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Closed (a, a + 1)-Knight's Tours on some Square Tubes

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Abstract. In this article, we present sufficient and necessary conditions for the (2a + 1, 2a + 1, k, a)-tube where k is a positive integer and $a \in \{2, 3, 4\}$ to have closed (a, a + 1)-knight's tours. Moreover, closed (a, a + 1)-knight's tours on the (2a + 1, 2a + 1, k, a)-tube are shown if they exist.

Keywords: A closed (a, b)-knight's tour, the ringboard of width r and the (m, n, k, r)-rectangular tube

1. Introduction

A knight is one of pieces in a chess game that moves in an L-shape, one square vertically or one square horizontally and then two squares at 90-degree angle. This move is called the *legal move*. For the standard chessboard, 64 squares arranged in 8 rows and 8 columns, the knight can move from a square to each square of the chessboard exactly once and return to the beginning square. These moves were presented by Euler (Watkins, 2004) and called a *closed knight's tour*. The standard chessboard is generalized to an *$m \times n$ chessboard*. It contains mn squares arranged in m rows and n columns. In 1991, Schwenk (Schwenk, 1991) obtained sufficient and necessary conditions for the $m \times n$ chessboard to have a closed knight's tour.

Theorem 1.1 The $m \times n$ chessboard with $m \leq n$ admits a closed knight's tour unless one or more of the following conditions hold: (i) m and n are both odd; or (ii) m = 1 or 2 or 4; or (iii) m = 3 and n = 4 or 6 or 8.

Moreover, Wiitala (Wiitala, 1996) modified the $m \times n$ chessboard to the *$m \times n$ ringboard of width r*. It is obtained from the $m \times n$ chessboard by removing the middle squares until the rim contains r rows and r columns. In 2005, Chia et.al. (Chia and Ong, 2005) generalized the legal knight's move to an (a, b)-*knight's move*, a square vertically or a square horizontally and then b squares at 90-degree angle. If the knight begins at a square of the $m \times n$ chessboard and moves to each square exactly once with an (a, b)-knight's move and returns to the beginning square, then these moves are called a *closed* (a, b)-*knight's tour*. The extension of the $m \times n$ chessboard in the 2-dimension to the 3-dimension can be see in Demaio, 2005; Demaio and Mathew 2011; and Singhun, Loykaew, Boonklurb and Srichote, 2021. One of them is the (m, n, k, r)-*rectangular tube* or (m, n, k, r)-*tube*. It is obtained from k copies of the $m \times n$ ringboard of width r by stacking each copy from the front. If $m = n$, then it is called the *square tube*. The example of a square tube is shown in Figure 1. By the legal move, (Singhun, Loykaew and Boonklurb, 2021) obtained sufficient and necessary conditions for some square tube to have closed knight's tours as in Theorems 1.2 - 1.5.

Theorem 1.2 The (3, 3, k, 1)-tube has a closed knight's tour for any positive integer k.
Theorem 1.3 The (4, 4, k, 1)-tube has no closed knight's tour if and only if k = 1 or k = 2.
Theorem 1.4 The (5, 5, k, 1)-tube has no closed knight's tour if and only if k = 1 or k = 2.
Theorem 1.5 Let m be an odd integer such that $m \geq 7$. Then, the (m, m, k, 1)-tube has a closed knight's tour when $k \equiv 0 \pmod 4$.

We see that if a = 1, by Theorem 1.2, the (2a + 1, 2a + 1, k, a)-tube has a closed (a, a + 1)-knight's tour for all positive integer k. In this article, we obtain sufficient and necessary conditions for the (2a + 1, 2a + 1, k, a)-tube where k is a positive integer and $a \in \{2, 3, 4\}$ to have closed (a, $a + 1$)-knight's tours. Moreover, closed (a, $a + 1$)-knight's tours on the ($2a + 1$, $2a + 1$, k, a)-tube are shown if they exist.

2. Results

For an (a, b)-knight's move on the (m, n, k, r)-rectangular tube, we consider as a graph by letting a square be a vertex and two vertices are adjacent if the knight can move between two squares by an (a, b)-knight's move. Such graph is called a (a, b)-*knight's graph*. Then, a closed (a, b)-knight's tour is a Hamiltonian cycle on the (a, b)-knight's graph. By the construction of the (m, n, k, r)-rectangular tube, the square in the ith row, the jth column and the lth level where $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 2, 3, \ldots, n\}$ and $l \in \{1, 2, 3, \ldots, k\}$ is the vertex (i, j, l) of the (a, b)-knight's graph. Note that (1, 1, 1) is a square in the upper left side conner of the front layer.

Theorem 2.1 Let k be a positive integer. The (5, 5, k, 2)-tube has no closed (2, 3)-knight's tour if and only if $k \leq 3$.

Proof. Let k be a positive integer. If k = 1 or 2, then we see that the degree of each conner vertex of the (2, 3)-knight's graph in the 1st level is 2. If the (2, 3)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 1)-(3, 4, 1), (1, 1, 1)-(4, 3, 1), (1, 5, 1)-(3, 2, 1), (1, 5, 1)-(4, 3, 1), (5, 1, 1)-(2, 3, 1), (5, 1, 1)-(3, 4, 1), (5, 5, 1)- (3, 2, 1) and (5, 5, 1)-(2, 3, 1) are in the Hamilton cycle. We see that such edges form an 8-cycle which is not a Hamiltonian cycle. Thus, the (5, 5, k, 2)-tube contains no closed (2, 3)-knight's tour.

If k = 3, then we see that the degree of each conner vertex of the (2, 3)-knight's graph in the 2nd level is 2. If the (2, 3)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 2)-(3, 4, 2), (1, 1, 2)-(4, 3, 2), (1, 5, 2)-(3, 2, 2), (1, 5, 2)-(4, 3, 2), (5, 1, 2)-(2, 3, 2), (5, 1, 2)-(3, 4, 2), (5, 5, 2)-(3, 2, 2) and (5, 5, 2)-(2, 3, 2) are in the Hamiltonian cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (5, 5, k, 2)-tube contains no closed (2, 3)-knight's tour.

Next, assume that $k \geq 4$. We construct a closed (2, 3)-knight's tour on the (5, 5, k, 2)-tube by using 2 disjoint cycles, a_1 - a_2 - a_3 - … - a_8 - a_1 and b_1 - b_2 - b_3 - … - b_{16} - b_1, on the (5, 5, 1, 2)-tube shown in Figure 2. Note that such 2 cycles contain all vertices of the (2, 3)-knight's graph.

The (5, 5, 4, 2)-tube is obtained from 4 copies of the (5, 5, 1, 2)-tube by stacking each of 4 copies from the front. Since each copy has 2 disjoint cycles, we combine 8 cycles to a single cycle containing all vertices of the (2, 3)-knight's graph.

(1) Delete the following edges: in the ith level for $i \in \{1, 2, 3, 4\}$, edges a_1 - a_2 and b_{12} - b_{13}, in the ith level for $i \in \{1, 4\}$, edges a_3 - a_4 and b_{15} - b_{16}; and in 4th level, edge b_6 - b_7. We see that, from the 8 cycles, we have 14 paths. Then, the end vertices of each path are connected by an edge as follows.

(2) Connect two vertices by the following edges. Join vertices in the 1st level to the 3rd level as follows: a_1 - b_{13}, a_2 - b_{12}, b_{12} - a_2 and b_{13} - a_1. Join vertices in the 1st level to the 4th level as follows: a_3 - b_{16}, a_4 - b_{15}, b_1 - b_7, b_2 - b_6, b_{15} - a_4 and b_{16} - a_3. Join vertices in the 2nd level to the 4th level as follows: a_1 - b_{13}, a_2 - b_{12}, b_{12} - a_2 and b_{13} - a_1. Then, each of 14 paths forms a single cycle containing all vertices of the (2, 3)-knight's graph. Therefore, the (5, 5, 4, 2)-tube has a closed (2, 3)-knight's tour.

The (5, 5, 5, 2)-tube is obtained from the (5, 5, 4, 2)-tube and the (5, 5, 1, 2)-tube by stacking the (5, 5, 4, 2)-tube in front of the (5, 5, 4, 1)-tube. Since the (5, 5, 4, 2)-tube contains a closed (2, 3)-knight's tour and the (5, 5, 1, 2)-tube contains 2 disjoint cycles, we combine the 3 cycles to a single cycle containing all vertices of the (2, 3)-knight's graph. Note that, by the construction of the (5, 5, 4, 2)-tube, the (5, 5, 5, 2)-tube contains 5 copies of (5, 5, 1, 2)-tubes.

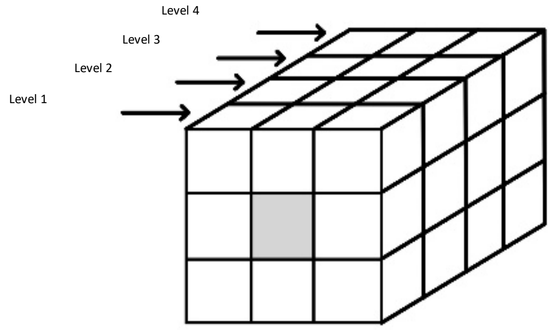

Figure 1. The square tube (the (3, 3, 4, 1)-tube).

a_1	b_3	b_8	b_{13}	a_7
b_1	b_{10}	a_4	b_6	b_{15}
b_{12}	a_6		a_2	b_4
b_7	b_{14}	a_8	b_2	b_9
a_3	b_5	b_{16}	b_{11}	a_5

Figure 2. The 2 disjoint cycles on the (5, 5, 1, 2)-tube.

(1) Delete the following edges: in the 3rd level, edges a_5 - a_6 and b_8 - b_9; and in the 5th level, edges a_7 - a_8 and b_4 - b_5. Now, we obtain 4 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by joining vertices in the 3rd level to the 5th level as follows: a_5 - b_5, a_6 - b_4, b_8 - a_8 and b_9 - a_7. Then, each of 4 paths forms a single cycle containing all vertices of the (2, 3)-knight's graph. Therefore, the (5, 5, 5, 2)-tube contains a closed (2, 3)-knight's tour. Note that edges a_5 - a_6 and b_8 - b_9 in the ith level for $i \in \{4, 5\}$ are in the closed (2, 3)-knight's tour.

For $k \geq 6$, we use the inductive process to obtain the (5, 5, k, 2)-tube by putting the (5, 5, k - 1, 2)-tube in the front of the (5, 5, 1, 2)-tube. Since the (5, 5, k - 1, 2)-tube contains a closed (2, 3)-knight's tour and the (5, 5, 1, 2)-tube contains 2 disjoint cycles, we combine the 3 cycles to a single cycle containing all vertices of the (2, 3)-knight's graph. Note that, by the construction of the (5, 5, k - 1, 2)-tube, the (5, 5, k, 2)-tube contains k copies of (5, 5, 1, 2)-tubes. (1) Delete the following edges: in (k - 2)th level, edges a_5 - a_6 and b_8 - b_9; and in kth level, edges a_7 - a_8 and b_4 - b_5. Now, we obtain 4 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by joining vertices in the (k - 2)th level to the kth level as follows: a_5 - b_5, a_6 - b_4, b_8 - a_8 and b_9 - a_7. Then, each of 4 paths forms a single cycle containing all vertices of the (2, 3)-knight's graph. Therefore, the (5, 5, k, 2)-tube contains a closed (2, 3)-knight's tour when $k \geq 4$. This completes the proof.

Theorem 2.2 Let k be a positive integer. The (7, 7, k, 3)-tube has no closed (3, 4)-knight's tour if and only if $k \leq 5$.

Proof. Let k be a positive integer. If k = 1, 2 or 3, then we see that the degree of each conner vertex of the (3, 4)-knight's graph in the 1st level is 2. If the (3, 4)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 1)-(5, 4, 1), (1, 1, 1)-(4, 5, 1), (1, 7, 1)-(5, 4, 1), (1, 7, 1)-(4, 3, 1), (7, 1, 1)-(3, 4, 1), (7, 1, 1)-(4, 5, 1), (7, 7, 1)-(3, 4, 1) and (7, 7, 1)-(4, 3, 1) are in the Hamilton cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (7, 7, k, 3)-tube contains no closed (3, 4)-knight's tour.

If $k = 4$, then we see that the degree of each conner vertex of the (3, 4)-knight's graph in the 2nd level is 2. If the (3, 4)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 2)-(5, 4, 2), (1, 1, 2)-(4, 5, 2), (1, 7, 2)-(5, 4, 2), (1, 7, 2)-(4, 3, 2), (7, 1, 2)-(3, 4, 2), (7, 1, 2)-(4, 5, 2), (7, 7, 2)-(3, 4, 2) and (7, 7, 2)-(4, 3, 2) are in the Hamilton cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (7, 7, k, 3)-tube contains no closed (3, 4)-knight's tour.

If $k = 5$, then we see that the degree of each conner vertex of the (3, 4)-knight's graph in the 3rd level is 2. If the (3, 4)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 3)-(5, 4, 3), (1, 1, 3)-(4, 5, 3), (1, 7 ,3)-(5, 4, 3), (1, 7, 3)-(4, 3, 3), (7, 1, 3)-(3, 4, 3), (7, 1, 3)-(4, 5, 3), (7, 7, 3)-(3, 4, 3) and (7, 7, 3)-(4, 3, 3) are in the Hamilton cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (7, 7, k, 3)-tube contains no closed (3, 4)-knight's tour.

Next, assume that $k \geq 6$. We construct a closed (3, 4)-knight's tour on the (7, 7, k, 3)-tube by using 3 disjoint cycles, $a_1 - a_2 - a_3 - \ldots - a_8 - a_1$, $b_1 - b_2 - b_3 - \ldots - b_{16} - b_1$, and $c_1 - c_2 - c_3 - \cdots - c_{24} - c_1$, on the (7, 7, 1, 3)-tube shown in Figure 3. Note that such 3 cycles contain all vertices of the (3, 4)-knight's graph.

The (7, 7, 6, 3)-tube is obtained from 6 copies of the (7, 7, 1, 3)-tube by stacking each of 6 copies from the front. Since each copy has 3 disjoint cycles, we combine 18 cycles to a single cycle containing all vertices of the (3, 4)-knight's graph. (1) Delete the following edges: in the ith level for $i \in \{1, 2, 3, 4, 5, 6\}$, edges $a_1 - a_8$, $b_{12} - b_{13}$, $c_7 - c_8$ and $c_9 - c_{10}$, in the ith level for $i \in \{1, 2, 5, 6\}$, edges $a_6 - a_7$ and $c_2 - c_3$, in the 1st level, edge $b_6 - b_7$; and in the 5th level, edges $b_{14} - b_{15}$. We see that, from the 18 disjoint cycles, we have 34 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by the following edges. Join vertices in the ith level to the $(i + 3)$th level for $i \in \{1,2,3\}$ as follows: $a_1 - c_7$, $a_8 - c_8$, $b_{12} - c_9$, $b_{13} - c_{10}$, $c_7 - a_1$, $c_8 - a_8$, $c_9 - b_{12}$ and $c_{10} - b_{13}$. Join vertices in the ith level to the $(i + 4)$th level for $i \in \{1,2\}$ as follows: $a_6 - c_3$, $a_7 - c_2$, $c_2 - a_7$ and $c_3 - a_6$. Join vertices in the 1st level to the 5th level as follows: $b_6 - b_{15}$ and $b_7 - b_{14}$. Then, each of 34 paths forms a single cycle containing all vertices of the (3, 4)-knight's graph. Therefore, the (7, 7, 6, 3)-tube contains a closed (3, 4)-knight's tour. Note that edges $a_3 - a_4$, $b_1 - b_{16}$, $c_{15} - c_{16}$ and $c_{20} - c_{21}$ in the ith level for $i \in \{4, 5, 6\}$ are in the closed (3, 4)-knight's tour.

a_1	b_{15}	c_{21}	c_{14}	c_7	b_5	a_3
b_1	c_{23}	c_{12}	b_{10}	c_{16}	c_5	b_3
c_1	c_{10}	b_8	a_6	b_{12}	c_{18}	c_3
c_8	b_6	a_4		a_8	b_{14}	c_{20}
c_{15}	c_6	b_4	a_2	b_{16}	c_{22}	c_{13}
b_{11}	c_{17}	c_4	b_2	c_{24}	c_{11}	b_9
a_7	b_{13}	c_{19}	c_2	c_9	b_7	a_5

Figure 3. The 3 disjoint cycles on the (7, 7, 1, 3)-tube.

Next, we construct closed (3, 4) knight's tours on the (7, 7, 7, 3)-tube, the (7, 7, 8, 3)-tube and the (7, 7, 9, 3)-tube, respectively. For $k \in \{7, 8, 9\}$, the (7, 7, k, 3)-tube is obtained from the (7, 7, k-1, 3)-tube and the (7, 7, 1, 3)-tube by stacking the (7, 7, k-1, 3)-tube in front of the (7, 7, 1, 3)-tube. Since the (7, 7, k-1, 3)-tube contains a closed (3, 4)-knight's tour and the (7, 7, 1, 3)-tube contains 3 disjoint cycles, we combine 4 cycles to a single cycle containing all vertices of the (3, 4)-knight's graph.

(1) Delete the following edges: in the (k - 3)th level, edges a_3 - a_4, c_{15} - c_{16} and c_{20} - c_{21}; and in the kth level, edges a_3 - a_4, b_1 - b_{16} and c_{20} - c_{21}. We see that, from the 4 disjoint cycles, we have 6 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by the following edges. Join vertices in the (k - 3)th level to the kth level as follows: a_3 - c_{21}, a_4 - c_{20}, c_{15} - b_{16}, c_{16} - b_1, c_{20} - a_4 and c_{21} - a_3. Then, each of 6 paths forms a single cycle containing all vertices of the (3, 4)-knight's graph. Therefore, the (7, 7, k, 3)-tube contains a closed (3, 4)-knight's tour. Note that edges a_1 - a_8, c_7 - c_8 and c_{15} - c_{16} in the ith level for $i \in \{7, 8, 9\}$ are in the closed (3, 4)-knight's tour.

For $k \geq 10$, we use the inductive process to obtain a closed (3, 4)-knight's tour on the (7, 7, k, 3)-tube by stacking the (7, 7, k - 1, 3)-tube in front of the (7, 7, 1, 3)-tube. Since the (7, 7, k - 1, 3)-tube contains a closed (3, 4)-knight's tour and the (7, 7, 1, 3)-tube contains 3 disjoint cycles, we combine 4 cycles to a single cycle containing all vertices of the (3, 4)-knight's graph. We consider k in two cases.

Case 1: $k \in \{i, i + 1, i + 2 \mid i \in \{10, 16, 22, \ldots\}\}$. (1) Delete the following edges: in the (k - 3)th level, edges a_1 - a_8, c_7 - c_8 and c_{15} - c_{16}; and in the kth level, edges a_1 - a_8, b_1 - b_{16} and c_7 - c_8. We see that, from the 4 disjoint cycles, we have 6 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by the following edges. Join vertices in the (k - 3)th level to the kth level as follows: a_1 - c_7, a_8 - c_8, c_7 - a_1, c_8 - a_8, c_{15} - b_{16} and c_{16} - b_1.

Case 2: $k \in \{i, i + 1, i + 2 \mid i \in \{13, 19, 25, \ldots\}\}$. (1) Delete the following edges: in the (k - 3)th level, edges a_3 - a_4, c_{15} - c_{16} and c_{20} - c_{21}; and in the kth level, edges a_3 - a_4, b_1 - b_{16} and c_{20} - c_{21}. We see that, from the 4 disjoint cycles, we have 6 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by the following edges. Join vertices in the (k - 3)th level to the kth level as follows: a_3 - c_{21}, a_4 - c_{20}, c_{15} - b_{16}, c_{16} - b_1, c_{20} - a_4 and c_{21} - a_3. In each case, the 6 paths form a single cycle containing all vertices of the (3, 4)-knight's graph. Therefore, the (7, 7, k, 3)-tube contains a closed (3, 4)-knight's tour for $k \geq 10$. This completes the proof.

Theorem 2.3 Let k be a positive integer. The (9, 9, k, 4)-tube has no closed (4, 5)-knight's tour if and only if $k \leq 7$.

Proof. Let k be a positive integer. If k = 1, 2, 3 or 4, then we see that the degree of each conner vertex of the (4, 5)-knight's graph in the 1st level is 2. If the (4, 5)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 1)-(6, 5, 1), (1, 1, 1)-(5, 6, 1), (1, 9, 1)-(5, 4, 1), (1, 9, 1)-(6, 5, 1), (9, 1, 1)-(4, 5, 1), (9, 1, 1)-(5, 6, 1), (9, 9, 1)-(5, 4, 1) and (9, 9, 1)-(4, 5, 1) are in the Hamilton cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (9, 9, k, 4)-tube contains no closed (4, 5)-knight's tour.

If k = 5, then we see that the degree of each conner vertex of the (4, 5)-knight's graph in the 2nd level is 2. If the (4, 5)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 2)-(6, 5, 2), (1, 1, 2)-(5, 6, 2), (1, 9, 2)-(5, 4, 2), (1, 9, 2)-(6, 5, 2), (9, 1, 2)-(4, 5, 2), (9, 1, 2)-(5, 6, 2), (9, 9, 2)-(5, 4, 2) and (9, 9, 2)-(4, 5, 2) are in the Hamilton cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (9, 9, k, 4)-tube contains no closed (4, 5)-knight's tour.

If k = 6, then we see that the degree of each conner vertex of the (4, 5)-knight's graph in the 3rd level is 2. If the (4, 5)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 3)-(6, 5, 3), (1, 1, 3)-(5, 6, 3), (1, 9, 3)-(5, 4, 3), (1, 9, 3)-(6, 5, 3), (9, 1, 3)-(4, 5, 3), (9, 1, 3)-(5, 6, 3), (9, 9, 3)-(5, 4, 3) and (9, 9, 3)-(4, 5, 3) are in the

Hamilton cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (9, 9, k, 4)-tube contains no closed (4, 5)-knight's tour.

If k = 7, then we see that the degree of each conner vertex of the (4, 5)-knight's graph in the 4th level is 2. If the (4, 5)-knight's graph has a Hamiltonian cycle, then edges (1, 1, 4)-(6, 5, 4), (1, 1, 4)-(5, 6, 4), (1, 9, 4)-(5, 4, 4), (1, 9, 4)-(6, 5, 4), (9, 1, 4)-(4, 5, 4), (9, 1, 4)-(5, 6, 4), (9, 9, 4)-(5, 4, 4) and (9, 9, 4)-(4, 5, 4) are in the Hamilton cycle. We see that such edges form a 8-cycle which is not a Hamiltonian cycle. Thus, the (9, 9, k, 4)-tube contains no closed (4, 5)-knight's tour.

Next, assume that $k \geq 8$. We construct a closed (4, 5)-knight's tour on the (9, 9, k, 4)-tube by using 4 disjoint cycles, a_1 - a_2 - a_3 - … - a_8 - a_1, b_1 - b_2 - b_3 - … - b_{16} - b_1, c_1 - c_2 - c_3 - ⋯ - c_{24} - c_1, and d_1 - d_2 - d_3 - ⋯ - d_{22} - d_{32} - d_1 on the (9, 9, 1, 4)-tube shown in Figure 4. Note that such 4 cycles contain all vertices of the (4, 5)-knight's graph.

a_1	b_{15}	c_{21}	d_{27}	d_{18}	d_9	c_7	b_5	a_3
b_1	c_{23}	d_{29}	d_{16}	c_{14}	d_{20}	d_7	c_5	b_3
c_1	d_{31}	d_{14}	c_{12}	b_{10}	c_{16}	d_{22}	d_5	c_3
d_1	d_{12}	c_{10}	b_8	a_6	b_{12}	c_{18}	d_{24}	d_3
d_{10}	c_8	b_6	a_4		a_8	b_{14}	c_{20}	d_{26}
d_{19}	d_8	c_6	b_4	a_2	b_{16}	c_{22}	d_{28}	d_{17}
c_{15}	d_{21}	d_6	c_4	b_2	c_{24}	d_{30}	d_{15}	c_{13}
b_{11}	c_{17}	d_{23}	d_4	c_2	d_{32}	d_{13}	c_{11}	b_9
a_7	b_{13}	c_{19}	d_{25}	d_2	d_{11}	c_9	b_7	a_5

Figure 4. The 4 disjoint cycles on the (9, 9, 1 4)-tube.

The (9, 9, 8, 4)-tube is obtained from 8 copies of the (9, 9, 1, 4)-tube by stacking each of 8 copies from the front. Since each copy has 4 disjoint cycles, we combine 32 cycles to a single cycle containing all vertices of the (4, 5)-knight's graph. (1) Delete the following edges: in the ith level for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, edges a_1 - a_8, b_5 - b_6, c_{20} - c_{21}, c_{23} - c_{24}, d_9 - d_{10} and d_{20} - d_{21}, in the ith level for $i \in \{1, 2, 3\}$, edges b_9 - b_{10} and c_1 - c_2, in the ith level for $i \in \{6, 7, 8\}$, edges b_9 - b_{10} and c_1 - c_2. We see that, from the 32 disjoint cycles, we have 62 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by the following edges. Join vertices in the ith level to the $(i + 4)$th level for $i \in \{1, 2, 3, 4\}$, as follows: a_1 - d_9, a_2 - d_{10}, b_5 - c_{21}, b_6 - c_{20}, c_{20} - b_6, c_{21} - b_5, c_{23} -d_{21}, c_{24} -d_{20}, d_9 -a_1, d_{10} -a_8, d_{20} -c_{24} and d_{21} - c_{23}. Join vertices in the ith level to the $(i + 5)$th level for $i \in \{1, 2, 3\}$ as follows: c_1 - b_{10} and c_2 - b_9. Join vertices in the 1st level to the 6th level as follows: d_1 - b_{11} and d_{32} - b_{12}. Then, each of 62 paths forms a single cycle containing all vertices of the (4, 5)-knight's graph. Therefore, the (9, 9, 8, 4)-tube contains a closed (4, 5)-knight's tour. Note that edges a_2 - a_3, b_1 - b_2, c_7 - c_8 and d_2 - d_3 in the ith level for $i \in \{5, 6, 7, 8\}$ are in the closed (4, 5)-knight's tour.

For $k \geq 9$, we use the inductive process to obtain a closed (4, 5)-knight's tour on the (9, 9, k, 4)-tube by stacking the (9, 9, k - 1, 4)-tube in front of the (9, 9, 1, 4)-tube. Since the (9, 9, k - 1, 4)-

tube contains a closed (4, 5)-knight's tour and the (9, 9, 1, 4)-tube contains 4 disjoint cycles, we combine 5 cycles to a single cycle containing all vertices of the (4, 5)-knight's graph. We consider k in two cases.

Case 1: $k \in \{i, i+1, i+2 \mid i \in \{9, 17, 25, \ldots\}\}$. (1) Delete the following edges: in the $(k - 4)$th level, edges a_2 - a_3, b_1 - b_2, c_7 - c_8 and d_2 - d_3; and in the kth level, edges a_5 - a_6, b_{14} - b_{15}, c_{14} - c_{15} and d_{17} - d_{18}. We see that, from the 5 disjoint cycles, we have 8 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by the following edges. Join vertices in the $(k - 4)$th level to the kth level as follows: a_2 - d_{17}, a_3 - d_{18}, b_1 - c_{15}, b_2 - c_{14}, c_7 - b_{15}, c_8 - b_{14}, d_2 - a_6 and d_3 - a_5.

Case 2: $k \in \{i, i+1, i+2 \mid i \in \{13, 21, 29, \ldots\}\}$. (1) Delete the following edges: in the $(k - 4)$th level and kth level, edges a_1 - a_8, b_5 - b_6, c_{20} - c_{21} and d_9 - d_{10}. We see that, from the 5 disjoint cycles, we have 8 paths. Then, the end vertices of each path are connected by an edge as follows. (2) Connect two vertices by the following edges. Join vertices in the $(k - 4)$th level to the kth level as follows: a_1 - d_9, a_8 - d_{10}, b_5 - c_{21}, b_6 - c_{20}, c_{20} - b_6, c_{21} - b_5, d_9 - a_1 and d_{10} - a_8. In each case, the 8 paths form a single cycle containing all vertices of the (4, 5)-knight's graph. Therefore, the (9, 9, k, 4)-tube contains a closed (4, 5)-knight's tour for $k \geq 9$. This completes the proof.

3. Conclusions

A closed (a, b)-knight's tour on the chessboard is the problem related to graph theory. The problem is finding a Hamiltonian cycle on a graph corresponding to the chessboard ((a, b)-knight's graph). In this article, we are interested in the 3-dimension chessboard called the (m, n, k, r)-rectangular tube. The sufficient and necessary conditions for the $(2a + 1, 2a + 1, k, a)$-tube where k is a positive integer and $a \in \{2, 3, 4\}$ to have closed $(a, a + 1)$-knight's tours are shown. Moreover, we present closed $(a, a + 1)$-knight's tours by using the inductive process. In future research, we obtain the sufficient and necessary conditions for the $(2a + 1, 2a + 1, k, a)$-tube where k is a positive integer and a is a positive integer greater than 4 to have closed $(a, a + 1)$-knight's tours.

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