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อัตราส่วนจำนวนเต็มของอนุกรมที่เรียงติดกัน

Integer Ratios of Some Consecutive Series

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บทคัดย่อ

ในบทความนี้ ผู้ประพันธ์ได้หาเงื่อนไขที่จำเป็นและเพียงพอที่ทำให้อัตราส่วนของอนุกรมที่เรียงติดกันต่อไปนี้เป็นจำนวนเต็ม โดยอนุกรมที่สนใจได้แก่

$$B_k(n)=\frac{1}{1(1+k)}+\frac{1}{2(2+k)}+\cdots+\frac{1}{n(n+k)} \quad (n,k\in\mathbb{N})$$

$$C_{a,\,d}(n)=\sum_{i=1}^{n}a+(n-i)d \quad (a,d,n\in\mathbb{N}) \text{ และ}$$

$$D_{a,\,r}(n)=a+ar+\cdots+ar^{n-1} \quad (a,n\in\mathbb{N},\ r\in\mathbb{Q}\setminus\{0\})$$

คำสำคัญ: อนุกรม อนุกรมที่เรียงติดกัน

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ABSTRACT

In this article, we establish necessary and sufficient conditions for the ratio of some consecutive series to be integers. The series that we investigate in the article are as follows:

$$B_k(n) = \frac{1}{1(1+k)} + \frac{1}{2(2+k)} + \cdots + \frac{1}{n(n+k)} \quad (n, k \in \mathbb{N}),$$

$$C_{a,\,d}(n) = \sum_{i=1}^{n} a + (n-i)d \quad (a, d, n \in \mathbb{N}) \text{ and}$$

$$D_{a,\,r}(n) = a + ar + \cdots + ar^{n-1} \quad (a, n \in \mathbb{N},\ r \in \mathbb{Q} \setminus \{0\}).$$

Keywords: Series, Consecutive series

1. Introduction

The ratios of two consecutive powers were studied by many mathematicians [1, 3]. One of the unsolved problems is to find all positive integers k and $n > 1$ such that the ratio of the power sum $\frac{1^k + 2^k + 3^k + \cdots + n^k}{1^k + 2^k + 3^k + \cdots + (n-1)^k}$ is an integer. In 2019, Ioulia N. Baoulina [2] studied a ratio for consecutive alternating power sums. Namely, he gave a necessary and sufficient condition for

$$\frac{1^k - 2^k + 3^k - \cdots + (-1)^{n+1} n^k}{1^k - 2^k + 3^k - \cdots + (-1)^n (n-1)^k} \quad (n, k \in \mathbb{N} \text{ and } n > 1)$$

to be an integer. From the mentioned result, it is natural to investigate the ratios for other series. In this paper, we establish necessary and sufficient conditions for the ratio of the following consecutive series to be integers:

$$B_k(n) = \frac{1}{1(1+k)} + \frac{1}{2(2+k)} + \cdots + \frac{1}{n(n+k)} \quad (n, k \in \mathbb{N}),$$

$$C_{a,\,d}(n) = \sum_{i=1}^{n} a + (n-i)d \quad (a, d, n \in \mathbb{N}) \text{ and}$$

$$D_{a,\,r}(n) = a + ar + \cdots + ar^{n-1} \quad (a, n \in \mathbb{N},\ r \in \mathbb{Q} \setminus \{0\}).$$

2. Main Results

Theorem 2.1 Let k and n be positive integers. Then $\dfrac{B_k(n+1)}{B_k(n)}$ is never an integer for all k and n.

Proof For positive integers k and n,

$$\begin{aligned}B_k(n) &= \frac{1}{1(1+k)}+\frac{1}{2(2+k)}+\cdots+\frac{1}{n(n+k)}\\ &= \frac{1}{k}\left(\frac{1}{1}-\frac{1}{1+k}\right)+\frac{1}{k}\left(\frac{1}{2}-\frac{1}{2+k}\right)+\cdots+\frac{1}{k}\left(\frac{1}{n}-\frac{1}{n+k}\right)\\ &= \frac{1}{k}\left(1-\frac{1}{1+k}+\frac{1}{2}-\frac{1}{2+k}+\cdots+\frac{1}{n}-\frac{1}{n+k}\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{B_k(n+1)}{B_k(n)} &= 1+\frac{\dfrac{1}{n+1}-\dfrac{1}{n+k+1}}{1-\dfrac{1}{1+k}+\dfrac{1}{2}-\dfrac{1}{2+k}+\cdots+\dfrac{1}{n}-\dfrac{1}{n+k}}\\ &= 1+\frac{\dfrac{1}{n+1}-\dfrac{1}{n+k+1}}{1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}-\dfrac{1}{k+1}-\dfrac{1}{k+2}-\cdots-\dfrac{1}{n+k}}.\end{aligned}$$

For $n=1$ and $k\geq 2$, we have $\dfrac{B_k(2)}{B_k(1)}=1+\dfrac{k+1}{2(k+2)}$.

Since k is a positive integer, we can see that $1<\dfrac{B_k(2)}{B_k(1)}<2$.

Therefore $\dfrac{B_k(2)}{B_k(1)}$ is not an integer.

For $n\geq 1$ and $k=1$,

$$\frac{2}{n+1}<1+\frac{1}{n+2}$$

$$\frac{1}{n+1}-\frac{1}{n+2}<1-\frac{1}{n+1}.$$

Since $\frac{1}{n+1}-\frac{1}{n+2}>0$, $1<\frac{B_1(n+1)}{B_1(n)}<2.$ Thus $\frac{B_1(n+1)}{B_1(n)}$ is not an integer.

For $n\geq 2$ and $k\geq 2$, it suffices to show that

$$\frac{1}{n+1}+\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{k+n}\right)<1+\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)+\frac{1}{n+k+1}.$$

We have

$$2n+k+1<n^2+nk+n+k$$
$$(2n+k+1)(n+k+1)<(n+1)(n+k)(n+k+2)$$
$$\frac{n+k+n+1}{(n+1)(n+k)}<\frac{n+k+2}{n+k+1}$$

Thus $\frac{1}{n+1}+\frac{1}{n+k}<1+\frac{1}{n+k+1}$.

Since $\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{k+n-1}<\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ and $\frac{1}{n+1}+\frac{1}{n+k}<1+\frac{1}{n+k+1}$, our claim holds.

Thus $0<\frac{1}{n+1}-\frac{1}{n+k+1}<1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{k+1}-\frac{1}{k+2}-\cdots-\frac{1}{n+k}$ and this implies that $1<\frac{B_k(n+1)}{B_k(n)}<2.$ Therefore $\frac{B_k(n+1)}{B_k(n)}$ is not an integer. □

We next consider arithmetic series.

Theorem 2.2 Let a,d and n be positive integers. Then $\frac{C_{a,d}(n+1)}{C_{a,d}(n)}$ is an integer if and only if one of the following holds:

1. $n=1$ and a divides d.
2. $n=2$ and $a=d$.

Proof For positive integers a,d and n, we have

$$\frac{C_{a,d}(n+1)}{C_{a,d}(n)}=\frac{\left(a+nd\right)+\left(a+(n-1)d\right)+\left(a+(n-2)d\right)+\cdots+(a+d)+a}{\left(a+(n-1)d\right)+\left(a+(n-2)d\right)+\left(a+(n-3)d\right)+\cdots+(a+d)+a}$$

$$=1+\frac{a+(n)d}{\frac{n}{2}\left(2a+(n-1)d\right)}=1+\frac{2a+2nd}{2na+nd(n-1)}.$$

For $n=1$, we have $\dfrac{C_{a,d}(2)}{C_{a,d}(1)}=1+\dfrac{2a+2d}{2a}=1+\dfrac{a+d}{a}=2+\dfrac{d}{a}$.

Thus $\dfrac{C_{a,d}(2)}{C_{a,d}(1)}$ is an integer if and only if a divides d.

For $n=2$, we have $\dfrac{C_{a,d}(3)}{C_{a,d}(2)}=1+\dfrac{2a+4d}{4a+2d}=1+\dfrac{a+2d}{2a+d}=2+\dfrac{d-a}{2a+d}$.

We first consider the case $d>a.$ Since $2a+d>d-a>0$, $2a+d$ divides $d-a$ if and only if $a=d.$

Now we assume that $d\le a$. If $2a+d$ divides $d-a$ then $d-a=m(2a+d)$ for some integer $m\le 0.$ Then $a=\dfrac{d(1-m)}{2m+1}.$ Since $m\le 0$ and $a,d>0,$ $2m+1>0.$ Thus $m=0$ and this implies that $d=a.$ For the converse, it is easy to see that if $d=a$ then $2a+d$ divides $d-a.$

For $n\ge 3$, $2na+nd(n-1)>2a+2nd.$ Therefore $\dfrac{C_{a,d}(n+1)}{C_{a,d}(n)}$ is never an integer. □

We now consider geometric series $D_{a,r}(n)$.

Theorem 3 Let $a\in\mathbb{N},$ $r\in\mathbb{Q}\setminus\{0\}$ and $n\in\mathbb{N}$. Then $\dfrac{D_{a,r}(n+1)}{D_{a,r}(n)}$ is an integer if and only if one of the following holds:

1. $n=1$ and $r=1.$.
2. n is odd and $r=-1.$
3. $n=1$ and r is an integer.
4. $n=2$ and $r=-2.$

Proof We first consider the case $r=1$ or $r=-1.$

If $r=1$ then $\dfrac{D_{a,1}(n+1)}{D_{a,1}(n)}=\dfrac{(n+1)a}{na}=1+\dfrac{1}{n}$ which is an integer if and only if $n=1.$

If $r=-1$ and n is odd, then $\dfrac{D_{a,-1}(n+1)}{D_{a,-1}(n)}$ is always an integer since $D_{a,-1}(n+1)=0$.

Thus we next consider the nontrivial case as follows.

It is easy to see that if $n=1$ and r is an integer then

$$\frac{D_{a,r}(n+1)}{D_{a,r}(n)}=\frac{a+ar}{a}=\frac{a(1+r)}{a}=1+r$$

is an integer.

If $n=2$ and $r=-2$ then $\dfrac{D_{a,r}(n+1)}{D_{a,r}(n)}=\dfrac{a+ar+ar^2}{a+ar}=\dfrac{1+r+r^2}{1+r}=-3$ is an integer.

Now let $a\in\mathbb{N},\ r\in\mathbb{Q}\setminus\{0,\pm1\}$ and $n\in\mathbb{N}$. Suppose $\dfrac{D_{a,r}(n+1)}{D_{a,r}(n)}$ is an integer.

Let $r=\dfrac{c}{d}$ where c and d are relatively prime. Then

$$\frac{D_{a,r}(n+1)}{D_{a,r}(n)}=1+\frac{\left(\frac{c}{d}\right)^n}{1+\frac{c}{d}+\cdots+\left(\frac{c}{d}\right)^{n-1}}=1+\frac{c^n}{d(d^{n-1}+cd^{n-2}+\cdots+c^{n-1})}.$$

Since c and d are relatively prime and $\dfrac{D_{a,r}(n+1)}{D_{a,r}(n)}$ is an integer, $d=1$ and this implies that r is an integer.

For $n=1$, $\dfrac{D_{a,r}(2)}{D_{a,r}(1)}=\dfrac{a+ar}{a}=\dfrac{a(1+r)}{a}=1+r$ is an integer.

For $n=2$, $\dfrac{D_{a,r}(3)}{D_{a,r}(2)}=\dfrac{a+ar+ar^2}{a+ar}=1+\dfrac{ar^2}{a(1+r)}=1+\dfrac{r^2}{1+r}=r+\dfrac{1}{r+1}$ is an integer if and only if $r=-2$.

For $n=3$, $\dfrac{D_{a,r}(4)}{D_{a,r}(3)}=\dfrac{a+ar+ar^2+ar^3}{a+ar+ar^2}=1+\dfrac{r^3}{1+r+r^2}=r+\dfrac{1}{r^2+r+1}$ is an integer if and only if $r^2+r+1=\pm1$. Solving $r^2+r+1=\pm1$ yields $r=\pm1$ or $r=\dfrac{-1\pm\sqrt{-7}}{2}$.

It leads to a contradiction on either case.

For $n \geq 4$, suppose $\dfrac{r^n}{1+r+\cdots+r^{n-1}} = m$ for some integer m.
Thus $r^n = m(1+r+\cdots+r^{n-1})$. If p is a prime divisor of r, p does not divide $1+r+\cdots+r^{n-1}$. Thus p divides m. Since all prime divisors of r divides m and $\gcd(r, 1+r+\cdots+r^{n-1}) = 1$, we have $r^n = \pm m$. Thus $1+r+\cdots+r^{n-1} = \pm 1$ and this implies that $r+r^2+\cdots+r^{n-1} = 0$ or $r+r^2+\cdots+r^{n-1} = -2$. If $r+r^2+\cdots+r^{n-1} = 0$ then $r(1+r+r^2+\cdots+r^{n-2}) = 0$. Thus $r = 0$ or $1+r+r^2+\cdots+r^{n-2} = 0$. The latter case implies that r divides ± 1 and this is a contradiction. If $r+r^2+\cdots+r^{n-1} = -2$ then $r = \pm 1$ or $r = \pm 2$. Since $r \neq \pm 1$ and $r+r^2+\cdots+r^{n-1} = -2 < 0$, $r = -2$. Thus $r+r^2+\cdots+r^{n-1} = r$. So $r^2+r^3+\cdots+r^{n-1} = 0$. Therefore, $r = 0$ or $1+r+\cdots+r^{n-3} = 0$. Since $n \geq 4$, the latter case implies that r divides 1 which is a contradiction. □

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