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The Solution Sets of $\lceil x\rceil^2 - c = 0$, $\lceil x^2\rceil - c = 0$ and $x\lceil x\rceil - c = 0$ where c is a Real Number

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Abstract. The ceiling function of a real number x, denoted by $\lceil x\rceil$, is the least integer greater than or equal to x. In this article, the solution sets of $\lceil x\rceil^2 - c = 0, \lceil x^2\rceil - c = 0$ and $x\lceil x\rceil - c = 0$, where c is a real number are shown.

Keywords: quadratic equation, solution set, the ceiling function

1. Introduction

Finding solutions of the equation is a basic study in mathematics. Starting with the linear equation $ax - b = 0$, where a and b are real number and $a \neq 0$, we see that $x = \frac{b}{a}$ is the solution. For the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, we see that if $b^2 - 4ac \geq 0$, then the equation has real solutions and $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ are the solution.

The floor function of a real number x, denoted by $\lfloor x\rfloor$, is the greatest integer less than or equal to x. The function is introduced in calculus. In 2020, Matsko [3] interested in the floor function and replaced x in the quadratic equation $x^2 + bx + c = 0$ by $\lfloor x\rfloor$. The behaviors of solutions of equations: (1) $\lfloor x\rfloor^2 + b\lfloor x\rfloor + c = 0$, (2)$\lfloor x\rfloor^2 + bx + c = 0$, (3) $x\lfloor x\rfloor + bx + c = 0$, (4) $\lfloor x^2\rfloor + bx + c = 0$, (5) $x\lfloor x\rfloor + b\lfloor x\rfloor + c = 0$, (6) $x^2 + b\lfloor x\rfloor + c = 0$, and (7) $\lfloor x^2\rfloor + b\lfloor x\rfloor + c = 0$, are posed. The solutions are depended on b and c. In 2024, W. Kitcharoensubdee et. al. [2] obtained the solution sets of the quadratic equations,

(Eq.1) $\lfloor x\rfloor^2 - c = 0$, (Eq.2) $\lfloor x^2\rfloor - c = 0$,
(Eq.3) $x\lfloor x\rfloor - c = 0$, (Eq.4) $\lfloor x\rfloor^2 + b\lfloor x\rfloor = 0$,
(Eq.5) $x\lfloor x\rfloor + bx = 0$, (Eq.6) $x\lfloor x\rfloor + b\lfloor x\rfloor = 0$,
(Eq.7) $\lfloor x\rfloor^2 + bx = 0$, (Eq.8) $\lfloor x\rfloor^2 + b\lfloor x\rfloor = 0$,

by taking $b = 0$ or $c = 0$ in the quadratic equations of Matsko. They said that it is easy to find the solution set of the linear equation $a\lfloor x\rfloor + b = 0$. The following theorem shows the solution sets of the first three equations, (Eq.1) - (Eq.3).

Theorem 1.1 [2]
(i) $\lfloor x\rfloor^2 - c = 0$ has a solution if and only if c is a square integer. In the case that it has a solution, the solution set is $[-\sqrt{c}, -\sqrt{c}+1) \cup [\sqrt{c}, \sqrt{c}+1)$.
(ii) $\lfloor x^2\rfloor - c = 0$ has a solution if and only if $c \in \mathbb{N} \cup \{0\}$. In the case that it has a solution, the solution set is $(-\sqrt{c+1}, -\sqrt{c}] \cup [\sqrt{c}, \sqrt{c+1})$.

Theorem 1.2 [2] Let $c \neq 0$ and S be a solution set of $x\lfloor x\rfloor - c = 0$.

(i) If there is an $n \in \mathbb{N}$ such that $c \in (n^2, n(n+1))$, then $S = \left\{\frac{c}{n}\right\}$.

(ii) If there is an $n \in \mathbb{N}$ such that $c \in ((n-1)n, n^2)$, then $S = \left\{-\frac{c}{n}\right\}$.

(iii) If there is an $n \in \mathbb{N}$ such that $c = n^2$, then $S = \{-n, n\}$.

(iv) If $c < 0$ or there is an $n \in \mathbb{N}$ such that $c = n(n+1)$, then $S = \emptyset$.

In Chapter 3 of calculus textbook [1], the authors presented the ceiling function of a real number x, denoted by $\lceil x\rceil$. It is the least integer greater than or equal to x. Some properties about the ceiling function are the following.

Let x be a real number and n be an integer.
1. $\lceil x \rceil = x$ if and only if x is an integer.
2. $x \leq \lceil x \rceil < x + 1$.
3. $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$.

We are interested in the solution sets of the linear equation and the quadratic equation when x is replaced by $\lceil x \rceil$. For the linear equation $a\lceil x \rceil + b = 0$, where a and b are real numbers and $a \neq 0$, we see that such equation has a solution if and only if a divides b. If it has a solution, then $\left(-\frac{b}{a} - 1, -\frac{b}{a}\right]$ is the solution set. In this article, the solution sets of equations, $\lceil x \rceil^2 - c = 0$, $\lceil x^2 \rceil - c = 0$ and $x\lceil x \rceil - c = 0$, where c is a real number, are shown.

2. Results

In this section, we obtain sufficient and necessary conditions for equations, $\lceil x \rceil^2 - c = 0$, $\lceil x^2 \rceil - c = 0$ and $x\lceil x \rceil - c = 0$, where c is a real number, to have solutions and then the solution sets are shown. First, we begin with $\lceil x \rceil^2 - c = 0$.

Theorem 2.1 Let c be a real number. Then, $\lceil x \rceil^2 - c = 0$ has a solution if and only if c is a square integer.

Proof. Let c be a real number. Assume that $\lceil x \rceil^2 - c = 0$ has a solution. Let a be a solution of $\lceil x \rceil^2 - c = 0$. Then, $\lceil a \rceil^2 - c = 0$. That is, $c = \lceil a \rceil^2$. Since $\lceil a \rceil$ is an integer, c is a square integer.

Next, assume that c is a square integer. There is an integer n such that $c = n^2$. Since $n^2 = \lceil n \rceil^2$, $\lceil n \rceil^2 - c = c - c = 0$. Therefore, $\lceil x \rceil^2 - c = 0$ has a solution.

Theorem 2.2 Let c be a real number. If $\lceil x \rceil^2 - c = 0$ has a solution, then the set

$$(-\sqrt{c} - 1, -\sqrt{c}] \cup (\sqrt{c} - 1, \sqrt{c}]$$

is the solution set.

Proof. Let c be a real number and S be the solution set of $\lceil x \rceil^2 - c = 0$. Assume that $\lceil x \rceil^2 - c = 0$ has a solution. By Theorem 2.1, c is a square integer. There is an integer n such that $c = n^2 \geq 0$. Assume that $a \in S$. Then, $\lceil a \rceil^2 - n^2 = 0$. Thus, $\lceil a \rceil = -n$ or $\lceil a \rceil = n$. This implies that

$$a \in (-n - 1, -n] \cup (n - 1, n].$$

Since $c = n^2$, $\sqrt{c} = n$. Then, $S \subseteq (-\sqrt{c} - 1, -\sqrt{c}] \cup (\sqrt{c} - 1, \sqrt{c}]$.

Next, assume that $a \in (-\sqrt{c} - 1, -\sqrt{c}] \cup (\sqrt{c} - 1, \sqrt{c}]$. Then, $\lceil a \rceil = -n$ or $\lceil a \rceil = n$. This implies that $\lceil a \rceil^2 - n^2 = n^2 - n^2 = 0$. Thus, $a \in S$.

Therefore, $S = (-\sqrt{c} - 1, -\sqrt{c}] \cup (\sqrt{c} - 1, \sqrt{c}]$ is the solution set of $\lceil x \rceil^2 - c = 0$.

Next, we consider $\lceil x^2 \rceil - c = 0$.

Theorem 2.3 Let c be a real number. Then, $\lceil x^2 \rceil - c = 0$ has a solution if and only if c is a non-negative integer.

Proof. Let c be a real number and a be a solution of $\lceil x^2 \rceil - c = 0$. Then, $\lceil a^2 \rceil - c = 0$. That is, $\lceil a^2 \rceil = c$. Since $a^2 \geq 0$ and $\lceil a^2 \rceil$ is an integer, $c = \lceil a^2 \rceil$ is a non-negative integer. Next, assume that c is a non-negative integer. Then, $\lceil (\sqrt{c})^2 \rceil - c = c - c = 0$. That is, $\sqrt{c}$ is a solution of $\lceil x^2 \rceil - c = 0$. Therefore, $\lceil x^2 \rceil - c = 0$ has a solution.

By Theorem 2.3, we see that if $c < 0$ or c is a real number such that c is not an integer, then $\lceil x^2 \rceil - c = 0$ has no solutions. The following theorem shows the solution set of $\lceil x^2 \rceil - c = 0$.

Theorem 2.4 Let c be a real number and S be the solution set of $\lceil x^2 \rceil - c = 0$.

1. If $c = 0$, then $S = \{0\}$.
2. If c is a positive integer, then $S = [-\sqrt{c}, -\sqrt{c - 1}) \cup (\sqrt{c - 1}, \sqrt{c}]$.

Proof. Let c be a real number and S be the solution set of $\lceil x^2 \rceil - c = 0$.

1. If $c = 0$, by Theorem 2.3, $\lceil x^2 \rceil - c = 0$ has a solution. Let $a \in S$. Then, $\lceil a^2 \rceil = 0$. That is, $a^2 = 0$. Then, $a = 0$. Thus, $S \subseteq \{0\}$. Since $\lceil 0^2 \rceil = 0$, $\{0\} \subseteq S$. Therefore, $S = \{0\}$ is the solution set of $\lceil x^2 \rceil - c = 0$.

2. If c is a positive integer, by Theorem 2.3, $\lceil x^2 \rceil - c = 0$ has a solution. Let

$a \in S$. Then, $\lceil a^2 \rceil - c = 0$. That is, $\lceil a^2 \rceil = c$. Then, $a^2 \in (c-1, c]$, i.e. $c - 1 < a^2 \leq c$. Thus, $\sqrt{c-1} < |a| \leq \sqrt{c}$. That is, $\sqrt{c-1} < |a|$ and $|a| \leq \sqrt{c}$.

Then, $(-\sqrt{c-1} < a$ or $\sqrt{c-1} > a)$ and $-\sqrt{c} \leq a \leq \sqrt{c}$. Thus, $a \in [-\sqrt{c}, -\sqrt{c-1}) \cup (\sqrt{c-1}, \sqrt{c}]$. This implies that $S \subseteq [-\sqrt{c}, -\sqrt{c-1}) \cup (\sqrt{c-1}, \sqrt{c}]$.

Next, assume that $a \in [-\sqrt{c}, -\sqrt{c-1}) \cup (\sqrt{c-1}, \sqrt{c}]$. We see that $c - 1 < a^2 \leq c$. Then, $\lceil a^2 \rceil = c$. That is, $\lceil a^2 \rceil - c = 0$. Thus, $[-\sqrt{c}, -\sqrt{c-1}) \cup (\sqrt{c-1}, \sqrt{c}] \subseteq S$.

Therefore, $S = [-\sqrt{c}, -\sqrt{c-1}) \cup (\sqrt{c-1}, \sqrt{c}]$ is the solution set of $\lceil x^2 \rceil - c = 0$.

Finally, we consider $x\lceil x \rceil - c = 0$ in Lamma 2.5 - 2.10 depending on the value c.

Lemma 2.5 Let c be a real number and S be the solution set of $x\lceil x \rceil - c = 0$. If $c < 0$, then $S = \emptyset$.

Proof. Let c be a real number such that $c < 0$. Suppose that $S \neq \emptyset$. Then, there is a real number $a \in S$. That is, $a\lceil a \rceil = c$. Since $c < 0$, $a\lceil a \rceil < 0$.

If $a > 0$, then $\lceil a \rceil > 0$. This implies that $a\lceil a \rceil > 0$, a contradiction.

If $a \leq 0$, then $\lceil a \rceil \leq 0$. This implies that $a\lceil a \rceil \geq 0$, a contradiction.

Therefore, $S = \emptyset$.

Lemma 2.6 Let c be a real number and S be the solution set of $x\lceil x \rceil - c = 0$. If $c = 0$, then $S = (-1,0]$.

Proof. Let $c = 0$ and S be the solution set of $x\lceil x \rceil - c = 0$.

1. We will show that $(-1,0] \subseteq S$. Assume that $x \in (-1,0]$. Then, $\lceil x \rceil = 0$. That is, $x\lceil x \rceil = 0$. Thus, $(-1,0] \subseteq S$.

2. We will show that $S \subseteq (-1,0]$. Suppose that there is a real number $a \in S$ such that $a \notin (-1,0]$. Then, $a\lceil a \rceil = 0$.

If $a > 0$, then $\lceil a \rceil > 0$. Thus, $a\lceil a \rceil > 0$, a contradiction.

If $a \leq -1$, then $\lceil a \rceil \leq -1$. Thus, $a\lceil a \rceil > 0$, a contradiction.

Therefore, $S = (-1,0]$ is the solution set of $x\lceil x \rceil = 0$.

Lemma 2.7 Let c be a real number and S be the solution set of $x\lceil x \rceil - c = 0$. If there is a positive integer n such that $c \in ((n-1)n, n^2)$, then $S = \left\{\frac{c}{n}\right\}$.

Proof. Let c be a real number and S be the solution set of $x\lceil x \rceil - c = 0$. Assume that there is a positive integer n such that $c \in ((n-1)n, n^2)$. That is, $0 \leq (n-1)n < c < n^2$. Then, $n - 1 < \frac{c}{n} < n$. This implies that $\left\lceil \frac{c}{n} \right\rceil = n$. We will show that $S = \left\{\frac{c}{n}\right\}$.

1. We will show that $\left\{\frac{c}{n}\right\} \subseteq S$. Since $\frac{c}{n}\left\lceil \frac{c}{n} \right\rceil - c = \frac{c}{n}(n) - c = 0$, $\frac{c}{n}$ is a solution of $x\lceil x \rceil - c = 0$. Then, $\left\{\frac{c}{n}\right\} \subseteq S$.

2. We will show that $S \subseteq \left\{\frac{c}{n}\right\}$. Suppose that there is a real number $a \in S$ such that $a \neq \frac{c}{n}$. Since $c \in ((n-1)n, n^2)$, we see that $n^2 - c > 0$, $(n-1)n - c < 0$ and $n - 1 < \frac{c}{n} < n$. Then, $\left\lceil \frac{c}{n} \right\rceil = n$.

Case 1: $a > \frac{c}{n}$. Then, $\lceil a \rceil \geq \left\lceil \frac{c}{n} \right\rceil$. Thus, $a\lceil a \rceil - c > \frac{c}{n}(n) - c = c - c = 0$, a contradiction.

Case 2: $0 \leq a < \frac{c}{n}$. Then, $\lceil a \rceil \leq \left\lceil \frac{c}{n} \right\rceil$. Thus, $a\lceil a \rceil - c < \frac{c}{n}(n) - c = c - c = 0$, a contradiction.

Case 3: $-n < a < 0$. Then, $-(n-1) \leq a < 0$. We see that, $-(n-1) \leq \lceil a \rceil \leq 0$. Thus, $a\lceil a \rceil - c < (-n)(-(n-1)) - c = n(n-1) - c < 0$, a contradiction

Case 4: $a \leq -n$. Then, $\lceil a \rceil \leq \lceil -n \rceil = -n$. Thus, $a\lceil a \rceil - c > (-n)(-n) - c = n^2 - c > 0$, a contradiction.

From cases 1, 2 and 3, we can conclude that $S \subseteq \left\{\frac{c}{n}\right\}$.

Therefore, we can conclude that $S = \left\{\frac{c}{n}\right\}$.

Lemma 2.8 Let c be a real number and S be the set of solution of $x\lceil x\rceil - c = 0$. If there is a positive integer n such that $c = n^2$, then $S = \{-n, n\}$.

Proof. Let c be a real number and assume that there is a positive integer n such that $c = n^2$. That is, $c > 0$. We will show that $S = \{-n, n\}$.

1. We will show that $\{-n, n\} \subseteq S$

If $x = -n$, then $x\lceil x\rceil - c = (-n)\lceil -n\rceil - n^2 = n^2 - n^2 = 0$. Thus, $-n$ is a solution of $x\lceil x\rceil - c = 0$.

If $x = n$, then $x\lceil x\rceil - c = (n)\lceil n\rceil - n^2 = n^2 - n^2 = 0$. Thus, n is a solution of$x\lceil x\rceil - c = 0$.

Thus, $\{-n, n\} \subseteq S$.

2. We will show that $S \subseteq \{-n, n\}$. Suppose that there is a real number $a \in S$ such that $a > n$ or $-n < a < n$ or $a < -n$.

Case 1: $a > n$. Then, $\lceil a\rceil > n$. We see that $a\lceil a\rceil - c > n(n) - n^2 = 0$, a contradiction.

Case 2: $-n < a < n$.

If $0 < a < n$, then $\lceil a\rceil \le n$. Thus, $a\lceil a\rceil - c < n(n) - n^2 = 0$, a contradiction.

If $-n < a \le 0$, then $\lceil a\rceil > -n$. Thus, $a\lceil a\rceil - c < (-n)(-n) - n^2 = 0$, a contradiction.

Case 3: $a < -n$. Then, $\lceil a\rceil \le -n$. Thus, $a\lceil a\rceil - c > (-n)(-n) - n^2 = 0$, a contradiction.

From cases 1 and 2, $S \subseteq \{-n, n\}$.

Therefore, we can conclude that $S = \{-n, n\}$.

Lemma 2.9 Let c be a real number and S be the solution set of $x\lceil x\rceil - c = 0$. If there is a positive integer n such that $c \in (n^2, n(n+1))$, then $S = \left\{-\frac{c}{n}\right\}$.

Proof. Let c be a real number and S be the set of solution of $x\lceil x\rceil - c = 0$. Assume that there is a positive integer n such that $c \in (n^2, n(n+1))$. That is, $0 < n^2 < c < n(n+1)$. Then, $n < \frac{c}{n} < n+1$ and $-(n+1) < -\frac{c}{n} < -n$. Thus, $\left\lceil -\frac{c}{n}\right\rceil = -n$. We will show that $S = \left\{-\frac{c}{n}\right\}$.

1. We will show that $\left\{-\frac{c}{n}\right\} \subseteq S$. Since $-\frac{c}{n}\left\lceil -\frac{c}{n}\right\rceil - c = -\frac{c}{n}(-n) - c = 0$, $-\frac{c}{n}$ is a solution of $x\lceil x\rceil - c = 0$. Then, $\left\{-\frac{c}{n}\right\} \subseteq S$.

2. We will show that $S \subseteq \left\{-\frac{c}{n}\right\}$. Suppose that there is a real number $a \in S$ such that $a \neq -\frac{c}{n}$.

Case 1: $a > n$. Then, $\lceil a\rceil > n$ and $a \ge n+1$. Thus,

$a\lceil a\rceil - c > (n+1)n - c > 0$,

a contradiction.

Case 2: $0 < a \le n$. Then, $\lceil a\rceil \le n$. Thus,

$a\lceil a\rceil - c \le (n)(n) - c = n^2 - c < 0$,

a contradiction.

Case 3: $-\frac{c}{n} < a < 0$. Then, $\left\lceil -\frac{c}{n}\right\rceil \le \lceil a\rceil \le 0$. Since $c \in (n^2, n(n+1))$, $-(n+1) < -\frac{c}{n} < -n$. Thus, $\lceil -\frac{c}{n}\rceil = -n$. We see that

$$a\lceil a\rceil - c < \left(-\frac{c}{n}\right)\left\lceil -\frac{c}{n}\right\rceil - c = \left(-\frac{c}{n}\right)(-n) - c = c - c = 0,$$

a contradiction.

Case 4: $a < -\frac{c}{n}$. Then, $\lceil a\rceil \le \left\lceil -\frac{c}{n}\right\rceil$. Since $c \in (n^2, n(n+1))$, $-(n+1) < -\frac{c}{n} < -n$. Thus, $\left\lceil -\frac{c}{n}\right\rceil = -n$. We see that

$$a\lceil a\rceil - c > \left(-\frac{c}{n}\right)\left\lceil -\frac{c}{n}\right\rceil - c = \left(-\frac{c}{n}\right)(-n) - c = c - c = 0,$$

a contradiction.

From cases 1, 2, 3 and 4, $S \subseteq \left\{-\frac{c}{n}\right\}$.

Therefore, we conclude that $S = \left\{-\frac{c}{n}\right\}$.

Lemma 2.10 Let c be a real number and S be the solution set of $x\lceil x\rceil - c = 0$. If there is a positive integer n such that $c = n(n+1)$, then $S = \emptyset$.

Proof. Let c be a real number and S be the solution set of $x\lceil x\rceil - c = 0$. Suppose that there is a positive integer n such that $c = n(n+1)$ and $S \neq \emptyset$. Let $a \in S$. Then, $a\lceil a\rceil - c = 0$. That is, $a\lceil a\rceil - n(n+1) = 0$. So $a\lceil a\rceil = n(n+1)$. If $a \in (-1,0]$, then $\lceil a\rceil = 0$. This implies that $a\lceil a\rceil = 0$ which is impossible. Then, $a \notin (-1,0]$.

Case 1: $a > 0$. Then, $0 \leq \lceil a\rceil - 1 < a \leq \lceil a\rceil$. This implies that

$(\lceil a\rceil - 1)^2 < \lceil a\rceil(\lceil a\rceil - 1) < a\lceil a\rceil \leq \lceil a\rceil^2$.

Since $a\lceil a\rceil = n(n+1)$, $(\lceil a\rceil - 1)^2 < \lceil a\rceil(\lceil a\rceil - 1) < n(n+1) \leq \lceil a\rceil^2$.

Then, $\lceil a\rceil - 1 < \sqrt{n(n+1)} \leq \lceil a\rceil$. Thus,

$\lceil\sqrt{n(n+1)}\rceil = \lceil a\rceil$. (2.1)

Since $n^2 < n(n+1) < (n+1)^2$, $n < \sqrt{n(n+1)} < n+1$. Thus,

$\lceil\sqrt{n(n+1)}\rceil = n+1$. (2.2)

From (2.1) and (2.2), we see that

$\lceil a\rceil = \lceil\sqrt{n(n+1)}\rceil = n+1$. (2.3)

Since $a\lceil a\rceil = n(n+1)$, $n(n+1) = a\lceil a\rceil = a(n+1)$. Thus, $a = n$. We see that

$\lceil a\rceil = n$. (2.4)

From (2.3) and (2.4), it is impossible.

Case 2: $a \leq -1$. Then $\lceil a\rceil - 1 < a \leq \lceil a\rceil \leq 0$. Since $a\lceil a\rceil = n(n+1) > 0$, $\lceil a\rceil \neq 0$. Then, $\lceil a\rceil - 1 < a \leq \lceil a\rceil < 0$. Thus, $(\lceil a\rceil - 1)^2 > \lceil a\rceil(\lceil a\rceil - 1) > a\lceil a\rceil \geq \lceil a\rceil^2$.

Since $a\lceil a\rceil = n(n+1) > 0$, $(\lceil a\rceil - 1)^2 > n(n+1) \geq \lceil a\rceil^2$. Then, $\lceil a\rceil - 1 < -\sqrt{n(n+1)} \leq \lceil a\rceil$. Thus,

$\lceil -\sqrt{n(n+1)}\rceil = \lceil a\rceil$. (2.5)

Since $n^2 < n(n+1) < (n+1)^2$, $n < \sqrt{n(n+1)} < n+1$. Then, $-(n+1) < -\sqrt{n(n+1)} < -n$. Thus,

$\lceil -\sqrt{n(n+1)}\rceil = -n$. (2.6)

From (2.5) and (2.6),

$\lceil a\rceil = \lceil -\sqrt{n(n+1)}\rceil = -n$. (2.7)

Since $a\lceil a\rceil = n(n+1)$, $n(n+1) = a\lceil a\rceil = a(-n)$. Thus,

$a = -(n+1)$ and $\lceil a\rceil = -(n+1)$ (2.8)

From (2.7) and (2.8), it is impossible.

Therefore, $S = \emptyset$.

Before concluding the solution set of $x\lceil x\rceil - c = 0$, we show Lemma 2.11 that is used in the proof of Theorem 2.12.

Lemma 2.11 Let $c > 0$ and $T = \{x \in \mathbb{N} \mid (x-1)x < c\}$. Then, there is the maximum positive number in T.

Proof. We will show that T is the finite set. Since $1 \in T$, $T \neq \emptyset$. Let $x \in T$. Then, $x \in \mathbb{N}$ and $(x-1)x < c$. Thus, $x^2 - x - c < 0$. This implies that

$$\left(x - \tfrac{1}{2}\right)^2 - \left(c + \tfrac{1}{4}\right) < 0,$$

$$\left(x - \frac{1}{2} - \sqrt{c + \frac{1}{4}}\right)\left(x - \frac{1}{2} + \sqrt{c + \frac{1}{4}}\right) < 0.$$

Thun,

$$\frac{1}{2} - \sqrt{c + \frac{1}{4}} < x < \frac{1}{2} + \sqrt{c + \frac{1}{4}}.$$

Hence, T is the finite set. Therefore, there is the maximum positive number in T.

From Lemmas 2.5 - 2.10, we conclude the solution sets of $x\lceil x\rceil - c = 0$ in the following theorem.

Theorem 2.12 Let c be a real number. Then, $x\lceil x\rceil - c = 0$ has a solution if and only if $c \geq 0$ and $c \neq n(n+1)$ for all positive integer n. Moreover, if $x\lceil x\rceil - c = 0$ has a solution and S is the solution set, then

1. $S = (-1,0]$ when $c = 0$;
2. $S = \left\{\frac{c}{n}\right\}$ when there is a positive integer n such that $c \in ((n-1)n, n^2)$;
3. $S = \{-n, n\}$ when there is a positive integer n such that $c = n^2$; and
4. $S = \left\{-\frac{c}{n}\right\}$ when there is a positive integer n such that $c \in (n^2, n(n+1))$.

Proof. Let c be a real number. If $c < 0$ or $c = n(n+1)$ for some positive integer n, then, by Lemma 2.5 and Lemma 2.10, $x\lceil x\rceil - c = 0$ has no solution.

Next, assume that $c \geq 0$ and $c \neq n(n+1)$ for all positive integer n. If $c = 0$, then, by Lemma 2.6, $S = (-1,0]$ is the set of solutions. In the case that $c > 0$, let

$$T = \{x \in \mathbb{N} | (x-1)x < c\}.$$

By Lemma 2.11, there is the maximum positive number $n_1 \in T$. Then, $n_1 + 1 > n_1$, $n_1 + 1 \notin T$ and $c \neq n_1(n_1+1)$. That is, $(n_1 - 1)n_1 < c < n_1(n_1+1)$.

If $c \in ((n_1 - 1)n_1, n_1^2)$, by Lemma 2.7, $S = \left\{\frac{c}{n_1}\right\}$ is the set of solutions.

If $c = n_1^2$, by Lemma 2.8, $S = \{-n_1, n_1\}$ is the set of solutions.

If $c \in (n_1^2, n_1(n_1+1))$, by Lemma 2.9, $S = \left\{-\frac{c}{n_1}\right\}$ is the set of solutions.

This completes the proof.

3. Conclusions

This article presents sufficient and necessary conditions for three quadratic equations to have solutions and their solution sets. For the equations, $\lceil x \rceil^2 - c = 0$ and $\lceil x^2 \rceil - c = 0$, where c is a real number, the conditions and the solution sets in, Theorems 2.1 - 2.4, are shown in Table 3.1.

For the equation $x\lceil x \rceil - c = 0$, in order to understand the solution set in Theorem 2.11, we present each positive integer n, the value c and the solution set of $x\lceil x \rceil - c = 0$ as Table 3.2. In the future work, we interest in the solution sets of the quadratic equations (Eq.4) - (Eq.8) replaced by the ceiling function.

Table 3.1 The conditions and the solution sets of $\lceil x \rceil^2 - c = 0$ and $\lceil x^2 \rceil - c = 0$, where c is a real number

Equation	**Condition**	**Solution Set**
$\lceil x \rceil^2 - c = 0$	c is a square integer	$S = (-\sqrt{c} - 1, -\sqrt{c}] \cup (\sqrt{c} - 1, \sqrt{c}]$
$\lceil x^2 \rceil - c = 0$	c is a non-negative integer	
	- $c = 0$	$S = \{0\}$
	- $c \in \mathbb{N}$	$S = [-\sqrt{c}, -\sqrt{c-1}) \cup (\sqrt{c-1}, \sqrt{c}]$

Table 3.2 Each positive integer n, the value c and the solution sets of $x\lceil x \rceil - c = 0$ obtained from Theorem 2.12.

The equation $\boldsymbol{x\lceil x \rceil - c = 0}$						
Condition	$c < 0$	$c = 0$				
$n = 1$			$c \in (0,1)$	$c = 1$	$c \in (1,2)$	$c = 2$
$n = 2$			$c \in (2,4)$	$c = 4$	$c \in (4,6)$	$c = 6$
$n = 3$			$c \in (6,9)$	$c = 9$	$c \in (9,12)$	$c = 12$
$n = 4$			$c \in (12,16)$	$c = 16$	$c \in (16,20)$	$c = 20$
$n = 5$			$c \in (20,25)$	$c = 25$	$c \in (25,30)$	$c = 30$
$n = 6$			$c \in (30,36)$	$c = 36$	$c \in (36,42)$	$c = 42$
$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$
n			$c \in ((n-1)n, n^2)$	$c = n^2$	$c \in (n^2, n(n+1))$	$c = n(n+1)$
Solution Set	$S = \emptyset$	$S = (-1,0]$	$S = \{\frac{c}{n}\}$	$S = \{-n, n\}$	$S = \{-\frac{c}{n}\}$	$S = \emptyset$

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