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ผลเฉลยของสมการไดโอแฟนไทน์เลขชี้กำลัง $7^x - 5^y = z^2$

The Solution of The Exponential Diophantine Equation

$$7^x - 5^y = z^2$$

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บทคัดย่อ

ในบทความนี้ นำเสนอการพิสูจน์ว่า สมการไดโอแฟนไทน์เลขชี้กำลัง $7^x - 5^y = z^2$ มีผลเฉลยเพียงผลเฉลยเดียว เมื่อ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ ในการพิสูจน์ เราได้ใช้ข้อความการณของคาคาลานและทฤษฎีบทต่าง ๆ ที่เกี่ยวข้องกับสมภาคมาช่วยในการพิสูจน์ ซึ่งพบว่า ผลเฉลยมีเพียงชุดเดียว คือ $(x, y, z) = (0, 0, 0)$

คำสำคัญ: สมการไดโอแฟนไทน์เลขชี้กำลัง ผลเฉลยจำนวนเต็ม ข้อคาดการณ์ของคาคาลาน

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ABSTRACT

In this paper, we prove that the exponential Diophantine equation $7^x - 5^y = z^2$ has one solution where x , y and z are non-negative integers. In the proof, we apply reasonably Catalan's conjecture and various theorems concerning the congruence to obtain the solution. The result reveals that the unique solution is $(x, y, z) = (0, 0, 0)$.

Keywords: Exponential Diophantine equation, Integer solution, Catalan's conjecture

1. Introduction

Diophantine equations have been interesting to many mathematicians for a long time. Over the last decade, a number of the exponential Diophantine equations have been studied since Catalan presented Catalan's conjecture [3] and Mihalescu proved this conjecture [11]. In 2007, Acu studied the equation $2^x + 5^y = z^2$ [1]. He proved that $(3, 0, 3)$ and $(2, 1, 3)$ are the only two solutions (x, y, z) where x , y and z are non-negative integers. In 2011, Suvarnamani et al. suggested that two equations which are $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no integer solutions [21]. In 2012, the Diophantine equation $4^x + p^y = z^2$ where x , y and z are non-negative integers and p is any positive prime number was studied by Chotchaisthit. The study revealed that the equation does not have solutions [5]. Meanwhile, Sroysang [15 - 16] proved two Diophantine equations including $3^x + 5^y = z^2$ and $31^x + 32^y = z^2$. In the same year, Peker and Cenberci presented that the Diophantine equation $8^x + 19^y = z^2$ has no solution [12]. After that several Diophantine equations have been studied by different mathematical researchers [4, 7, 13, 14, 17, 18, 19, 20]. In 2017, Jayakumar and Shankarakalidoss proved that the Diophantine equation $47^x + 2^y = z^2$ has a unique non-negative solution $(x, y, z) = (0, 3, 3)$ [6] while Asthana and Singh showed that $(1, 0, 2)$, $(1, 1, 4)$, $(3, 2, 14)$ and $(5, 1, 16)$ are solutions (x, y, z) to the Diophantine equation $3^x + 13^y = z^2$ [2]. After that, Kumar et al. [8] studied two equations including $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They could show that the two equations have

no solution. Recently, Laipaporn et al. proved the solution of $3^x + p5^y = z^2$ where p is a prime number. They found that the equations has infinitely many solutions [9]. While Makate et al. proved that $8^x + 61^y = z^2$ and $8^x + 67^y = z^2$ have same unique solution $(x, y, z) = (1, 0, 3)$ [10].

In this paper, we present a different Diophantine equation $7^x - 5^y = z^2$ and prove that the equation has only one trivial solution.

2. Preliminaries

Proposition 2.1 [11] (Catalan's conjecture) $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.

Lemma 2.2 The Diophantine equation $7^x - z^2 = 1$ has no solution (x, z) where x and z are integers and $\min\{x, z\} > 1$.

Proof. Suppose that x and z are integers where $\min\{x, z\} > 1$ such that $7^x - z^2 = 1$. By Proposition 2.1, it is sufficient to conclude that the equation has no solution. \square

3. Main Result

Theorem 3.1 The Diophantine equation $7^x - 5^y = z^2$ has a unique solution $(x, y, z) = (0, 0, 0)$ where x, y and z are non-negative integers.

Proof. Let x, y and z be non-negative integers such that

$$7^x - 5^y = z^2. \tag{1}$$

First, we consider two cases including $x = 0$ and $x \geq 1$.

Case 1 If $x = 0$, then from (1) we have $z^2 + 5^y = 1$. Obviously, the equation has only one solution when z and $y = 0$.

Case 2 If $x \geq 1$, then we separate y in two subcases including $y = 0$ and $y \geq 1$.

Subcase 1 If $y = 0$, then $7^x - z^2 = 1$. By Lemma 2.2, it is sufficient to consider only $x = 1$ or $z \leq 1$. Hence, we consider x and z which are as follow. For $x = 1$, we have

$z^2 = 6$. This is impossible. For $z = 0$, it follows that $7^x = 1$. It has no solution. For $z = 1$, we have $7^x = 2$. This is impossible.

Subcase 2 If $y \geq 1$, then z^2 is even which implies that $z^2 \equiv 0 \pmod{4}$. From (1), it follows that $z^2 \equiv (-1)^x - 1 \pmod{4}$. Then, we have $(-1)^x - 1 \equiv 0 \pmod{4}$. This implies that x is positive even, so we let $x = 2m$ where $m \in \mathbb{Z}^+$. From (1), we have $5^y = 7^{2m} - z^2$. It is written as

$$5^y = (7^m + z)(7^m - z). \quad (2)$$

From (2), it yields (3) and (4):

$$5^\alpha = 7^m - z \quad (3)$$

and

$$5^\beta = 7^m + z, \quad (4)$$

where $0 \leq \alpha < \beta \leq y$ and $\alpha + \beta = y$. From (3) and (4), we have

$$2 \cdot 7^m = 5^\alpha(1 + 5^{\beta-\alpha}). \quad (5)$$

We separate α into $\alpha = 0$ and $\alpha \geq 1$.

If $\alpha = 0$, then (5) becomes $2 \cdot 7^m = 1 + 5^\beta$. Thus, we have $2 \equiv 1 + (-1)^\beta \pmod{3}$.

This implies that β is even. Let $\beta = 2t$ where $t \in \mathbb{Z}^+$, we have

$$2 \cdot 7^m = 1 + 25^t. \quad (6)$$

From (6), we have $2 \cdot (-1)^m \equiv 1 + 1 \pmod{8}$. This implies that m is even. Suppose $m = 2l$ where $l \in \mathbb{Z}^+$. Then, we have

$$2 \cdot 49^l = 1 + 25^t. \quad (7)$$

From (7), we have $2 \cdot (-1)^l \equiv 1 \pmod{5}$ which is impossible.

If $\alpha \geq 1$, (5) implies that $5 \mid 2 \cdot 7^m$ which contradicts the fact that 2, 5 and 7 are relatively primes. Hence, the theorem is proved. \square

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