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A Jacobian Matrix Approach to Transform Vector Calculus
Operators from Cartesian to Spherical Coordinates

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บทคัดย่อ

บทความนี้จะสำรวจเมทริกซ์จาโคเบียนจากทั้งมุมมองทางเรขาคณิตและเชิงวิเคราะห์ โดยเน้นที่การเกิดขึ้นของเมทริกซ์จากการแปลงองค์ประกอบปริมาตรกณิกนันต์ หลังจากนั้นจะสาธิตวิธีการใช้เมทริกซ์จาโคเบียน เพื่อแปลงตัวดำเนินการแคลคูลัสเวกเตอร์หลักจากพิกัดคาร์ทีเซียนเป็นพิกัดทรงกลม

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ABSTRACT

This paper will explore the Jacobian matrix from both geometric and analytical perspective, focusing on how it arises from the transformation of infinitesimal volume elements. Then, as a practical application, we will demonstrate how the Jacobian can be used to transform key vector calculus operators from Cartesian to spherical coordinates.

Keywords: Jacobian matrix, Vector field, Gradient, Divergence, Curl

1. Introduction

The Jacobian matrix is a mathematical tool that shows how a set of variables changes with respect to another set. Each entry in the matrix corresponds to a partial derivative. Its structure allows powerful ways to handle coordinate transformations and analyze local behavior of functions [1].

One particular application of the Jacobian is in vector calculus, where there are several key operators such as the gradient, divergence, and curl. While these operators are typically expressed in Cartesian coordinates, many problems—especially where spherical symmetry occurs—are easier to solve using spherical coordinates [3]. Therefore, to work effectively in spherical coordinates, these operators need to be transformed accordingly.

While it is possible to derive these transformations using the multivariable chain rule, such derivations are rarely fully shown in most calculus textbooks. The steps are often abbreviated or spread out across different topics, making it difficult for readers to grasp the full transformation process. The Jacobian approach therefore aims to offer a more systematic and unified method that ties together geometry, linear algebra, and differential calculus [4].

2. Spherical Coordinates

Before going into the derivations, let us review about the spherical coordinates. A problem featuring a spherical shape (a sphere, or a hemispherical shell) is often easier to solve using spherical coordinates [3].

Compared to the Cartesian coordinates of (x, y, z) , spherical coordinates are expressed as (r, θ, φ) where r is the radial distance from the origin, θ is the polar angle, and φ is the azimuthal angle [2]. Note that in this paper, we will use the physics convention as specified by the ISO standard 80000-2:2019.

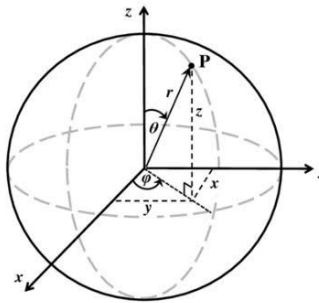


Figure 2.1 3D diagram of spherical coordinates.

From Figure 2.1, we can use the Cartesian coordinates to express spherical coordinates [2].

$$r = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

$$\cos \theta = \frac{z}{r} \quad (2)$$

$$\tan \varphi = \frac{y}{x} \quad (3)$$

On the contrary, we can also express Cartesian coordinates in terms of spherical coordinates by the following expressions:

$$x = r \cos \varphi \sin \theta \quad (4)$$

$$y = r \sin \varphi \sin \theta \quad (5)$$

$$z = r \cos \theta \quad (6)$$

In spherical coordinates, the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ point in directions of increasing r , θ , and φ , respectively. Unlike Cartesian unit vectors, these are not fixed

and vary with position. They are defined relative to the geometry of a sphere. However, their orientation can be expressed in terms of Cartesian unit vectors as follows [3]:

$$\hat{\mathbf{r}} = \hat{\mathbf{i}} \cos \varphi \sin \theta + \hat{\mathbf{j}} \sin \varphi \sin \theta + \hat{\mathbf{k}} \cos \theta \quad (7)$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{i}} \cos \varphi \cos \theta + \hat{\mathbf{j}} \sin \varphi \cos \theta - \hat{\mathbf{k}} \sin \theta \quad (8)$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{i}} \sin \varphi + \hat{\mathbf{j}} \cos \varphi \quad (9)$$

Geometrically, the radial unit vector $\hat{\mathbf{r}}$ points directly outward away from the origin. The polar unit vector $\hat{\boldsymbol{\theta}}$ is tangent to a vertical circle centered at the origin with a diameter on the z -axis. The azimuthal unit vector $\hat{\boldsymbol{\phi}}$ is tangent to a horizontal circle in the xy plane and centered on the z -axis in an anticlockwise sense (just as in 2D polar coordinates) [3]. Remember that these unit vectors are of magnitude one.

3. The Jacobian

The Jacobian matrix and determinant are used when transitioning between coordinate systems occurs. Geometrically, they can represent the factor by which areas or volumes are scaled when changing coordinates. Suppose we have a density function $f(x, y, z)$ and want to calculate its mass. We have to use an integral as the density of each point can vary. To simplify our calculation, we can switch to a different coordinate system u, v, w with a scale factor $\mathbf{J}$ (Jacobian). This can be represented below as:

$$\iiint f(x, y, z) dx dy dz = \iiint f(\alpha(u, v, w), \beta(u, v, w), \gamma(u, v, w)) \mathbf{J} du dv dw \quad (10)$$

Here, we have expressed (x, y, z) in terms of our new coordinates as functions of (u, v, w) denoted below.

$$x = \alpha(u, v, w)$$

$$y = \beta(u, v, w)$$

$$z = \gamma(u, v, w)$$

We know that $dx dy dz$ represents the infinitesimal volume element (a rectangular prism) in Cartesian coordinates. However, after applying the transformation, this volume element is distorted and becomes a shape called a parallelepiped.

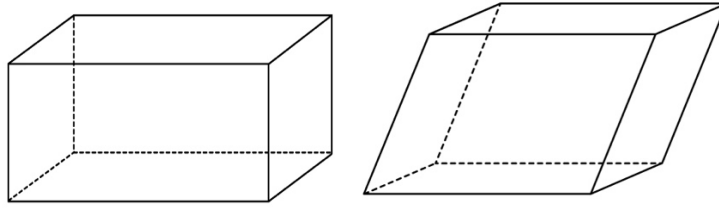


Figure 3.1 A rectangular prism and a parallelepiped.

We can label each side of the parallelepiped with a vector.

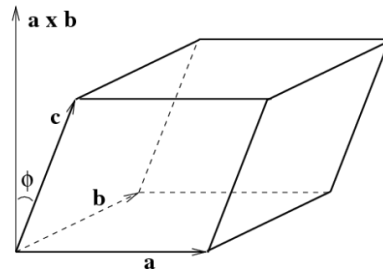


Figure 3.2 Parallelepiped labelled with vectors **a**, **b**, **c**.

We know that the volume of the parallelepiped is calculated by:

$$\begin{aligned} V &= \text{area of base} \cdot \text{height} \\ &= (|\mathbf{a}||\mathbf{b}| \sin \theta) \cdot |\mathbf{c}| \cos \phi \\ &= |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| \end{aligned}$$

In order to find the vectors **a**, **b**, **c**, we can use the relationship between our coordinates system. In Figure 3.1, we consider the sides of the infinitesimal volume element of the rectangular prism to be vectors $dx\hat{i}$, $dy\hat{j}$, $dz\hat{k}$ respectively. Once we have changed into our new coordinates, the vectors will be shifted. Given that x, y, z are functions of u, v, w , the transformed vectors could have components in the $\hat{i}, \hat{j}, \hat{k}$ directions after applying the transformation. We can represent these changes with the following expressions.

$$\begin{aligned} dx &\rightarrow \frac{\partial x}{\partial u} \delta u \hat{i} + \frac{\partial x}{\partial v} \delta v \hat{j} + \frac{\partial x}{\partial w} \delta w \hat{k} \\ dy &\rightarrow \frac{\partial y}{\partial u} \delta u \hat{i} + \frac{\partial y}{\partial v} \delta v \hat{j} + \frac{\partial y}{\partial w} \delta w \hat{k} \\ dz &\rightarrow \frac{\partial z}{\partial u} \delta u \hat{i} + \frac{\partial z}{\partial v} \delta v \hat{j} + \frac{\partial z}{\partial w} \delta w \hat{k} \end{aligned}$$

These new vectors now represent the vectors **a**, **b**, **c** respectively. To find the volume of the parallelepiped, we will take the triple product of **a**, **b** and **c**. This can be computed by finding the determinant of a 3x3 matrix (Jacobian matrix) arranged in the order below. Note that we have factored out $\delta u \delta v \delta w$.

$$dx dy dz \rightarrow \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix} \delta u \delta v \delta w = \mathbf{J} \delta u \delta v \delta w$$

We have now shown how the volume element has been transformed under our new coordinates. Furthermore, as the infinitesimal quantities $\delta u, \delta v, \delta w$ approaches zero, they will converge to their differential forms of du, dv, dw . Hence, our new volume integral can be written as:

$$\iiint dx dy dz = \iiint \mathbf{J} du dv dw$$

Going back to equation (10), since we have already parameterized our density function using the new coordinates, we can now calculate the mass using the transformed volume integral. Thus, we have finally showed how the Jacobian (the determinant of the 3x3 matrix in the previous page) behaves like a factor when changing between coordinate systems.

Lastly, the Jacobian matrix can be generally expressed as:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

4. The Gradient

The gradient operator defined by the symbol $\vec{\nabla}$ (del, nabla) and has partial derivatives as its components [2]. In Cartesian coordinates, it is represented as:

$$\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \quad (11)$$

We can apply the gradient operator to a multivariable function, resulting in the gradient vector.

$$\vec{\nabla}f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Similar to how taking the derivative of a function would give the rate of change at any point, the gradient operator not only gives the rate of change but also indicates the direction of the fastest increase for a multivariable function.

5. The Divergence

The divergence of a vector function is the scalar product between the gradient operator and a vector function [2]. In Cartesian coordinates, it is represented as:

$$\begin{aligned}\vec{\nabla} \cdot \vec{F} &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\end{aligned}\tag{12}$$

You can think of the divergence as a way to represent the rate at which a vector field spreads out or converges into a given point in space. A value greater than zero would mean that the field lines (flux) are radiating outwards, while a value less than zero would mean that the field lines (flux) are converging inwards [2].

6. The Curl

The curl of a vector function is the vector product between the gradient operator and a vector function [2]. We can carry out this operation by setting up a 3×3 matrix and calculating its determinant. The result of calculating the curl is a vector. In Cartesian coordinates, it is represented as:

$$\begin{aligned}\vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_y & F_z \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_x & F_z \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_x & F_y \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}\end{aligned}$$

You can think of the curl as a measure of the net circulation of field lines at a point. For example, when an electric current is running through a wire, it creates a magnetic field around the wire. By calculating the curl, we can determine both the direction in which the magnetic field rotates and its rate (magnitude) at any given point.

7. Gradient Operator in Spherical Coordinates

Now that we have familiarize ourselves with the Jacobian and all the vector calculus operators, let's see how the Jacobian matrix can be used to transform the gradient from Cartesian to spherical.

Firstly, the gradient operator in Cartesian is represented by equation (11). However, we want it to be in terms of r, θ, φ . We can use the chain rule which will give us:

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial}{\partial z} \frac{\partial z}{\partial r} \quad (13)$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial}{\partial z} \frac{\partial z}{\partial \theta} \quad (14)$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \varphi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \varphi} + \frac{\partial}{\partial z} \frac{\partial z}{\partial \varphi} \quad (15)$$

We can now use matrix notation to represent equations (13), (14), and (15) altogether. The matrix on the right side is the Jacobian matrix (**J**).

$$\begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix}$$

Going back to equations (4), (5), and (6). We can take the partial derivative with respect to each variables, giving us:

$$\begin{aligned} & \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \cos \varphi \sin \theta & r \cos \varphi \cos \theta & -r \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & r \sin \varphi \cos \theta & r \cos \varphi \sin \theta \\ \cos \theta & -r \sin \theta & 0 \end{bmatrix} \\ & \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \mathbf{J} = \vec{\nabla} \end{aligned}$$

Here, the Jacobian is acting as a scaling factor to the gradient in Cartesian. Solving for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, we can take the inverse of the Jacobian matrix.

$$\begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

Although the detailed steps for computing the inverse of a 3×3 matrix especially one involving trigonometric functions are complicated and requires lots of algebraic manipulation, the final result is provided for completeness. The detailed calculations are omitted to maintain focus on the primary analysis.

$$\mathbf{J}^{-1} = \begin{bmatrix} \frac{\cos \varphi \sin \theta}{r} & \frac{\sin \varphi \sin \theta}{r} & \frac{\cos \theta}{r} \\ \frac{\cos \varphi \cos \theta}{r} & \frac{\sin \varphi \cos \theta}{r} & -\frac{\sin \theta}{r} \\ -\frac{\sin \varphi}{r \sin \theta} & \frac{\cos \varphi}{r \sin \theta} & 0 \end{bmatrix}$$

One property of the dot product is that when two perpendicular vectors are dotted together, the answer will be zero. We can utilize this property to find the coefficients of each of the components by multiplying the transformed gradient operator with the spherical unit vectors $\hat{r}, \hat{\theta}, \hat{\varphi}$ respectively.

$$\begin{aligned}\vec{\nabla} \cdot \hat{r} &= \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{\partial}{\partial r} \\ \vec{\nabla} \cdot \hat{\theta} &= \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} \cos \varphi \cos \theta \\ \sin \varphi \cos \theta \\ -\sin \theta \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ r \end{bmatrix} = \frac{1}{r} \frac{\partial}{\partial \theta} \\ \vec{\nabla} \cdot \hat{\varphi} &= \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ r \sin \theta \end{bmatrix} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}\end{aligned}$$

We have finally derived the gradient operator in spherical coordinates using the Jacobian matrix and applying its inverse. Thus, the gradient in spherical coordinates is expressed as:

$$\vec{\nabla} = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \hat{\varphi}$$

Compare to just using the chain rule, using the Jacobian matrix provides a more structured and organized approach. By encapsulating all the partial derivatives into a single matrix, it shows how each component changes across coordinate systems.

8. The Divergence in Spherical Coordinates

We know that the divergence in Cartesian coordinates is defined by equation (12). It is the dot product between the gradient operator and a vector field. In order to transform it into spherical coordinates, we can first take the same approach as we did for the gradient (equations (13), (14), and (15)).

$$\begin{bmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{bmatrix} \begin{bmatrix} \frac{\cos \varphi \sin \theta}{r} & \frac{\sin \varphi \sin \theta}{r} & -\frac{\cos \theta}{r} \\ \frac{\cos \varphi \cos \theta}{r} & \frac{\sin \varphi \cos \theta}{r} & \frac{\sin \theta}{r} \\ -\frac{\sin \varphi}{r \sin \theta} & \frac{\cos \varphi}{r \sin \theta} & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

Here, we have used the inverse of the Jacobian matrix to obtain $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in terms of r, θ, φ . Next, let us express F_x, F_y, F_z in terms of F_r, F_θ, F_φ . This can be done by manipulating the unit vectors.

Firstly, writing equations (7), (8), and (9) in matrix notation.

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \cos \varphi \sin \theta & \sin \varphi \sin \theta & \cos \theta \\ \cos \varphi \cos \theta & \sin \varphi \cos \theta & -\sin \theta \\ -\sin \varphi & \cos \varphi & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix}$$

As before, in order to express $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, we can take the inverse of the matrix. However, since this is an orthogonal matrix, the inverse is simply its transpose.

$$\begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} = \begin{bmatrix} \cos \varphi \sin \theta & \cos \varphi \cos \theta & -\sin \varphi \\ \sin \varphi \sin \theta & \sin \varphi \cos \theta & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}$$

We can now use this relationship to express F_x, F_y, F_z in terms of F_r, F_θ, F_ϕ .

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \cos \varphi \sin \theta & \cos \varphi \cos \theta & -\sin \varphi \\ \sin \varphi \sin \theta & \sin \varphi \cos \theta & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix}$$

Multiplying out the matrix will give us:

$$\begin{aligned} F_x &= \cos \varphi \sin \theta F_r + \cos \varphi \cos \theta F_\theta - \sin \varphi F_\phi \\ F_y &= \sin \varphi \sin \theta F_r + \sin \varphi \cos \theta F_\theta + \cos \varphi F_\phi \\ F_z &= \cos \theta F_r - \sin \theta F_\theta \end{aligned}$$

Putting this altogether will give us:

$$\vec{\nabla} \cdot \vec{F} = \left[\frac{\partial}{\partial r} \quad \frac{\partial}{\partial \theta} \quad \frac{\partial}{\partial \varphi} \right] \mathbf{J}^{-1} \begin{bmatrix} \cos \varphi \sin \theta F_r & \cos \varphi \cos \theta F_\theta & -\sin \varphi F_\phi \\ \sin \varphi \sin \theta F_r & \sin \varphi \cos \theta F_\theta & \cos \varphi F_\phi \\ \cos \theta F_r & -\sin \theta F_\theta & 0 \end{bmatrix}$$

Again, we shall omit the algebraic manipulation in order to maintain focus on the primary analysis. Thus, after all of the tedious algebraic manipulation, we are left with:

$$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial F_r}{\partial r} + \frac{2}{r} F_r \right) + \left(\frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\cos \theta}{r \sin \theta} F_\theta \right) + \left(\frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \varphi} \right)$$

Notice that the first two terms come from the product rule for derivatives. We can reverse this to help simplifying the first two terms. Hence, the divergence in spherical coordinates is given as:

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 F_r + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} F_\theta \sin \theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} F_\phi$$

9. The Curl in Spherical Coordinates

Before we start deriving the curl in spherical coordinates, let us look at another notation of the curl involving unit vectors first in Cartesian coordinates.

$$\vec{\nabla} \times \vec{F} = \left(\hat{i} \times \frac{\partial \vec{F}}{\partial x} \right) + \left(\hat{j} \times \frac{\partial \vec{F}}{\partial y} \right) + \left(\hat{k} \times \frac{\partial \vec{F}}{\partial z} \right)$$

This notation helps us visualize how each component of the curl is related to the changes in the vector field in the directions of the coordinate axes. It highlights how the curl is constructed from partial derivatives in different directions.

Now, suppose we have a vector field in spherical coordinate expressed as:

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi}$$

From Section 7, we have already found the coefficients of the unit vectors when it is transformed into spherical coordinates. Thus, we can write the following:

$$\vec{\nabla} \times \vec{F} = \left(\hat{r} \times \frac{\partial \vec{F}}{\partial r} \right) + \left(\frac{\hat{\theta}}{r} \times \frac{\partial \vec{F}}{\partial \theta} \right) + \left(\frac{\hat{\phi}}{r \sin \theta} \times \frac{\partial \vec{F}}{\partial \phi} \right)$$

In order to make our derivation easier, let us work with each component separately. Starting with $\hat{r}$ component:

$$\hat{r} \times \frac{\partial \vec{F}}{\partial r} = \hat{r} \times \frac{\partial (F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi})}{\partial r}$$

We shall use the product rule on the second term. This will give us:

$$\frac{\partial (F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi})}{\partial r} = \hat{r} \frac{\partial F_r}{\partial r} + \hat{\theta} \frac{\partial F_\theta}{\partial r} + \hat{\phi} \frac{\partial F_\phi}{\partial r}$$

Note that we have referred to equations (7), (8), and (9) for the partial derivatives of $\hat{r}, \hat{\theta}, \hat{\phi}$ with respect to r . Next, we will cross it with the unit vector $\hat{r}$.

$$\begin{aligned} \hat{r} \times \frac{\partial \vec{F}}{\partial r} &= \hat{r} \times \left(\hat{r} \frac{\partial F_r}{\partial r} + \hat{\theta} \frac{\partial F_\theta}{\partial r} + \hat{\phi} \frac{\partial F_\phi}{\partial r} \right) \\ &= \hat{\phi} \frac{\partial F_\theta}{\partial r} - \hat{\theta} \frac{\partial F_\phi}{\partial r} \end{aligned} \tag{16}$$

Moving on to the $\hat{\theta}$ component.

$$\frac{\hat{\theta}}{r} \times \frac{\partial \vec{F}}{\partial \theta} = \frac{\hat{\theta}}{r} \times \frac{\partial (F_r \hat{r} + F_\theta \hat{\theta} + F_\varphi \hat{\varphi})}{\partial \theta}$$

Using the product rule again on the second term.

$$\frac{\partial (F_r \hat{r} + F_\theta \hat{\theta} + F_\varphi \hat{\varphi})}{\partial \theta} = \hat{r} \frac{\partial F_r}{\partial \theta} + F_r \hat{\theta} + \hat{\theta} \frac{\partial F_\theta}{\partial \theta} - F_\theta \hat{r} + \hat{\varphi} \frac{\partial F_\varphi}{\partial \theta}$$

Note that we have referred to equations (7), (8), and (9) again for the partial derivatives of $\hat{r}, \hat{\theta}, \hat{\varphi}$ with respect to θ . Next, we will cross it with the unit vector $\hat{\theta}$.

$$\begin{aligned} \frac{\hat{\theta}}{r} \times \frac{\partial \vec{F}}{\partial \theta} &= \frac{\hat{\theta}}{r} \times \left(\hat{r} \frac{\partial F_r}{\partial \theta} + F_r \hat{\theta} + \hat{\theta} \frac{\partial F_\theta}{\partial \theta} - F_\theta \hat{r} + \hat{\varphi} \frac{\partial F_\varphi}{\partial \theta} \right) \\ &= -\frac{\hat{\varphi}}{r} \frac{\partial F_r}{\partial \theta} + \frac{\hat{\varphi}}{r} F_\theta + \frac{\hat{r}}{r} \frac{\partial F_\varphi}{\partial \theta} \end{aligned} \quad (17)$$

Lastly, consider the $\hat{\varphi}$ component. Since this requires a lot more algebraic manipulation and space, we shall omit the working out and skip to the final answer.

$$\frac{\hat{\varphi}}{r \sin \theta} \times \frac{\partial \vec{F}}{\partial \varphi} = \frac{\hat{\theta}}{r \sin \theta} \frac{\partial F_r}{\partial \varphi} - \frac{\hat{r}}{r \sin \theta} \frac{\partial F_\theta}{\partial \varphi} - F_\varphi \frac{\hat{\theta}}{r} + F_\varphi \cos \theta \frac{\hat{r}}{r \sin \theta} \quad (18)$$

Combine equations (16), (17), and (18) together and group all the $\hat{r}, \hat{\theta}, \hat{\varphi}$ together, respectively.

$$\frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} \sin \theta F_\varphi - \frac{\partial F_\theta}{\partial \varphi} \right] \hat{r} - \frac{1}{r} \left[\frac{\partial}{\partial r} r F_\varphi - \frac{1}{\sin \theta} \frac{\partial F_r}{\partial \varphi} \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} r F_\theta - \frac{\partial F_r}{\partial \theta} \right] \hat{\varphi}$$

Here we have taken out the common factors and use the product rule for derivatives to simplify things. To finalize our derivation, we can write it in the conventional matrix form (determinant of a 3x3 matrix).

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\varphi} \\ \frac{r^2 \sin \theta}{\partial} & \frac{r \sin \theta}{\partial} & \frac{r}{\partial} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & r F_\theta & r \sin \theta F_\varphi \end{vmatrix} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r \hat{\theta} & r \sin \theta \hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & r F_\theta & r \sin \theta F_\varphi \end{vmatrix}$$

Thus, we have finally derived the curl in spherical coordinates.

10. Conclusion

In this paper, we have explored the Jacobian and its role in deriving vector calculus operators from Cartesian to spherical coordinates [3]. By visualizing how the Jacobian arises from the deformation of infinitesimal volume elements, we provided a clear and intuitive framework for understanding coordinate transformations [1]. This approach allowed us to derive the gradient, divergence, and curl in spherical coordinates, offering an alternative approach to the abbreviated derivations often seen in traditional calculus textbooks. Future work could extend this framework to more complex coordinate systems such as the curvilinear systems, or integrate them with numerical methods for even broader applications [4].

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