \frac{rx_1\sin 2\theta - x_2\sin\theta}{r^2\sin\theta}\\ &= \frac{rx_1\sin 2\theta - x_1\left(cz_1+d\right)\sin\theta}{r^2\sin\theta}\\ &= \frac{x_1}{r^2\sin\theta}\left[r\sin 2\theta-\left(cz_1+d\right)\sin\theta\right]\end{aligned}$$

and

$$\begin{aligned}k_2 &= \frac{x_2\cos\theta - rx_1\cos 2\theta}{r^2\sin\theta}\\ &= \frac{x_1\left(cz_1+d\right)\cos\theta - rx_1\cos 2\theta}{r^2\sin\theta}\\ &= \frac{x_1}{r^2\sin\theta}\left[\left(cz_1+d\right)\cos\theta-r\cos 2\theta\right].\end{aligned}$$

Substituting k_1, k_2 and (13) in (5), we get

$$\begin{aligned}z_n &= \frac{r}{c}\left[\frac{\left(\cos\left(n+1\right)\theta\right)\left[r\sin 2\theta-\left(cz_1+d\right)\sin\theta\right]+\left(\sin\left(n+1\right)\theta\right)\left[\left(cz_1+d\right)\cos\theta-r\cos 2\theta\right]}{\left(\cos n\theta\right)\left[r\sin 2\theta-\left(cz_1+d\right)\sin\theta\right]+\left(\sin n\theta\right)\left[\left(cz_1+d\right)\cos\theta-r\cos 2\theta\right]}\right]-\frac{d}{c}\\ &= \frac{r}{c}\left[\frac{\left(cz_1+d\right)\left[\sin\left(n+1\right)\theta\cos\theta-\cos\left(n+1\right)\theta\sin\theta\right]}{\left(cz_1+d\right)\left[\sin n\theta\cos\theta-\cos n\theta\sin\theta\right]+r\left[\cos n\theta\sin 2\theta-\sin n\theta\cos 2\theta\right]}\right.\\ &\qquad\left.+\frac{r\left[\cos\left(n+1\right)\theta\sin 2\theta-\sin\left(n+1\right)\theta\cos 2\theta\right]}{\left(cz_1+d\right)\left[\sin n\theta\cos\theta-\cos n\theta\sin\theta\right]+r\left[\cos n\theta\sin 2\theta-\sin n\theta\cos 2\theta\right]}\right]-\frac{d}{c}\\ &= \frac{r}{c}\left[\frac{\left(cz_1+d\right)\sin n\theta-r\sin\left(n-1\right)\theta}{\left(cz_1+d\right)\sin\left(n-1\right)\theta-r\sin\left(n-2\right)\theta}\right]-\frac{d}{c}.\end{aligned}$$

3. Examples

We illustrate the results in Theorem 1 by the following examples.

1. Let $a=1,\ b=1,\ c=0,\ d=1$ and $z_1=1.$

   Then, (1) becomes $z_{n+1}=z_n+1.$ By Theorem 1 (1), we get $z_n=n$ for all $n\in\mathbb{N}.$

2. Let $a=2,\ b=1,\ c=0,\ d=1$ and $z_1=2.$

   Then, (1) becomes $z_{n+1}=2z_n+1.$ By Theorem 1 (2), we get $z_n=3\cdot 2^{n-1}-1$ for all $n\in\mathbb{N}.$

3. Let $a=1,\ b=4,\ c=1,\ d=1$ and $z_1=1.$

   Then, (1) becomes $z_{n+1}=\dfrac{z_n+4}{z_n+1}.$ By Theorem 1 (3), we get $z_n=\dfrac{2\cdot 3^n+2(-1)^n}{3^n-(-1)^n}$ for all $n\in\mathbb{N}.$

4. Let $a=3,\ b=-1,\ c=1,\ d=1$ and $z_1=2.$

   Then, (1) becomes $z_{n+1}=\dfrac{3z_n-1}{z_n+1}.$ By Theorem 1 (4), we get $z_n=\dfrac{n+3}{n+1}$ for all $n\in\mathbb{N}.$

5. Let $a=0,\ b=1,\ c=-1,\ d=1$ and $z_1=2.$

   Then, (1) becomes $z_{n+1}=\dfrac{1}{-z_n+1}.$ By Theorem 1 (5), we get

$$z_n=\frac{\sin\frac{(n-2)\pi}{3}-\sin\frac{n\pi}{3}}{\sin\frac{(n-2)\pi}{3}+\sin\frac{(n-1)\pi}{3}}$$ for all $n\in\mathbb{N}.$

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References

[1] De Pree, J. D. and Thron, W. J. (1962). On Sequences of Moebius Transformations. *Mathematische Zeitschrift*, 80 (1), p. 184 - 193.

[2] Eljoseph, N. (1968). On The Iteration of Linear Fractional Transformations. *The American Mathematical Monthly*, 75 (4), p. 362 - 366.

[3] Karlsson, J., Wallin, H., and Gelfgren, J. (1991). Iteration of Möbius Transforms and Continued Fractions. *The Rocky Mountain Journal of Mathematics*, 21 (1), p. 451 - 472.

[4] Liebeck, H . (1961). The Convergence of Sequences with Linear Fractional Recurrence Relation. *The American Mathematical Monthly*, 68 (4), p. 353 - 355.

[5] Piranian, G. and Thron, W. J. (1957). Convergence Properties of Sequences of Linear Fractional Transformations. *The Michigan Mathematical Journal*, 4 (2), p. 129 - 135.