ORIGINAL PAPER

Hamilton-Connectedness of Some Cubic 3-connected Plane Graph

Adthasit Sinna*

Department of Mathematics, Faculty of Science, Ramkhamhaeng University

*Corresponding author: *adthasit@rumail.ru.ac.th*

Received: 18 November 2025 / Revised: 16 December 2025 / Accepted: 21 December 2025

Abstract. We define a Pupa graph $PP(k, [r_2, s_2], [r_3, s_3], \ldots, [r_{n-1}, s_{n-1}], l)$ which is a cubic 3-connected plane graph. In this paper, we show that a Pupa graph is hamilton-connected if $n \geq 2$, both k and l are even, r_i is odd, and $s_i = 0$ for every $i \in \{2, 3, \ldots, n-1\}$.

Keywords: Hamilton-connected, plane graph

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph, and $p, q \in V(G)$. A path from p to q is denoted by $pq - path$. A *hamilton path* (respectively, A *hamilton cycle*) is a path (respectively, a cycle) passing through all vertices of G. Note that A *hamilton* $pq - path$ is a hamilton path from p to q. A graph G is *hamiltonian* if G has a hamilton cycle. Moreover, G is *hamilton-connected* if for every $p, q \in V(G)$, there is a hamilton $pq - path$. Note that a hamilton-connected graph is a hamiltonian graph. A *degree* of a vertex u, denoted by $deg\, u$, is a number of edges that have u as an endvertex. A graph is *cubic* if all vertices have degree three. A graph G is *3-connected* if for any $u, v \in V(G)$, $G - \{u, v\}$ is connected.

A hamiltonian problem is a well-known problem in graph theory that applies to computer graphics and logistics. Tait (Tait 1884) conjectured that a cubic 3-connected graph is hamiltonian. Later, Tutte (Tutte 1946) disproved this by showing a nonhamiltonian cubic 3-connected plane graph called *Tutte graph.* However, there is open problems about cubic 3-connected plane graphs such that Barnette's conjecture (Barnette 1969): a cubic 3-connected bipartite graph is hamiltonian. In this paper, we show a family of 3-connected cubic plane graph that is a hamilton-connected.

A *leaf* is a vertex of degree 1. Let *CP* be a *Caterpillar graph* embedded on a plane. A spine S of a caterpillar *CP* is *CP* - X where X is a set of all leaves of *CP*. Note that S is a path with vertices $v_1, v_2, \ldots, v_n$. Assume that a caterpillar *CP* has no vertex of degree two and its spine S has at least two vertices. A *Pupa graph PP* is a plane graph obtained from a Caterpillar graph *CP* with its spine S by

(1) construct the outer cycle passing through all leaves of *CP*, and

(2) for each $v_i \in V(S)$, replacing it by a cycle D_i of size $deg\, v_i$.

By this construction, all vertices of *PP* have degree three and removing any two vertices from *PP,* the remaining graph is still connected. Then *PP* is a cubic 3-connected plane graph and cycles D_1, D_2, …, and D_n corresponding to vertices v_1, v_2, …, and, v_n, respectively. Define a *Head graph* H_1 (respectively, H_n) as a graph induced by all vertices of D_1 (respectively, D_n) and all neighbors of D_1 (respectively, D_n) in the outer cycle.

Moreover, for $i \in \{2, 3, \ldots, n-1\}$, a *Body graph* B_i is a graph induced by all vertices of D_i and all neighbors of D_i in the outer cycle.

For $n \geq 2$, we let a Pupa graph $PP = PP(k, [r_2, s_2], [r_3, s_3], \ldots, [r_{n-1}, s_{n-1}], l)$ where $k, l, r_2, r_3, \ldots, r_{n-1}, s_2, s_3, \ldots, s_{n-1}$ are nonnegative integer. Note that $k \geq 2, l \geq$

2, and $r_i + s_i \geq 1$. We label all vertices of PP as in Figure 2.

From this labeling, we have $V(D_1) = \{y_1, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k\}$ and $V(D_n) = \{x_n, \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_l\}$. Then $|V(D_1)| = k + 1$ and $|V(D_n)| = l + 1$. Note that for $i \in \{1, 2, \ldots, n-1\}$, an edge $y_i x_{i+1}$ joins between D_i and D_{i+1}. For each cycle D_i, $i \in \{2, \ldots, n-1\}$, a path P_1^i from x_i to y_i in clockwise direction passing through $\bar{u}_1^i, \bar{u}_2^i, \ldots, \bar{u}_{r_i}^i$ and a path P_2^i from y_i to x_i in clockwise direction passing through $\bar{w}_1^i, \bar{w}_2^i, \ldots, \bar{w}_{s_i}^i$. Then $r_i = |V(P_1^i)|$ and $s_i = |V(P_2^i)|$. Note that r_i and s_i represent the number of vertices on D_i which has a neighbor in upper side and lower side of the outer cycle, respectively. Note that $|V(D_i)| = r_i + s_i + 2$ for all $i \in \{2, \ldots, n-1\}$. Moreover, for each $\bar{x} \in V(D_i)$, its neighbor in outer cycle is x. We show examples of Pupa graphs that $|V(D_i)|$ is even for some $i \in \{1, 2, \ldots, n\}$, and there is no hamilton *xy*-path as in Figure 3

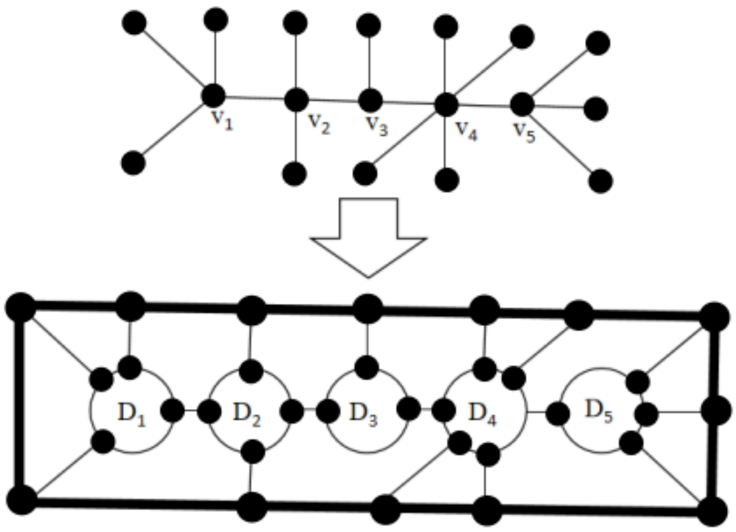

Figure 1. An example of a Pupa graph obtained from a caterpillar graph.

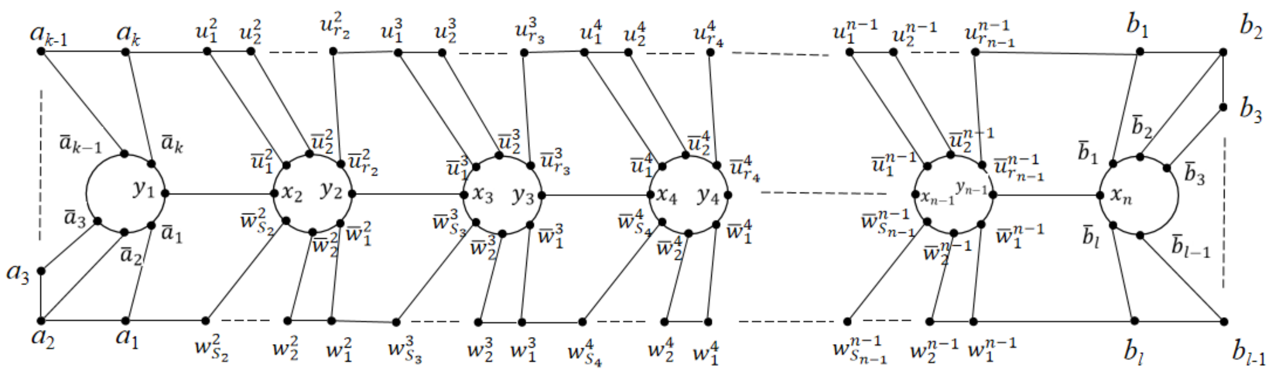

Figure 2. A Pupa graph $PP(k, [r_2, s_2], [r_3, s_3], \ldots, [r_{n-1}, s_{n-1}], l)$.

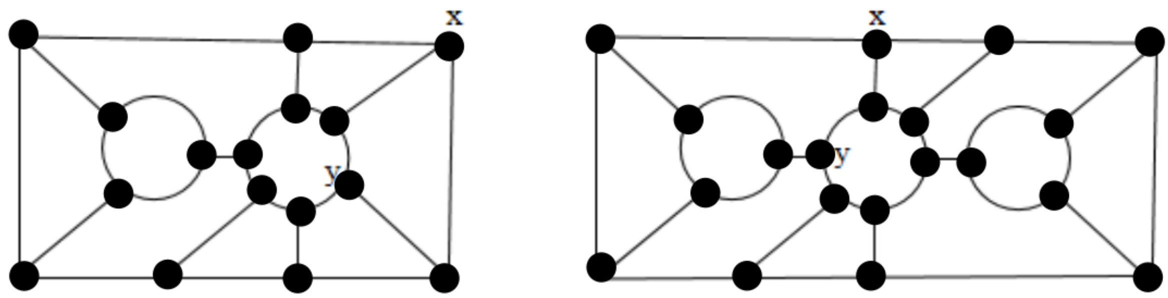

Figure 3. Pupa graphs $PP(2,5)$ and $PP(3,[2,2],3)$ have no hamilton *xy*-path.

Then we assume that $|V(D_i)|$ is odd for every $i \in \{1,2,\ldots,n\}$. Note that both k and l are even, and $r_i + s_i$ is odd for every $i \in \{2,3,\ldots,n-1\}$. Furthermore, we also assume that $s_i = 0$ for every $i \in \{2,3,\ldots,n-1\}$. Then the main result of this paper is as follows.

Theorem 1.1: For every natural number $n \geq 2$, even natural numbers *k, l,* and odd natural numbers $r_2, r_3, \ldots, r_{n-1}$, a Pupa graph $PP(k,[r_2,0],[r_3,0],\ldots,[r_{n-1},0],l)$is hamilton-connected.

We prove Theorem 1.1 for the case that $n = 2$ and $n \geq 3$ in Theorem 2.4, and Theorem 4.6, respectively.

2. Head graph

We recall a Head graph H as in figure 4.

Note that if $H = H_1$ (respectively $H = H_n$), then $m = k$ and $z = y_1$ (respectively, $m = l$ and $z = x_m$). We also let $Z^{odd} = \{c_1, c_3, c_5, \ldots, c_{m-1}\}$, $Z^{even} = \{c_2, c_4, c_6, \ldots, c_m\}$,

$\bar{Z}^{odd} = \{\bar{c}_1, \bar{c}_3, \bar{c}_5, \ldots, \bar{c}_{m-1}\}$ and $\bar{Z}^{even} = \{\bar{c}_2, \bar{c}_4, \bar{c}_6, \ldots, \bar{c}_m\}$. We generate all possible cases and then get hamilton paths on a Head graph as follows.

Lemma 2.1: Let H be a Head graph and $p \in V(H)$. Then

(1) if $p \neq z$, then there is a hamilton pz −path,

(2) if $p \in Z^{odd} \cup \bar{Z}^{even}$, then there is a hamilton pc_m −path, and

(3) if $p \in Z^{even} \cup \bar{Z}^{odd}$, then there is a hamilton pc_1 −path.

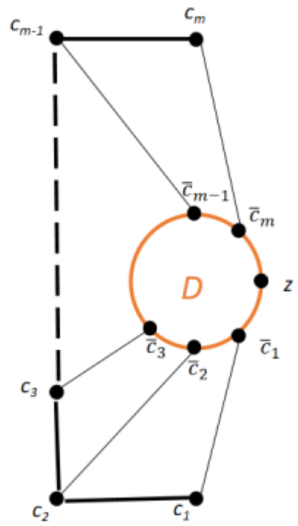

Figure 4. A Head graph *H*.

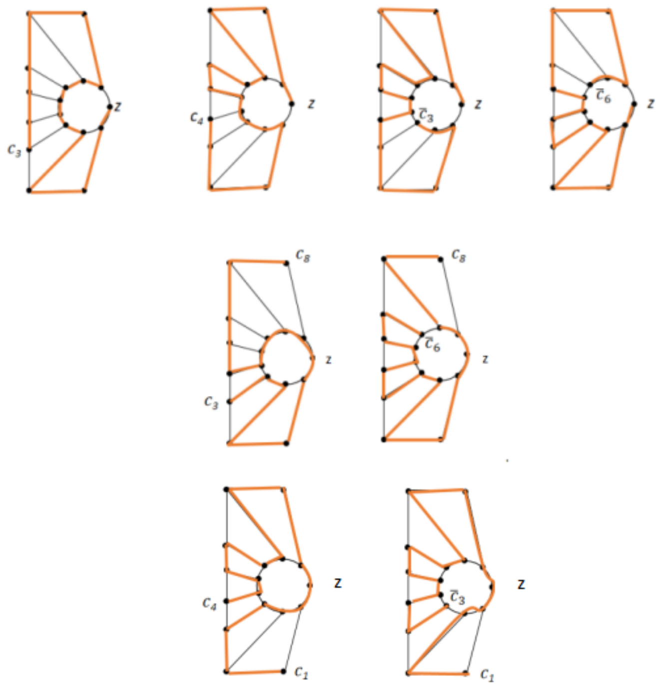

Figure 5. Examples of paths in Lemma 2.1 (1), (2), and (3) in the first row, the second row, and the third row, respectively.

Next, we construct two disjoint paths, S and T, and show that $V(S) \cup V(T) = V(H)$, and $V(S) \cap V(T) = \emptyset$ in the following lemmas.

Lemma 2.2: Let $p, q \in V(H), p \neq q$. Then there is a pp' −path S and a $qq' - path$ T, such that $p', q' \in \{z, c_1, c_m\}, V(S) \cup V(T) = V(H)$, and $V(S) \cap V(T) = \emptyset$.

Lemma 2.3: Let $p \in V(H) - \{z\}$. Then there are paths S and T such that $V(S) \cup V(T) = V(H)$, and $V(S) \cap V(T) = \emptyset$ as follows.

(1) if $p = c_i \in Z^{odd}$, then S is a pc_1 −path and T *is* a $zc_m - path$,

(2) if $p = c_i \in Z^{even}$, then S is a pc_n −path and T *is* a $zc_1 - path$, and

(3) if $p = \bar{c}_i \in \bar{Z}^{odd} \cup \bar{Z}^{even}$, then S is a pz −path and T *is* a $c_1c_m - path$.

For a case that $n = 2$, a Pupa graph $PP = PP(k,l)$ has no a Body graph. Then we show the first part of main theorem as follows.

Theorem 2.4: For every even natural numbers k and l, a pupa graph $PP(k,l)$ is hamilton-connected.

Proof Let p, q be distinct vertices of a Pupa graph PP. We will show a hamilton pq −path P_H in the following cases.

Case 1: both p and q are in $V(H_1)$. (respectively, $V(H_2)$)
By symmetry, we will show only the case that $p, q \in V(H_1)$. From Lemma 2.2, there is a pu −path S and a $qv -$ path T such that $u, v \in \{y_1, a_1, a_k\}, V(S) \cup V(T) = V(H_1)$, and $V(S) \cap V(T) = \emptyset$. Then we have the following subcases.

Subcase 1.1: $u = y_1$ and $v = a_1$.
From Lemma 2.1(1), there is a hamilton $b_l x_2 -$ path M in H_2. Then
P_H : pSy_1, x_2Mb_l, a_1Tq.

Subcase 1.2: $u = y_1$ and $v = a_k$.
From Lemma 2.1(1), there is a hamilton $b_1 x_2 -$ path M in H_2. Then
P_H : pSy_1, x_2Mb_1, a_kTq.

Subcase 1.3: $u = a_1$ and $v = a_k$.

From Lemma 2.1(2), there is a hamilton b_1b_l- path M in H_2. Then $P_H: pSa_1, b_lMb_1, a_kTq$.

Case 2: $p \in V(H_1)$ and $q \in V(H_2)$. Define subsets of $V(H_1)$ Z_1^{odd}, Z_1^{even}, $\bar{Z}_1^{odd}$, $\bar{Z}_1^{even}$ in the same way as Z^{odd}, Z^{even}, $\bar{Z}^{odd}$, $\bar{Z}^{even}$. Note that $V(H_1) = \{y_1\} \cup Z_1^{odd} \cup Z_1^{even} \cup \bar{Z}_1^{odd} \cup \bar{Z}_1^{even}$. Similary, Z_2^{odd}, Z_2^{even}, $\bar{Z}_2^{odd}$ and $\bar{Z}_2^{even}$ are defined as subsets of $V(H_2)$. Note that $V(H_2) = \{x_2\} \cup Z_2^{odd} \cup Z_2^{even} \cup \bar{Z}_2^{odd} \cup \bar{Z}_2^{even}$. Then we have the following subcases.

Subcase 2.1: $p \neq y_1$ and $q \neq x_2$. From Lemma 2.1(1), there is a hamilton py_1- path M in H_1, and there is a hamilton qx_2- path N in H_2. Then $P_H: pMy_1, x_2Nq$. *Subcase 2.2*: $p \in Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q = x_2$ (respectively, $p = y_1$ and $q \in Z_2^{odd} \cup \bar{Z}_2^{even}$).

By symmetry, we will show only the case that $p \in Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q = x_2$. From Lemma 2.1(2), there is a hamilton pa_k- path M in H_1 and from Lemma 2.1(1), there is a hamilton x_2b_1- path N in H_2. Then $P_H: pMa_k, b_1Nx_2$. *Subcase 2.3*: $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q = x_2$ (respectively, $p = y_1$ and $q \in \{x_2\} \cup Z_2^{even} \cup \bar{Z}_2^{odd}$).

By symmetry, we will show only the case that $p \in Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q = x_2$.

From Lemma 2.1(1, 3), there is a hamilton pa_1- path M in H_1 and from Lemma 2.1(1), there is a hamilton x_2b_l- path N in H_2. Then $P_H: pMa_1, b_lNx_2$.

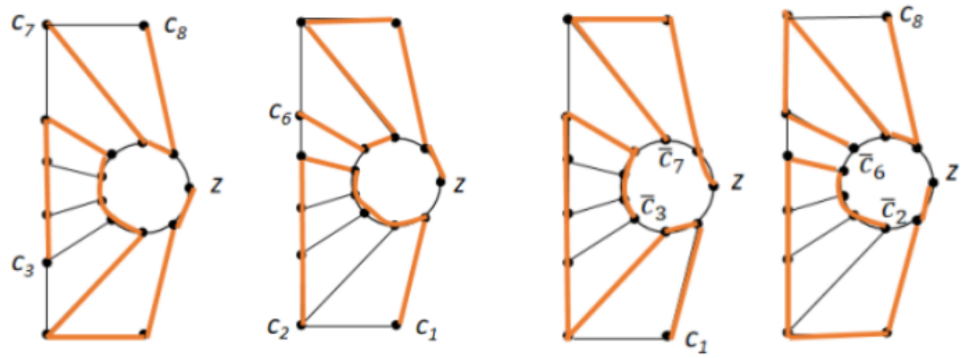

Figure 6. Examples of paths *S* and *T* in Lemma 2.2.

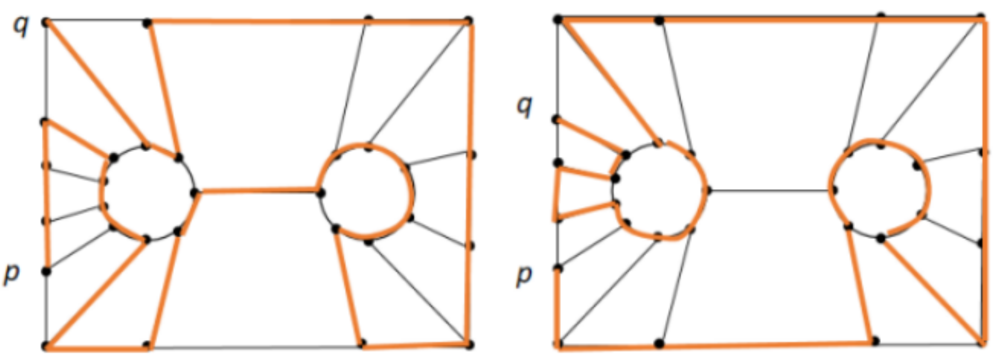

Figure 7. Examples of hamilton $pq-$paths of *PP*(8, 6) in subcase 1.2 and 1.3 of Theorem 2.4.

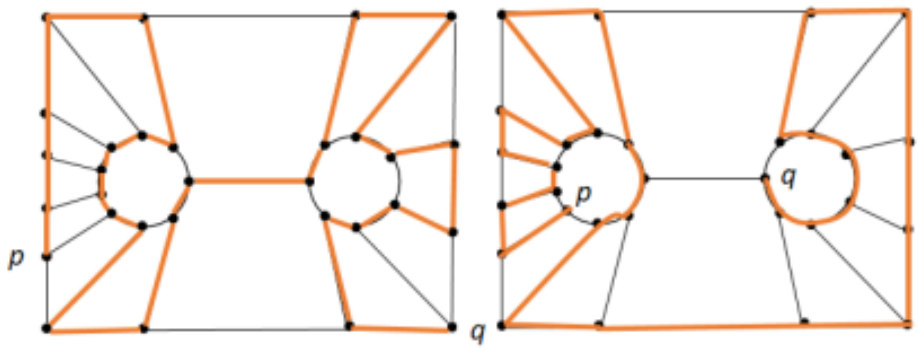

Figure 8. Examples of hamilton pq −paths of PP(8, 6) in subcase 2.1 and 2.3 in Theorem 2.4.

3. Body graph

Recall a Body graph B of $PP(k,[r_2,0],[r_3,0],\ldots,[r_{n-1},0],l)$ as in Figure 9.

Note that if $B = B_i$ for $i \in \{2,3,\ldots,n-1\}$, then $m = r_i$ and $D = D_i$. We also let $U^{odd} = \{u_1,u_3,u_5,\ldots,u_{m-1}\}$, $U^{even} = \{u_2,u_4,u_6,\ldots,u_m\}$, $\bar{U}^{odd} = \{\bar{u}_1,\bar{u}_3,\bar{u}_5,\ldots,\bar{u}_{m-1}\}$ and $\bar{U}^{even} = \{\bar{u}_2,\bar{u}_4,\bar{u}_6,\ldots,\bar{u}_m\}$. We generate all possible cases and then get hamilton paths on a Body graph as follows.

Lemma 3.1: Let B be a Body graph. If $p \in \{x,y\} \cup U^{odd} \cup \bar{U}^{even}$ and $q \in \{x,y,u_1,u_m\} - \{p\}$, then there is a hamilton pq −path.

Next, we construct two disjoint paths, S and T, and show that $V(S) \cup V(T) = V(B)$, and $V(S) \cap V(T) = \emptyset$ in the following lemmas.

Lemma 3: There are paths S and T such that $V(S) \cup V(T) = V(B)$, and $V(S) \cap V(T) = \emptyset$ as follows.

(1) If $p \in U^{odd} \cup \bar{U}^{even}$, then S is a $pp_1 -$ path and T *is* a $yp_2 - path$,

where $\{p_1,p_2\} = \{u_1,x\}$ (respectively, S is a $pp_1 -$ path and T *is* a $xp_2 - path$, where $\{p_1,p_2\} = \{u_m,y\}$).

(2) If $p \in U^{even} \cup \bar{U}^{odd}$, then S is a pp_1 −path and T *is* a $u_m p_2 - path$,

where $\{p_1,p_2\} = \{u_1,x\}$ (respectively, S is a pp_1 −path and T *is* a $u_1 p_2 - path$, where $\{p_1,p_2\} = \{u_m,y\}$).

(3) $S = \{x\}$ and T *is* a $u_1 y - path$ (respectively, $S = \{y\}$ and T *is* a $u_m x - path$).

(4) S is a xy −path and T *is* a $u_1 u_m - path$.

Lemma 3.3: Let B be a Body graph and $p,p',q,q' \in V(B)$. Then there are paths S and T such that $V(S) \cup V(T) = V(B)$, and $V(S) \cap V(T) = \emptyset$ as follows.

(1) If $p = x$ and $q = y$, then $S = \{x\}$ and T is a $yu_m - path$.

(2) If $p \in \{x,y\}$ and $q \in U^{odd} \cup \bar{U}^{even}$, then $S = \{p\}$ and T is a qq' − path where $q' \in \{x,y\} - \{p\}$.

(3) If $p \in \{x,y\}$ and $q \in U^{even} \cup \bar{U}^{odd}$, then S: x,y and T is a qu_1 − path .

(4) If $p,q \in U^{odd} \cup \bar{U}^{even}$, then S is a pp' −path and T is a qq' −path where $\{p',q'\} = \{x,y\}$.

(5) If $p \in U^{odd} \cup \bar{U}^{even}$ and $q \in U^{even} \cup \bar{U}^{odd}$, then S is a pp' − path and T is a

$qq'-$path where $z_1 \in \{x, u_1\}$, $z_2 \in \{y, u_m\}$, and $\{p', q'\} = \{z_1, z_2\}$.

Finally, we also construct two disjoint paths, S and T, but $V(S) \cup V(T) = V(H) - \{x, y\}$ in the following.

Lemma 3.3: Let B be a Body graph and $p, q \in U^{even} \cup \bar{U}^{odd}$. Then there are a $pp'-$path and a $qq'-$path T such that $\{p', q'\} = \{u_1, u_m\}$, $V(S) \cup V(T) = V(B) - \{x, y\}$, and $V(S) \cap V(T) = \emptyset$.

4. Proof of Main result

To prove Theorem 1.1 for the case that $n \geq 3$, we split that theorem into five lemmas depending on hamilton $pq-$ path P_H as follows.

(i) $p, q \in V(H_1 \cup H_2)$ in Lemma 4.1.

(ii) $p \in V(U_i)$ and $q \in V(U_j)$, $i \neq j$, in Lemma 4.2.

(iii) $p, q \in V(U_i)$ in Lemma 4.3.

(iv) n is odd, $p \in V(H_1 \cup H_2)$, and $q \in V(U_i)$ in Lemma 4.4.

(v) n is even, $p \in V(H_1 \cup H_2)$ and $q \in V(U_i)$ in Lemma 4.5.

Next, let B_i be a Body graph for $i \in \{2, 3, \ldots, n-1\}$, by using Lemma 3.1, B_i have a hamilton $x_i u^i_{r_i}-$ path X_i, and a hamilton $y_i u^i_1-$ path Y_i. Moreover, from lemma 3.2(1), B_i have an $x_i y_i-$ path D^*_i and a $u^i_1 u^i_{r_i}-$ path P^*_i such that $V(D^*_i) \cup V(P^*_i) = V(B_i)$, and $V(D^*_i) \cap V(P^*_i) = \emptyset$. Then we will use these paths in the following lemmas.

Lemma 4.1: Let $n \geq 3$ and a Pupa graph $PP = PP(k, [r_2, 0], \ldots, [r_{n-1}, 0], l)$. If $p, q \in V(H_1) \cup V(H_n)$, then there is a hamilton $pq-$path in PP.

Proof We will show a hamilton $pq-$path P_H in the following cases.

Case 1: $p, q \in V(H_1)$ (respectively, $p, q \in V(H_n)$). By symmetry, we will show only the case that $p, q \in V(H_1)$. From Lemma 2.2, there is a $pu-$path S and a $qv-$ path T, such that $u, v \in \{y_1, a_1, a_k\}$, $V(S) \cup V(T) = V(H_1)$, and $V(S) \cap V(T) = \emptyset$. Then we have the following subcases.

Subcase 1.1: $u = y_1$ and $v = a_1$.

- For n is odd, from Lemma 2.1(2), there is a hamilton $b_1 b_l-$ path M in H_n. Then $P_H : pSy_1, X_2, Y_3, X_4, \ldots, X_{n-1}, b_1 M b_l, a_1 T q$.
- For n is even, from Lemma 2.1 (1), there is a hamilton $x_n b_l-$ path M in H_n. Then P_H : $pSy_1, X_2, Y_3, X_4, \ldots, Y_{n-1}, x_n M b_l, a_1 T q$.

Subcase 1.2: $u = y_1$ and $v = a_k$.
From Lemma 2.1(1), there is a hamilton $b_1 x_n-$ path M in H_n. Then $P_H : pSy_1, D^*_2, D^*_3, \ldots, D^*_{n-1}, x_n M b_1, P^*_{n-1}, P^*_{n-2}, \ldots, P^*_2, a_k T q$.

Subcase 1.3: $u = a_1$ and $v = a_k$.

- For n is odd, from Lemma 2.1(1), there is a hamilton $x_n b_l-$ path M in H_n. Then P_H : $pSa_1, b_l M x_n, Y_{n-1}, X_{n-2}, Y_{n-3}, \ldots, Y_2, a_k T q$.
- For n is even, from Lemma 2.1(2), there is a hamilton $b_1 b_l-$ path M in H_n. Then P_H : $pSa_1, b_l M b_1, X_{n-1}, Y_{n-2}, X_{n-3}, \ldots, X_2, a_k T q$.

Case 2: $p \in V(H_1)$ and $q \in V(H_n)$ (respectively, $p \in V(H_n)$ and $q \in V(H_1)$). By symmetry, we will show only the case that $p \in V(H_1)$ and $q \in V(H_n)$. Recall $Z^{odd}_i, Z^{even}_i, \bar{Z}^{odd}_i$ และ $\bar{Z}^{even}_i$ when $i \in \{1, n\}$. Then we have the following cases.

Subcase 2.1: n is odd, $p \neq y_1$ and $q \in \{x_n\} \cup Z^{even}_n \cup \bar{Z}^{odd}_n$

- For n is odd, from Lemma 2.1(1), there is a hamilton py_1- path M in H_1, and from Lemma 2.1(1, 3), there is a hamilton qb_1- path N in H_n. Then P_H : $pMy_1, X_2, Y_3, X_4, \ldots, X_{n-1}, b_1 N q$.

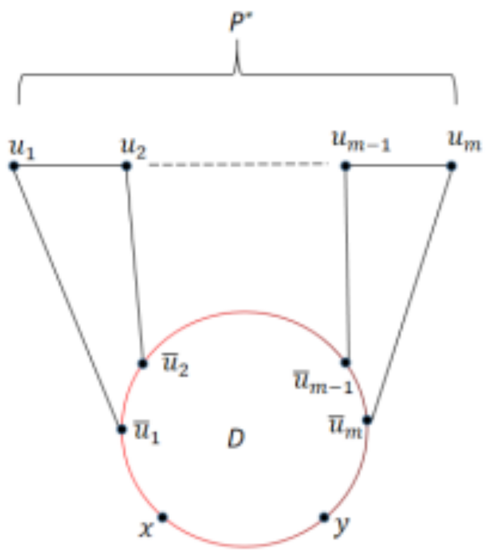

Figure 9. A Body graph B.

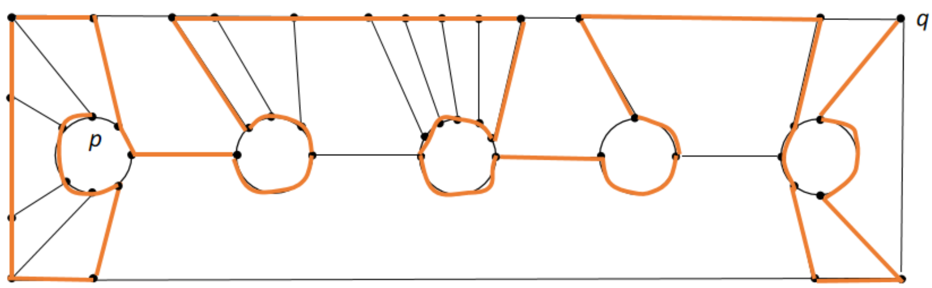

Figure 10. Example of a hamilton $pq-$path in subcase 2.1 in Lemma 4.1.

Subcase 2.2: n is odd, $p \in \{y_1\} \cup Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q \neq x_n$.

From Lemma 2.1(1, 2), there is a hamilton $pa_k -$ path M in H_1, and from Lemma 2.1(1), there is a hamilton $qx_n -$ path N in H_n. Then $P_H : pMa_k, Y_2, X_3, Y_4, \ldots, Y_{n-1}x_nNq$.

Subcase 2.3: n is odd, $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q \in \{x_n\} \cup Z_n^{odd} \cup \bar{Z}_n^{even}$.

From Lemma 2.2 and 2.3(2, 3), there is a $pp_1 -$ path S and a $p_2a_1 -$ path T, such that $\{p_1, p_2\} = \{y_1, a_k\}$, $V(S) \cup V(T) = V(H_1)$, and $V(S) \cap V(T) = \emptyset$.
From Lemma 2.1(1, 2), there is a hamilton $qb_l -$ path N in H_n. By Lemma 3.1, there is a hamilton $x_{n-1}\bar{u}_1^{n-1} -$ path K in B_{n-1}. If $p_1 = y_1$, then $P_H : pSy_1, x_2, D_2^*, D_3^*, \ldots, D_{n-2}^*, K, P_{n-2}^*, P_{n-2}^*, \ldots, P_2^*, u_1^2, a_kTa_1, b_lNq$

Otherwise, $P_H : qSb_l, a_1Ty_1, x_2, D_2^*, D_3^*, \ldots, D_{n-2}^*, K, P_{n-2}^*, P_{n-2}^*, \ldots, P_2^*, u_1^2, a_kSp$.

Subcase 2.4: n is even, $p \neq y_1$ and $q \neq x_n$. From Lemma 2.1(1), there is a hamilton $py_1 -$ path M in H_1, and there is a hamilton $qx_n -$ path N in H_n. Then $P_H : pMy_1, X_2, Y_3, X_4, \ldots, Y_{n-1}, x_nNq$.

Subcase 2.5: n is even, $p \in Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q = x_n$ (respectively, $p = y_1$ and $q \in Z_n^{odd} \cup \bar{Z}_n^{even}$). By symmetry, we will show only the case that $p \in Z_1^{odd} \cup \bar{Z}_1^{even}$

and $q = x_n$. From Lemma 2.1(2), there is a hamilton $pa_k -$ path M in H_1, and from Lemma 2.1(1), there is a hamilton $b_1x_n -$ path N in H_n. Then
$P_H: pMa_k, Y_2, X_3, Y_4, \ldots, X_{n-1}, b_1Nx_n$.

Subcase 2.6: n is even, $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q = x_n$ (respectively, $p = y_1$ and $q \in \{x_n\} \cup Z_2^{even} \cup \bar{Z}_2^{odd}$). By symmetry, we will show only the case that $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q = x_n$.From Lemma 2.2 and 2.3(2, 3), there is a $pp_1 -$ path S and a $p_2a_1 -$ path T, such that $\{p_1, p_2\} = \{y_1, a_k\}$, $V(S) \cup V(T) = V(H_1)$, and $V(S) \cap V(T) = \emptyset$. From Lemma 2.1(1, 2), there is a hamilton $x_nb_l -$ path N in H_n. By Lemma 3.1, there is a hamilton $x_{n-1}\bar{u}_1^{n-1} -$ path K in B_{n-1}. If $p_1 = y_1$, then P_H : $pSy_1, x_2, D_2^*, D_3^*, \ldots, D_{n-2}^*, K, P_{n-2}^*, P_{n-2}^*, \ldots, P_2^*, u_1^2, a_kTa_1, b_lNqx_n$. Otherwise, P_H : $x_nSb_l, a_1Ty_1, x_2, D_2^*, D_3^*, \ldots, D_{n-2}^*, K, P_{n-2}^*, P_{n-2}^*, \ldots, P_2^*, u_1^2, a_kSp$.

Lemma 4.2: Let $n \geq 4$ and a Pupa graph $PP = PP(k, [r_2, 0], \ldots, [r_{n-1}, 0], l)$. If $p \in V(U_i)$ and $q \in V(B_j), 2 \leq i < \mathrm{j} \leq n-1$ then there is a hamilton $pq -$ path in PP.

Proof We will use Lemma 3.2 to construct paths $S_\mathrm{i}, S_j, T_i, T_i$ and then use them to construct a hamilton path P_H as follows.

(1) There is an pp_1 –path S_i and a $p_2p' -$ path T_i, such that $\{p_1, p_2\} = \{u_{r_i}^i, y_i\}, V(S_i) \cup V(T_i) = V(B_i), V(S_i) \cap V(T_i) = \emptyset$, and

$$p' = \begin{cases} x_i, p \in \{y_i\} \cup U_i^{odd} \cup \bar{U}_i^{even} \\ u_1^i, p \in \{x_i\} \cup U_i^{even} \cup \bar{U}_i^{odd} \end{cases}$$

(2) There is an $\mathrm{q}q_1$ –path S_j and a $q_2q' -$ path T_j, such that $\{q_1, q_2\} = \{u_1^j, x_j\}, V(S_i) \cup V(T_i) = V(B_i), V(S_i) \cap V(T_i) = \emptyset$, and

$$q' = \begin{cases} y_j, q \in \{x_j\} \cup U_j^{odd} \cup \bar{U}_j^{even} \\ u_{r_j}^j, q \in \{y_j\} \cup U_j^{even} \cup \bar{U}_j^{odd} \end{cases}$$

Next, we construct paths P' : $u_{r_i}^i, u_1^{i+1}, \bar{u}_1^{i+1}, x_{i+1}, y_i$ and Q' : $u_1^j, P_{j-1}^*, P_{j-2}^*, \ldots, P_{i+2}^*, u_{r_{i+1}}^{i+1} P_{i+1}^* u_2^{i+1}, \bar{u}_2^{i+1} D_{i+1}^* \bar{u}_{r_{i+1}}^{i+1}, y_{i+1}, D_{i+1}^*, D_{i+2}^*, \ldots, D_{j-1}^*, x_j$.

Then we have a path P'' : pS_ip_1, P', p_2T_ip' and Q'' : qS_jq_1, Q', q_2T_jq' .

Let $p'' \in \{a_k, y_1\} \subseteq V(H_1)$. Define a path $P(p', p'')$ as one of the following paths depending on vertices p', p'' and integer *i* :
$p', Y_{i-1}, X_{i-2}, Y_{i-1}, \ldots, Y_2, p''$,
$p', Y_{i-1}, X_{i-2}, Y_{i-1}, \ldots, X_2, p''$,
$p', X_{i-1}, Y_{i-2}, X_{i-1}, \ldots, Y_2, p''$, or
$p', X_{i-1}, Y_{i-2}, X_{i-1}, \ldots, X_2, p''$. Moreover, we give the following.
- if *i* is odd, then $P(x_i, p'') = P(x_i, a_k)$ and $P(u_1^i, p'') = P(u_1^i, y_1)$.
- if *i* is even, then $P(x_i, p'') = P(x_i, y_1)$ and $P(x_i, p'') = P(u_1^i, a_k)$.

Similarly, let $q'' \in \{b_1, x_n\} \subseteq V(H_n)$. Define a path $Q(q', q'')$ as one of the following paths depending on vertices p', p'' and integer *j,n* :
$q', Y_{j+1}, X_{j+2}, Y_{j+3}, \ldots, Y_{n-1}, q''$,
$q', Y_{j+1}, X_{j+2}, Y_{j+3}, \ldots, X_{n-1}, q''$,
$q', X_{j+1}, Y_{j+2}, X_{j+3}, \ldots, Y_{n-1}, q''$, or
$q', X_{j+1}, Y_{j+2}, X_{j+3}, \ldots, X_{n-1}, q''$. Moreover, we give the following.
- if *j*, n is odd (or even), then $\mathrm{Q}(\mathrm{y_j}, \mathrm{q}'') = \mathrm{Q}(\mathrm{y_j}, \mathrm{b_1})$ and $\mathrm{Q}(\mathrm{u_{r_j}^j}, \mathrm{q}'') = \mathrm{P}(\mathrm{u_{r_j}^j}, \mathrm{x_n})$.
- if j is odd and n is even (or j is even and n is odd), then $\mathrm{Q}(\mathrm{y_j}, \mathrm{q}'') = \mathrm{Q}(\mathrm{y_j}, \mathrm{x_n})$ and $\mathrm{Q}(\mathrm{u_{r_j}^j}, \mathrm{q}'') = \mathrm{Q}(\mathrm{u_{r_j}^j}, \mathrm{b_1})$.

By Lemma 2.1, there is a hamilton $\mathrm{p''a_1} -$ path M in $\mathrm{H_1}$, and, a hamilton $\mathrm{q''b_l} -$ path N in $\mathrm{H_n}$. Combining all paths, we have $\mathrm{P_H}$:
$\mathrm{pP''p'}, \mathrm{P(p', p'')}, \mathrm{p''Ma_1}, \mathrm{b_lNq''}, \mathrm{Q(q', q'')}, \mathrm{q'Q''q}$

Define subsets of $\mathrm{V(B_i)}$ $\mathrm{U_i^{odd}}$, $\mathrm{U_i^{even}}$, $\mathrm{\bar{U}_i^{odd}}$, $\mathrm{\bar{U}_i^{even}}$ in the same way as $\mathrm{U^{odd}}$, $\mathrm{U^{even}}$,

$\bar{U}^{odd}, \bar{U}^{even}$. Note that $V(B_i) = \{x_i, y_i\} \cup U_i^{odd} \cup U_i^{even} \cup \bar{U}_i^{odd} \cup \bar{U}_i^{even}$. Then we show the following.

Lemma 4.3 : Let $n \geq 3$ and a Pupa graph $PP = PP(k, [r_2, 0], \ldots, [r_{n-1}, 0], l)$. If $p, q \in V(B_i), 2 \leq i \leq n-1$, then there is a hamilton $pq-$path in PP. Proof We will show a hamilton $pq-$path P_H in the following cases.

Case 1: $p, q \notin U_i^{even} \cup \bar{U}_i^{odd}$

By Lemma 3.3, there is a $pp'-$path S and a $qq'-$ path T, such that $V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$, $z_1 \in \{x_i, u_1^i\}, z_2 \in \{y_i, u_m^i\}$ and $\{p', q'\} = \{z_1, z_2\}$. Assume without the loss of generality that $p' = z_1$ and $q' = z_2$. Let $p'' \in \{a_k, y_1\} \subseteq V(H_1)$ and $q'' \in \{b_1, x_n\} \subseteq V(H_2)$. We define paths $P(p', p'')$ and $Q(q', q'')$ as in Lemma 4.2.

By Lemma 2.1, there is a hamilton $p''a_1-$path M in H_1, and, a hamilton $q''b_l-$path N in H_n. Combining all paths, we have P_H : $pSp', P(p', p''), p''Ma_1, b_lNq'', Q(q', q''), q'Tq$.

Case 2: $p, q \in U_i^{even} \cup \bar{U}_i^{odd}$

By Lemma 3.3, there is a $pp'-$path S and a $qq'-$ path T, such that $V(S) \cup V(T) = V(B_i) - \{x_i, y_i\}$, $V(S) \cap V(T) = \emptyset$, and $\{p', q'\} = \{u_1^i, u_m^i\}$. Assume without the loss of generality that $p' = u_1^i$ and $q' = u_m^i$.

By Lemma 2.1, there is a hamilton $y_1 a_k-$ path M in H_1, and, a hamilton $x_m b_1-$path N in H_n. Combining all paths, we have P_H :

$$pSu_1^i, P_{i-1}^*, P_{i-2}^*, \ldots, P_2^*, a_k M y_1, D_2^*, D_3^*, \ldots, D_{i-1}^*, x_i, y_i, D_{i+1}^*, D_{i+2}^*, \ldots, D_{n-1}^* x_n N b_1, P_{n-1}^*, P_{n-2}^*, \ldots, P_{i+1}^*, u_m^i T q.$$

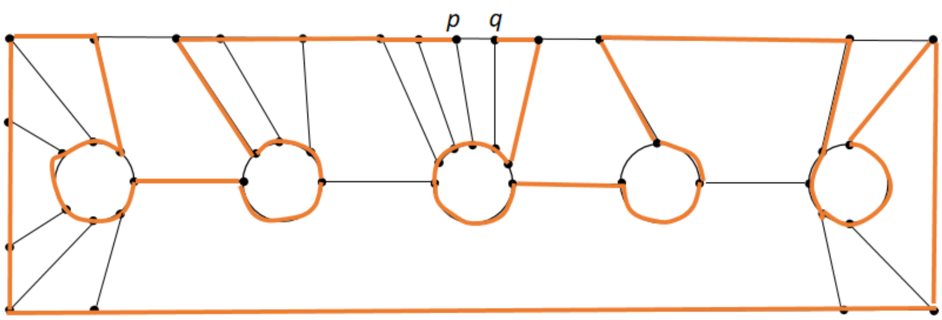

Figure 11. Example of a hamilton $pq-$path in case 1 in Lemma 4.3.

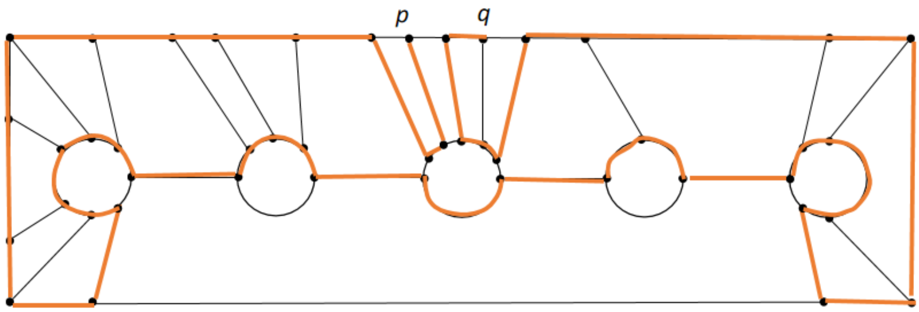

Figure 12. Example of a hamilton $pq-$path in case 2 in Lemma 4.3.

Lemma 4.4: Let $n \geq 3$ be odd and $PP = PP(k, [r_2, 0], \ldots, [r_{n-1}, 0], l)$. If $p \in V(H_1) \cup V(H_n)$ and $q \in V(B_i), 2 \leq i \leq n-1$, then there is a hamilton $pq-$ path in PP.

Proof By symmetry, we will show only the case that $p \in V(H_1)$. We will show a hamilton pq −path P_H in the following cases.

Case 1: *i* is odd.
According to vertices of H_1 and B_i, we have the following subcases.

Subcase 1.1: $p \in \{y_1\} \cup Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q \in \{y_i\} \cup U_i^{odd} \cup \bar{U}_i^{even}$. By Lemma 2.1, there is a hamilton pa_k- path M in H_1, and, a hamilton $x_n b_1-$ path N in H_n. By Lemma 3.2(1, 3), there is an qp_1 −path S and a p_2x_i- path T, such that $\{p_1, p_2\} = \{u_{r_i}^i, y_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Moreover, we define a path Q_H : $y_i, D_{i+1}^*, D_{i+2}^*, \ldots, D_{n-1}^*, x_n N b_1, P_{n-1}^*, P_{n-2}^*, \ldots, D_{i+1}^*, u_{r_i}^i$. Then P_H : $pMa_k, Y_2, X_3, Y_4, \ldots, Y_{i-1}, x_i T p_2, Q_H, p_1 S q$.

Subcase 1.2: $p \neq y_1$ and $q \in \{x_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1), there is a hamilton py_1- path M in H_1, and, a hamilton $x_n b_1-$ path N in H_n. By Lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2u_1^i-$ path T, such that $\{p_1, p_2\} = \{u_{r_i}^i, y_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. We use a path Q_H in subcase 1.1 and then P_H : $pMy_1, X_2, Y_3, X_4, \ldots, X_{i-1}, u_1^i T p_2, Q_H, p_1 S q$.

Subcase 1.3: $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q \in \{y_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 3), there is a hamilton pa_1- path M in H_1, and, a hamilton $x_n b_l-$ path N in H_n. By Lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2u_{r_i}^i-$ path T, such that $\{p_1, p_2\} = \{u_1^i, x_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Moreover, we define a path Q'_H: $x_i, D_{i-1}^*, D_{i-2}^*, \ldots, D_3^*, y_2, x_2, \bar{u}_1^2 \bar{A}_2 u_{r_2}^2, P_3^*, P_4^*, \ldots, D_{i-1}^*, u_1^i$. Then P_H : $pMa_1, b_l N x_n, Y_{n-1}, X_{n-2}, Y_{n-3}, \ldots, Y_{i+1}, u_{r_i}^i T p_2, Q'_H, p_1 S q$.

Subcase 1.4: $p = y_1$ and $q = x_i$. By Lemma 2.1(1, 2), there is a hamilton $y_1 a_1-$ path M in H_1, and, a hamilton $b_1 b_l-$ path N in H_n. By Lemma 3.2(3), there is $S = \{x_i\}$ and a $y_1 u_1^i-$ path T, such that $V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. We use a path Q'_H in subcase 1.3 and then P_H :

$$y_1 M a_1, b_l N b_1, X_{n-1}, Y_{n-2}, X_{n-3}, \ldots, X_{i+1}, y_i T u_1^i, Q'_H, x_i.$$

Case 2: *i* is even.
We use Q_H, Q'_H defined in case 1 and have the following subcases.

Subcase 2.1: $p \neq y_1$ and $q \in \{y_i\} \cup U_i^{odd} \cup \bar{U}_i^{even}$. By Lemma 2.1(1), there is a hamilton py_1- path M in H_1, and a hamilton $x_n b_1-$ path N in H_n. By Lemma 3.2(1, 3), there is an qp_1 −path S and a p_2x_i- path T, such that $\{p_1, p_2\} = \{u_{r_i}^i, y_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Then P_H : $pMy_1, X_2, Y_3, X_4, \ldots, Y_{i-1}, x_i T p_2, Q_H, p_1 S q$.

Subcase 2.2: $p \in \{y_1\} \cup Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q \in \{x_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 2), there is a hamilton pa_k- path M in H_1, and , a hamilton $x_n b_1-$ path N in H_n. By lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2u_1^i-$ path T, such that $\{p_1, p_2\} = \{u_{r_i}^i, y_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Then P_H : $pMa_k, Y_2, X_3, Y_4, \ldots, X_{i-1}, u_1^i T p_2, Q_H, p_1 S q$.

Subcase 2.3: $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q \in \{y_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 3), there is a hamilton pa_1- path M in H_1, and , a hamilton $b_1 b_l-$ path N in H_n. By Lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2u_{r_i}^i-$ path T, such that $\{p_1, p_2\} = \{u_1^i, x_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Then P_H : $pMa_1, b_l N b_1, X_{n-1},$

$Y_{n-2}, X_{n-3}, \ldots, Y_{i+1}, u_{r_i}^i T p_2, Q'_H, p_1 S q$.

Subcase 2.4: $p = y_1$ and $q = x_i$. By Lemma 2.1(1), there is a hamilton $y_1 a_1 -$ path M in H_1, and , a hamilton $x_n b_l -$ path N in H_n. By Lemma 3.2(3), there is $S = \{x_i\}$ and a $y_1 u_1^i -$ path T, such that $V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then P_H : $y_1 M a_1, b_l N x_n$, $Y_{n-1}, X_{n-2}, Y_{n-3}, \ldots, X_{i+1}, y_i T u_1^i, Q'_H, x_i$

Lemma 4.5: Let $n \geq 4$ be even and $PP = PP(k, [r_2, 0], \ldots, [r_{n-1}, 0], l)$. If $p \in V(H_1) \cup V(H_n)$ and $q \in V(B_i), 2 \leq i \leq n-1$, then there is a hamilton $pq -$ path in PP. *Proof* By symmetry, we will show only the case that $p \in V(H_1)$. We will use paths Q_H, Q'_H defined in Lemma 4.4 and show a hamilton $pq -$path P_H in the following cases.

Case 1: *i* is odd.
we have the following subcases.

Subcase 1.1: $p \in \{y_1\} \cup Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q \in \{y_i\} \cup U_i^{odd} \cup \bar{U}_i^{even}$. Use the same proof as subcase 1.1 in Lemma 4.4.

Subcase 1.2: $p \neq y_1$ and $q \in \{x_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. Use the same proof as subcase 1.2 in Lemma 4.4.

Subcase 1.3: $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q \in \{y_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 3), there is a hamilton $pa_1 -$ path M in H_1, and , a hamilton $b_1 b_l -$ path N in H_n. By Lemma 3.2(2, 4), there is an $qp_1 -$path S and $ap_2 u_{r_i}^i -$path T, such that $\{p_1, p_2\} = \{u_1^i, x_i\}, V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then P_H : $pMa_1, b_l N b_1, X_{n-1}$, $Y_{n-2}, X_{n-3}, \ldots, Y_{i+1}, u_{r_i}^i T p_2, Q'_H, p_1 S q$.

Subcase 1.4: $p = y_1$ and $q = x_i$.By Lemma 2.1(1, 2), there is a hamilton $y_1 a_1 -$ path M in H_1, and, a hamilton $x_n b_l -$ path N in H_n. By Lemma 3.2(3), there is $S = \{x_i\}$ and a $y_i u_1^i -$ path T, such that $V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then P_H : $y_1 M a_1$, $b_l N x_n, Y_{n-1}, X_{n-2}, Y_{n-3}, \ldots, X_{i+1}, y_i T u_1^i, Q'_H, x_i$.

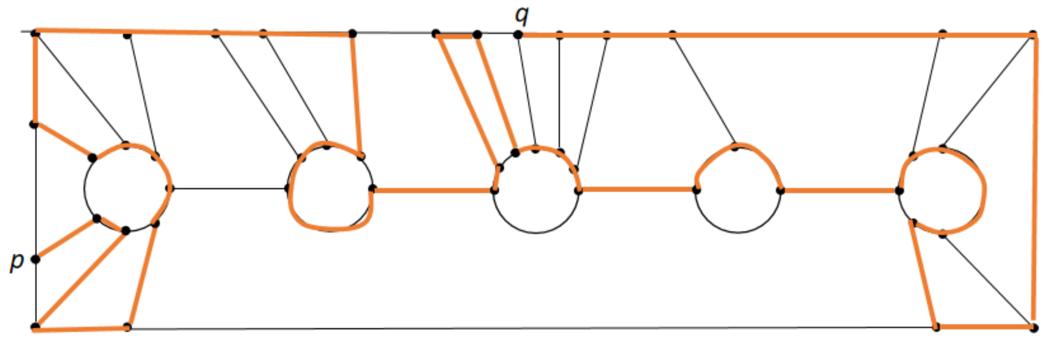

Figure 13. Example of a hamilton $pq -$ path in subcase 1.1 in Lemma 4.4.

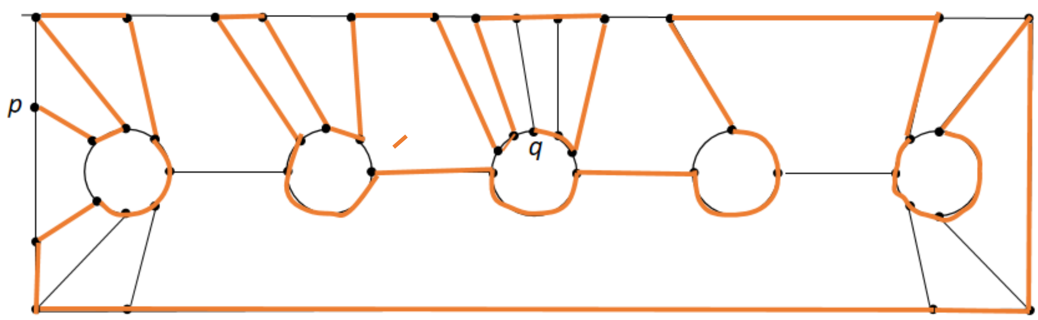

Figure 14. Example of a hamilton $pq -$ path in subcase 1.3 in Lemma 4.4.

Case 2: i is even. We use Q_H, Q'_H defined in case 1 and have the following subcases.

Subcase 2.1: $p \neq y_1$ and $q \in \{y_i\} \cup U_i^{odd} \cup \bar{U}_i^{even}$. By Lemma 2.1(1), there is a hamilton $py_1 -$ path M in H_1, and a hamilton $x_n b_1 -$ path N in H_n. By Lemma 3.2(1, 3), there is an qp_1 −path S and a $p_2 x_i -$ path T, such that $\{p_1, p_2\} = \{u_{r_i}^i, y_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Then P_H : $pMy_1, X_2, Y_3, X_4, \ldots, Y_{i-1}, x_i T p_2, Q_H, p_1 S q$.

Subcase 2.2: $p \in \{y_1\} \cup Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q \in \{x_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 2), there is a hamilton $pa_k -$ path M in H_1, and, a hamilton $x_n b_1 -$ path N in H_n. By lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2 u_1^i -$ path T, such that $\{p_1, p_2\} = \{u_{r_i}^i, y_i\}, V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Then P_H : $pMa_k, Y_2, X_3, Y_4, \ldots, X_{i-1}, u_1^i T p_2, Q_H, p_1 S q$.

Subcase 2.3: $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q \in \{y_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 3), there is a hamilton $pa_1 -$ path M in H_1, and, a hamilton $b_1 b_l -$ path N in H_n. By Lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2 u_{r_i}^i -$ path T, such that $\{p_1, p_2\} = \{u_1^i, x_i\}, V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then P_H : $pMa_1, b_l N b_1, X_{n-1}, Y_{n-2}, X_{n-3}, \ldots, Y_{i+1}, u_{r_i}^i T p_2, Q'_H, p_1 S q$.

Subcase 2.4: $p = y_1$ and $q = x_i$.By Lemma 2.1(1), there is a hamilton $y_1 a_1 -$ path M in H_1, and, a hamilton $x_n b_l -$ path N in H_n. By Lemma 3.2(3), there is $S = \{x_i\}$ and a $y_1 u_1^i -$ path T, such that $V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then P_H : $y_1 M a_1, b_l N x_n, Y_{n-1}, X_{n-2}, Y_{n-3}, \ldots, X_{i+1}, y_i T u_1^i, Q'_H, x_i$

Lemma 4.5: Let $n \geq 4$ be even and $PP = PP(k, [r_2, 0], \ldots, [r_{n-1}, 0], l)$. If $p \in V(H_1) \cup V(H_n)$ and $q \in V(B_i), 2 \leq i \leq n-1$, then there is a hamilton $pq -$ path in PP.

Proof By symmetry, we will show only the case that $p \in V(H_1)$. We will use paths Q_H, Q'_H defined in Lemma 4.4 and show a hamilton pq −path P_H in the following cases.

Case 1: i is odd.we have the following subcases.

Subcase 1.1: $p \in \{y_1\} \cup Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q \in \{y_i\} \cup U_i^{odd} \cup \bar{U}_i^{even}$. Use the same proof as subcase 1.1 in Lemma 4.4.

Subcase 1.2: $p \neq y_1$ and $q \in \{x_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. Use the same proof as subcase 1.2 in Lemma 4.4.

Subcase 1.3: $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q \in \{y_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 3), there is a hamilton $pa_1 -$ path M in H_1, and, a hamilton $b_1 b_l -$ path N in H_n. By Lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2 u_{r_i}^i$ −path T, such that $\{p_1, p_2\} = \{u_1^i, x_i\}, V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then P_H : $pMa_1, b_l N b_1, X_{n-1}, Y_{n-2}, X_{n-3}, \ldots, Y_{i+1}, u_{r_i}^i T p_2, Q'_H, p_1 S q$.

Subcase 1.4: $p = y_1$ and $q = x_i$. By Lemma 2.1(1, 2), there is a hamilton $y_1 a_1 -$ path M in H_1, and, a hamilton $x_n b_l -$ path N in H_n. By Lemma 3.2(3), there is $S = \{x_i\}$ and a $y_i u_1^i -$ path T, such that $V(S) \cup V(T) = V(B_i), V(S) \cap V(T) = \emptyset$. Then P_H : $y_1 M a_1, b_l N x_n, Y_{n-1}, X_{n-2}, Y_{n-3}, \ldots, X_{i+1}, y_i T u_1^i, Q'_H, x_i$.

Case 2: i is even.we have the following subcases.

Subcase 2.1: $p \neq y_1$ and $q \in \{y_i\} \cup U_i^{odd} \cup \bar{U}_i^{even}$. Use the same proof as subcase 2.1 in Lemma 4.4.

Subcase 2.2: $p \in \{y_1\} \cup Z_1^{odd} \cup \bar{Z}_1^{even}$ and $q \in \{x_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. Use the same proof as subcase 2.2 in Lemma 4.4.

Subcase 2.3: $p \in \{y_1\} \cup Z_1^{even} \cup \bar{Z}_1^{odd}$ and $q \in \{y_i\} \cup U_i^{even} \cup \bar{U}_i^{odd}$. By Lemma 2.1(1, 3), there is a hamilton $pa_1 -$ path M in H_1, and , a hamilton $x_n b_l -$ path N in H_n. By Lemma 3.2(2, 4), there is an qp_1 −path S and a $p_2 u_{r_i}^i -$ path T, such that $\{p_1, p_2\} = \{u_1^i, x_i\}$, $V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then $P_H : pMa_1, b_l N x_n, X_{n-1}, Y_{n-2}, X_{n-3}, \ldots, Y_{i+1}, u_{r_i}^i T p_2, Q'_H, p_1 S q$.

Subcase 2.4: $p = y_1$ and $q = x_i$. By Lemma 2.1(1), there is a hamilton $y_1 a_1 -$ path M in H_1, and , a hamilton $b_1 b_l -$ path N in H_n. By Lemma 3.2(3), there is $S = \{x_i\}$ and a $y_i u_1^i -$ path T, such that $V(S) \cup V(T) = V(B_i)$, $V(S) \cap V(T) = \emptyset$. Then $P_H : y_1 M a_1, b_l N b_1, Y_{n-1}, X_{n-2}, Y_{n-3}, \ldots, X_{i+1}, y_i T u_1^i, Q'_H, x_i$.

Combining all Lemmas 4.1, 4.2, 4.3, 4.4, and 4.5, we get the following. **Theorem 4.6 :** For every natural number $n \geq 3$, even natural numbers *k, l,* and odd natural numbers r_2, $r_3, \ldots, r_{n-1}$, a Pupa graph $PP(k, [r_2, 0], [r_3, 0], \ldots, [r_{n-1}, 0], l)$ is hamilton-connected.

5. Conclusion and Open Problems

A In this paper, we already show that a Pupa graph PP(k,[r_2,0],[r_3,0],..., 〖[r〗 _(n-1),0],l) is hamilton-connected. Then this can be improved to general case so we give the following conjecture. Conjecture 5.1 : For all natural numbers n≥3, even natural numbers k, l, and odd natural numbers r_2,r_3,...,r_(n-1),s_2,s_3,...,s_(n-1), a Pupa graph PP(k,[r_2,s_2],[r_3,s_3],..., 〖[r〗 _(n-1),s_(n-1)],l) is hamilton-connected.

References

Barnette D (1969) Conjecture 5. In Tutte WT (ed.) Recent Progress in Combinatorics. Academic Press, New York, pp 343

Tait PG (1884) Listing´s topologie. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 17: 30–46

Tutte WT (1946) On Hamiltonian circuits. Journal of the London Mathematical Society 21 : 98–101