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ค่าพื่นของค่าเฉลี่ยของรากที่สี่ของจำนวนเต็ม

The Floor of The Average for The Fourth Roots of Integers

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บทคัดย่อ

จุดมุ่งหมายของบทความนี้คือการสร้างลำดับโดยประมาณสำหรับลำดับของค่าเฉลี่ยของรากที่สี่ของจำนวนเต็ม n พจน์แรก และเพื่อที่จะพิสูจน์ว่าลำดับทั้งสองมีค่าพื่นเดียวกัน ในอีกทางหนึ่งได้แสดงว่าการสร้างลำดับในรูปแบบเดียวกันนี้ไม่สามารถใช้ได้ในกรณีของรากที่ห้าและรากที่หก

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ABSTRACT

The aim of this paper is to construct an estimated sequence for the sequence of the mean of the fourth roots of the first n integers and to prove that both sequences share the same floor. On the other hand, we show that the same pattern of sequence construction is not applicable for the fifth-root and the sixth-root cases.

Keywords: Floor, Average, Fourth roots

1. Introduction

It has been a long time that many researchers have studied for sum of square roots of non-negative integers. For instance, a mathematical formula of this sum is proposed by Ramanujan [2] as

$$\sum_{k=1}^n \sqrt{k} = \frac{2}{3}n\sqrt{n} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}} + \frac{1}{6} \sum_{k=0}^{\infty} \frac{1}{(\sqrt{n+k} + \sqrt{n+k+1})^3}.$$

Later, Merca [1] focused on the arithmetic mean of the square roots of the first n integers and provided bounds of this mean as follows:

$$\left(\frac{2}{3} + \frac{1}{8n} - \frac{1}{8n\sqrt{n+1}} \right) \sqrt{n+1} < \frac{1}{n} \sum_{k=1}^n \sqrt{k} < \left(\frac{2}{3} + \frac{1}{6n} - \frac{1}{6n\sqrt{n+1}} \right) \sqrt{n+1}.$$

Also he conjectured that the sequence has the same floor as a certain approximated sequence, that is,

$$\left\lfloor \frac{1}{n} \sum_{k=1}^n \sqrt{k} \right\rfloor = \left\lfloor \left(\frac{2}{3} + \frac{1}{6n} \right) \sqrt{n+1} \right\rfloor.$$

In Zacharias [5], this conjecture was proved by Zacharias. The key of the proof is to find the partition sets in which both sequences have the same floor. Zacharias presented the partition sets in the modulo 2 pattern as, for $k \in \mathbb{N} \cup \{0\}$,

$$B_{1,k} = \left[\frac{9(2k+1)^2 - 5}{4}, \frac{9(2k+2)^2 - 8}{4} \right] \cap \mathbb{N}$$

and

$$B_{2,k} = \left[\frac{9(2k+2)^2 - 4}{4}, \frac{9(2k+3)^2 - 9}{4} \right] \cap \mathbb{N},$$

where $\bigcup_{k=0}^{\infty} B_{1,k} \cup B_{2,k} = \mathbb{N}$ forms a partition of \mathbb{N} .

Afterwards Wihler [4] derived a formula for the arithmetic mean of the r -th roots of the first n integers where $r \geq 1$ as

$$\sum_{k=v}^n \sqrt[r]{k} = \frac{r}{r+1} \sqrt[r]{n+1} \left(n + \frac{1-\frac{1}{r}}{2} \right) - \frac{r}{r+1} \sqrt[r]{v} \left(v - \frac{1+\frac{1}{r}}{2} \right) - \frac{\delta_{v,n,r}}{12r} \quad (*)$$

with $\delta_{v,n,1} = 0$ and $\sigma_r(v+2, n+2) < \delta_{v,n,r} < \sigma_r(v, n)$ for $r > 1$, where

$$\sigma_r(v, n) = \begin{cases} 2 - \frac{1}{r} - n^{-1+\frac{1}{r}} & \text{if } v=1, \\ (v-1)^{-1+\frac{1}{r}} - n^{-1+\frac{1}{r}} & \text{if } v \geq 2. \end{cases}$$

Recently, Sriponpaew and Intep [3] defined the estimated sequence for average of the cube roots of integers by picking up the first two terms from the Wihler's formula as

$$\left(\frac{3}{4} + \frac{1}{4n} \right) \sqrt[3]{n+1} - \frac{1}{4n}.$$

Furthermore, we generated the partition sets in the modulo 9 pattern as follows: for $k \in \mathbb{N} \cup \{0\}$,

$$B_{1,k} = \begin{cases} \left[\frac{64(9k+1)^3 - 37}{27}, \frac{64(9k+2)^3 - 53}{27} \right] \cap \mathbb{N} & \text{if } k = 0, 1, 2, \\ \left[\frac{64(9k+1)^3 - 37}{27}, \frac{64(9k+2)^3 - 80}{27} \right] \cap \mathbb{N} & \text{if } k \geq 3, \end{cases}$$

$$B_{2,k} = \begin{cases} \left[\frac{64(9k+2)^3 - 26}{27}, \frac{64(9k+3)^3 - 54}{27} \right] \cap \mathbb{N} & \text{if } k = 0, 1, 2, \\ \left[\frac{64(9k+2)^3 - 53}{27}, \frac{64(9k+3)^3 - 54}{27} \right] \cap \mathbb{N} & \text{if } k \geq 3, \end{cases}$$

and for $j = 3, 4, \dots, 9$, define $B_{j,k}$ as

$$B_{j,k} = \left[\frac{64(9k+j)^3 - b_j}{27}, \frac{64(9k+j+1)^3 - b_{j+1} - 27}{27} \right] \cap \mathbb{N},$$

where

$$b_j = \begin{cases} 27 & \text{if } j \equiv 0 \pmod{3}, \\ 2(25-i) & \text{if } j \not\equiv 0 \pmod{3} \text{ and } j = 2^i, \\ j(5 + (-1)^i i) & \text{if } j \text{ is prime and } j \equiv i \pmod{3}, 0 < i < 3, \end{cases}$$

and $b_{10} = 37$. Consequently, $\bigcup_{k=0}^{\infty} \bigcup_{j=1}^9 B_{j,k} = \mathbb{N}$ forms a partition of \mathbb{N} . The result shown in [3] is that both sequences share the same floor in each set.

In this article, we concentrate on the floor of arithmetic mean value S_n of the fourth roots of the first n integers. In the process of proof, we construct the sequence $A(n)$ from the formula of Wihler which share the same floor with S_n . Also, we define the highly sophisticated partition sets which are applicative for the fourth roots in order to prove the following main theorem.

Theorem 1.1 (Main Theorem) For any positive integer n ,

$$\left\lfloor \frac{1}{n} \sum_{k=1}^n \sqrt[4]{k} \right\rfloor = \left\lfloor \left(\frac{4}{5} + \frac{3}{10n} \right) \sqrt[4]{n+1} - \frac{3}{10n} \right\rfloor.$$

Moreover, based on some numerical results, we can show that the same technique of construction for the approximated sequence is not suitable for the cases of the fifth roots and the sixth roots.

2. Preliminaries

The goal of this article is to derive the floor of mean value of the fourth roots of the first n integers into the floor of a simpler sequence. In particular, we define the sequence

$$S_n = \frac{1}{n} \sum_{k=1}^n \sqrt[4]{k}$$

and prove that

$$\lfloor S_n \rfloor = \left\lfloor \left(\frac{4}{5} + \frac{3}{10n} \right) \sqrt[4]{n+1} - \frac{3}{10n} \right\rfloor.$$

To simplify the process, we define the functions

$$A(x) = \left(\frac{4}{5} + \frac{3}{10x} \right) \sqrt[4]{x+1} - \frac{3}{10x}$$

and

$$L(x) = \left(\frac{4}{5} + \frac{3}{10x} \right) \sqrt[4]{x+1} - \frac{323}{960x},$$

for all real $x \geq 1$. Firstly, we verify that both functions are strictly increasing.

Lemma 2.1 For $x \geq 1$, $A(x)$ and $L(x)$ are strictly increasing functions.

Proof. To show the strict monotonicity of $A(x)$, we consider its derivative as

$$A'(x) = \frac{8x^2 - 9x - 12 + 12(x+1)^{\frac{3}{4}}}{40x^2(x+1)^{\frac{3}{4}}}.$$

After we simplify its numerator, we have

$$8x^2 - 9x - 12 + 12(x+1)^{\frac{3}{4}} = 8 \left(x - \frac{9}{16} \right)^2 + 12 \left((x+1)^{\frac{3}{4}} - \frac{155}{128} \right) > 0$$

for all $x \geq 1$. This implies that $A(x)$ is a strictly increasing function for all $x \geq 1$.

Similarly, the numerator of derivative of $L(x)$ are algebraically simplified to be

$$192 \left(x - \frac{9}{16} \right)^2 + 323 \left((x+1)^{\frac{3}{4}} - \frac{1395}{1292} \right) > 0 \text{ for all } x \geq 1.$$

Hence, $L(x)$ is also a strictly increasing function for all $x \geq 1$. □

We need to prove that S_n is increasing in order to consider only boundaries of the partition sets when we deal with the integer part of S_n .

Lemma 2.2 $S_n = \frac{1}{n} \sum_{k=1}^n \sqrt[4]{k}$ is a strictly increasing sequence.

Proof. It is well-known that $f(x) = x^{\frac{1}{4}}$ for $x \geq 1$ is a strictly increasing function.

Therefore, we have

$$\sum_{k=1}^n \sqrt[4]{k} < \sum_{k=1}^n \sqrt[4]{n+1} = n \sqrt[4]{n+1}.$$

After adding $n \sum_{k=1}^n \sqrt[4]{k}$ on the both sides of the inequality, we obtain that

$$(n+1) \sum_{k=1}^n \sqrt[4]{k} < n \sum_{k=1}^{n+1} \sqrt[4]{k}.$$

That is $S_n = \frac{1}{n} \sum_{k=1}^n \sqrt[4]{k} < \frac{1}{n+1} \sum_{k=1}^{n+1} \sqrt[4]{k} = S_{n+1}$, which implies that S_n is a strictly increasing sequence. \square

We observe the values of both (S_n) and $(A(n))$ calculated by using MATLAB and define the partition sets where both sequences share the same floor. Let $B_{1,0} = [1,37] \cap \mathbb{N}$ and $B_{2,0} = [38,195] \cap \mathbb{N}$. For $k \in \mathbb{N} \cup \{0\}$ and $j = 1, 2, \dots, 64$ with $(j, k) \neq (1, 0), (2, 0)$, we define R_j as the remainder of division of $|32 - j|$ by 32, i.e. $R_j \equiv |32 - j| \pmod{32}$ where $0 \leq R_j < 32$ and define the set $B_{j,k}$ as follows:

$$B_{j,k} = \left[\frac{625(64k + j)^4 - b_j}{256}, \frac{625(64k + j + 1)^4 - b_{j+1} - 256}{256} \right] \cap \mathbb{N},$$

where

$$b_j = \begin{cases} 512 & \text{if } R_j \equiv 0 \pmod{4}, \\ 528 & \text{if } R_j \equiv 2 \pmod{4}, \\ 497 & \text{if } R_j = 1, \\ 577 & \text{if } R_j = 3, \\ 609 & \text{if } R_j = 5, \\ 593 & \text{if } R_j = 7, \\ 401 & \text{if } R_j = 9, \\ 545 & \text{if } R_j = 11, \\ 385 & \text{if } R_j = 13, \\ 433 & \text{if } R_j = 15, \\ 561 & \text{if } R_j = 17, \\ 513 & \text{if } R_j = 19, \\ 417 & \text{if } R_j = 21, \\ 529 & \text{if } R_j = 23, \\ 465 & \text{if } R_j = 25, \\ 481 & \text{if } R_j = 27, \\ 449 & \text{if } R_j = 29, \\ 625 & \text{if } R_j = 31, \end{cases}$$

and $b_{65} = 625$.

Notice that boundary points of $B_{j,k}$ are all integers satisfying $1 + \max B_{j,k} = \min B_{j+1,k}$ and $1 + \max B_{64,k} = \min B_{1,k+1}$. In conclusion, we have that $\{B_{j,k}\}$ is the partition of \mathbb{N} .

From (*), we obtain that

$$\frac{1}{n} \sum_{k=1}^n \sqrt[4]{k} = \left(\frac{4}{5} + \frac{3}{10n} \right) \sqrt[4]{n+1} - \frac{3}{10n} - \frac{\delta_{1,n,4}}{48n}$$

where $2^{-\frac{3}{4}} - n^{-\frac{3}{4}} < \delta_{1,n,4} < \frac{7}{4} - n^{-\frac{3}{4}}$. Note that for any $n > 2$, $0 < \delta_{1,n,4} < \frac{7}{4}$.

Consequently,

$$A(n) > \frac{1}{n} \sum_{k=1}^n \sqrt[4]{k} > L(n) \quad \text{for } n > 2.$$

The aim of the main theorem is to show that for any $n \in B_{j,k}$ the integer part of $A(n)$ and S_n is $64k + j$. More precisely, $64k + j \leq S_n < A(n) < 64k + j + 1$. Notice

that $L(n)$ and $A(n)$ are strictly increasing and the value of S_n is between $L(n)$ and $A(n)$. It is enough to show that $L(r) > 64k + j$ and $A(s) < 64k + j + 1$ for any

$$r = \min B_{j,k} = \frac{625(64k + j)^4 - b_j}{256} \text{ and } s = \max B_{j,k} = \frac{625(64k + j + 1)^4 - b_{j+1}}{256} - 1.$$

To verify these inequalities, we investigate signs of coefficients of some Taylor expansions. Nevertheless, for some basic numbers n , we estimate the floor of S_n and $A(n)$ by simple calculation.

3. Proof of Main Theorem

Proof. Let $n \in \mathbb{N}$.

Case 1 $n \in B_{1,k}$

Subcase 1.1 $k = 0$

For $n \in B_{1,0}$ that is $1 \leq n \leq 37$, the bounds of S_n and $A(n)$ are the following:

$$1 = S_1 < S_n < S_{37} \approx 1.9978$$

and

$$1.0081 \approx A(1) < A(n) < A(37) \approx 1.9983.$$

Hence $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 1$ for $1 \leq n \leq 37$.

Subcase 1.2 $k \geq 1$

$$\text{Let } r = \frac{625(64k + 1)^4 - b_1}{256} = \frac{625(64k + 1)^4 - 625}{256}.$$

We will prove that $L(r) > 64k + 1$. By using MATLAB, we expand the following expression to be the Taylor series about 1 as

$$\begin{aligned}
 & (768r + 288)^4(r + 1) - (960r(64k + 1) + 323)^4 \\
 &= 47547546119427481762032000826199731966895 \\
 &+ 774640203119425658720785007239699135200000(k - 1) \\
 &+ 5909344435099731237832022406397662720000000(k - 1)^2 \\
 &+ 28020765127418531234936412187585855488000000(k - 1)^3 \\
 &+ 92444324128312396503878409379251565363200000(k - 1)^4 \\
 &+ 225017389924911730934110727749382229196800000(k - 1)^5 \\
 &+ 418034724196383328296638346040442880000000000(k - 1)^6 \\
 &+ 6046723082822828669527351424031129600000000000(k - 1)^7 \\
 &+ 6882532487035574340988103685801246720000000000(k - 1)^8 \\
 &+ 618527192560207135388140144951296000000000000(k - 1)^9 \\
 &+ 4374449368210858330089188037859737600000000000(k - 1)^{10} \\
 &+ 240916714292533524653917863936000000000000000(k - 1)^{11} \\
 &+ 1012914734961035836478598414336000000000000000(k - 1)^{12} \\
 &+ 314305350420754699187650560000000000000000000(k - 1)^{13} \\
 &+ 678834970707310676520468480000000000000000000(k - 1)^{14} \\
 &+ 91178035341302248892478259200000000000000000(k - 1)^{15} \\
 &+ 57376752766866189346406400000000000000000000(k - 1)^{16} > 0.
 \end{aligned}$$

Then we derive the above inequality into $(768r + 288)\sqrt[4]{r+1} > 960r(64k + 1) + 323$.

We have $L(r) = \left(\frac{4}{5} + \frac{3}{10r}\right)\sqrt[4]{r+1} - \frac{323}{960r} > 64k + 1$ by direct calculation.

Let $s = \frac{625(64k + 2)^4 - b_2 - 256}{256} = \frac{625(64k + 2)^4 - 784}{256}$. We will show that

$A(s) < 64k + 2$. Similarly, the following expression are written as the Taylor expansion about 1 as

$$\begin{aligned}
 & (10s(64k+2)+3)^4 - (8s+3)^4(s+1) \\
 &= 10953709046418827757109091072488404 \\
 & \quad +169615890158725058399687231063750520(k-1) \\
 & \quad +1231151431305453417400028832235382400(k-1)^2 \\
 & \quad +5560325905943677952822429251120128000(k-1)^3 \\
 & \quad +17488957096089532314859379642818560000(k-1)^4 \\
 & \quad +40621294930019226943217203622707200000(k-1)^5 \\
 & \quad +99643568429174106784851547914240000000(k-1)^7 \\
 & \quad +108486397100314238227268920934400000000(k-1)^8 \\
 & \quad +9332425620134618869593538560000000000(k-1)^9 \\
 & \quad +63221184162282908838568919040000000000(k-1)^{10} \\
 & \quad +33372558966296612654442086400000000000(k-1)^{11} \\
 & \quad +13456894494578024210497536000000000000(k-1)^{12} \\
 & \quad +4007064398391505059840000000000000000(k-1)^{13} \\
 & \quad +8309647967456644300800000000000000000(k-1)^{14} \\
 & \quad +1072216999284367687680000000000000000(k-1)^{15} \\
 & \quad +648518346341351424000000000000000000(k-1)^{16} > 0.
 \end{aligned}$$

By the same calculation, we have

$$64k+2 > \left(\frac{4}{5} + \frac{3}{10s}\right)^4 \sqrt[4]{s+1} - \frac{3}{10s} = A(s).$$

Since $A(n)$ and $L(n)$ are increasing, for $r \leq n \leq s$, we have

$$64k+1 < L(r) < L(n) < S_n < A(n) < A(s) < 64k+2.$$

Hence $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 64k+1$ for $k \geq 1$.

Case 2 $n \in B_{2,k}$

Subcase 2.1 $k = 0$

We consider $n \in B_{2,0}$ so that $38 \leq n \leq 195$. Since S_n and $A(n)$ are strictly increasing sequences,

$$2.0105 \approx S_{38} < S_n < S_{195} \approx 2.9974$$

and

$$2.0110 \approx A(38) < A(n) < A(195) \approx 2.9975.$$

Hence $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 2$ for $38 \leq n \leq 195$.

Subcase 2.2 $k \geq 1$

$$\text{Let } r = \frac{625(64k+2)^4 - b_2}{256} = \frac{625(64k+2)^4 - 528}{256}.$$

We will show that $L(r) > 64k + 2$. Again we consider that the Taylor expansion about 1 of $(768r+288)^4(r+1) - (960r(64k+2)+323)^4$ in terms of k is greater than zero for $k \geq 1$ as the following.

$$\begin{aligned} & (768r+288)^4(r+1) - (960r(64k+2)+323)^4 \\ &= 668277994873531131546094004060653155415727 \\ & \quad +10400114950554878959352664454883733411164160(k-1) \\ & \quad +75867603840583500904679025425153247200870400(k-1)^2 \\ & \quad +344362702240926543282997452509306552057856000(k-1)^3 \\ & \quad +1088550155865703890324832023114330710999040000(k-1)^4 \\ & \quad +2541000229559109600433119988914263477452800000(k-1)^5 \\ & \quad +4530928649775465308350805797614611595264000000(k-1)^6 \\ & \quad +6295443882813605556499487033900075581440000000(k-1)^7 \\ & \quad +6888301792756014127024691114710715596800000000(k-1)^8 \\ & \quad +595510011493061963451662767006679040000000000(k-1)^9 \\ & \quad +4054270203976961665595455761348034560000000000(k-1)^{10} \\ & \quad +2150765292754342181898829079720755200000000000(k-1)^{11} \\ & \quad +871564931093289201074214436601856000000000000(k-1)^{12} \\ & \quad +26081380974292471618423750656000000000000000(k-1)^{13} \\ & \quad +54354342663695314594163589120000000000000000(k-1)^{14} \\ & \quad +70482240618293614395229470720000000000000000(k-1)^{15} \\ & \quad +428413087325934213786501120000000000000000(k-1)^{16} > 0. \end{aligned}$$

In the similar process to Case 1, we have $L(r) > 64k + 2$.

$$\text{Let } s = \frac{625(64k+3)^4 - b_3 - 256}{256} = \frac{625(64k+3)^4 - 705}{256}.$$

The same technique of the Taylor expansion about 1 is used for the following algebraic expression

$$\begin{aligned}
 & (10s(64k+3)+3)^4 - (8s+3)^4(s+1) \\
 & = 6522182691627129366698257394526050 \\
 & \quad +99271809888737786023175689174466525(k-1) \\
 & \quad +708260082620689530531688729770735000(k-1)^2 \\
 & \quad +3144109142755200329883445563177100000(k-1)^3 \\
 & \quad +9720116809480917161028739153319050000(k-1)^4 \\
 & \quad +22190424775092205530308079652435200000(k-1)^5 \\
 & \quad +38697441968427874104549561320960000000(k-1)^6 \\
 & \quad +52583764932346668227730894848000000000(k-1)^7 \\
 & \quad +56268307020125628881185701888000000000(k-1)^8 \\
 & \quad +47573280080701744289624883200000000000(k-1)^9 \\
 & \quad +31674149437471491100183101440000000000(k-1)^{10} \\
 & \quad +16432337223384313110724608000000000000(k-1)^{11} \\
 & \quad +6512021529840412367257600000000000000(k-1)^{12} \\
 & \quad +1905689081114326466560000000000000000(k-1)^{13} \\
 & \quad +3883797885918747033600000000000000000(k-1)^{14} \\
 & \quad +492491137251100590080000000000000000(k-1)^{15} \\
 & \quad +29273397577908224000000000000000000(k-1)^{16} > 0.
 \end{aligned}$$

By positivity of coefficients of the Taylor series of $(10s(64k+3)+3)^4 - (8s+3)^4(s+1)$ in term of k about 1, we get $A(s) < 64k+3$.

Thus, for $r \leq n \leq s$, we obtain that

$$64k+2 < L(n) < S_n < A(n) < 64k+3.$$

$$\text{Hence } \lfloor S_n \rfloor = \lfloor A(n) \rfloor = 64k+2.$$

Case 3 $n \in B_{3,k}$

Let $r = \frac{625(64k+3)^4 - 449}{256}$. By using the Maclaurin series expansion, we gain

$$\begin{aligned}
 & (768r+288)^4(r+1) - (960r(64k+3)+323)^4 \\
 & = 156625164231863950511 + 58569627496076052622080k \\
 & \quad + 10194329686304320962969600k^2 \\
 & \quad + 1097187308676744111194112000k^3 \\
 & \quad + 81785824949797735225098240000k^4
 \end{aligned}$$

$$\begin{aligned}
 &+44799260872772250771652608000000k^5 \\
 &+1866370868767347099744337920000000k^6 \\
 &+60352908843588287865657753600000000k^7 \\
 &+1531601983605763075385327616000000000k^8 \\
 &+3061598521606783995425587200000000000k^9 \\
 &+480622656354369212671170969600000000000k^{10} \\
 &+586475431819508949402098073600000000000k^{11} \\
 &+5454620023724177977994379264000000000000k^{12} \\
 &+3738874287354554345324544000000000000000k^{13} \\
 &+17815929989992942934163456000000000000000k^{14} \\
 &+527356109875019020303859712000000000000000k^{15} \\
 &+730597318564762811010908160000000000000000k^{16} > 0.
 \end{aligned}$$

Consequently, we gain $L(r) > 64k + 3$.

Let $s = \frac{625(64k + 4)^4 - 768}{256}$. In the same process, we obtain $A(s) < 64k + 4$ and

conclude that $\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 64k + 3$.

For the remaining cases, to show that

$$\lfloor S_n \rfloor = \lfloor A(n) \rfloor = 64k + j, \text{ for } j = 4, 5, \dots, 64,$$

we use similar arguments to those in case 3, i.e., we expand

$$(768r + 288)^4 (r + 1) - (960r(64k + j) + 323)^4$$

and

$$(10s(64k + j + 1) + 3)^4 - (8s + 3)^4 (s + 1)$$

into the Maclaurin series in the following form

$$c_{j,0} + c_{j,1}k + c_{j,2}k^2 + \dots + c_{j,16}k^{16}$$

where $c_{j,k} > 0$. □

In an analogous way, we define approximated sequences as

$$A_5(n) = \left(\frac{5}{6} + \frac{1}{3n} \right) \sqrt[5]{n+1} - \frac{1}{3n}$$

and

$$A_6(n) = \left(\frac{6}{7} + \frac{5}{14n} \right) \sqrt[6]{n+1} - \frac{5}{14n}$$

for the fifth roots and the sixth roots respectively. Nonetheless, the numerical results show that

$$\lfloor A_5(146930) \rfloor = 9 \neq 8 = \left\lfloor \frac{1}{146930} \sum_{k=1}^{146930} \sqrt[5]{k} \right\rfloor$$

and

$$\lfloor A_6(661026) \rfloor = 8 \neq 7 = \left\lfloor \frac{1}{661026} \sum_{k=1}^{661026} \sqrt[6]{k} \right\rfloor.$$

These imply that the similar pattern of formulation for the estimated sequences $A_5(n)$ and $A_6(n)$ is not applicable.

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