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Bayesian Analysis of Correlated Regressors in Seemingly Unrelated Regression Model

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Abstract

Multicollinearity is known to result in inefficiency of estimators. This study introduces the use of Bayesian estimator to handle the problem of multicollinearity in Seemingly Unrelated Regression (SUR) model. Results of Bayesian method of estimation were compared with classical methods of estimation namely; SUR and Ordinary Least Squares (OLS) estimators when the regressors are correlated through a M. The Mean Squared Error (MSE) criterion was used to facilitate comparison among these estimators. The results revealed that the Bayesian method outperformed both SUR and OLS estimators for all sample sizes and levels of correlation considered.

Keywords: Bayesian, Correlated, MSE, Multicollinearity, SUR

1. Introduction

In most practical situations, economics relationships are often described using multiple equation models. For example, an interest may centre on estimating different kinds of factors that affect production in a demand scenario with several equations. If the regressors of each equation are correlated, the problem of multicollinearity emerges.

The issue of multicollinearity in Seemingly Unrelated Regression (SUR) model has not been taken seriously by researchers. However, some of the suggested solutions to the problem of multicollinearity in SUR model are the addition of sample information by Rao, et al. [1] and use of biased estimator by Liu and Wang [2]. Few researches on the issue of multicollinearity in SUR model recorded in literature are the works of Liu [3], Alkhamasi and Shukur [4], Qiu [5], Yahya, et al. [6], Roozbeh, et al. [7], Wu [8], Rana and Al-Alim [9], Erdugan and Akdeniz [10].

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A new biased estimator for SUR parameters when the explanatory variables are affected by multicollinearity was proposed by Alkhamisi and Shukur [4]. The ridge parameters were compared in terms of trace mean squared error with the SUR estimators. It was observed that the proposed SUR ridge parameters performed well than existing ridge parameters while the SUR estimator performed better than the other ridge parameters in large samples.

The relative gain and loss in efficiency of SUR estimators when one or more pair of independent variable(s) in the SUR model is non-orthogonal was demonstrated by Yahya, et al. [6]. The Tolerable Non-orthogonal Correlation Points (TNCP) was determined among the independent variables and results from their simulation studies revealed that SUR estimators are efficient up to the range of TNCP. Comparisons of SUR with OLS approach when there is a non-orthogonal explanatory variable also showed that the SUR estimators are more efficient than OLS.

Some new biased estimators of SUR parameters developed by Alkhmisi and Shukur [4] was modified by Zeebari et al. [11] to solve the problem of multicollinearity. These new modified estimators were compared in terms of Trace Mean Squared Error (TMSE) and Proportion of Replication (PR) criteria through simulation study. Their results showed that the performance of the multivariate ridge estimators under certain conditions is superior to other estimators in terms of both TMSE and PR.

Wu [8] considered two seemingly unrelated regression systems by proposing a Liu-type estimator in order to overcome multicollinearity. The superiority of this new estimator above existing estimators and admissibility of Liu-type estimator were demonstrated.

Rana and Al-Amin [9] proposed a transformed method for SUR model. This method provided an unbiased estimation for the case where there are two and three equations of both high and low collinearity for large and small datasets. The proposed method provided an unbiased estimator with lower Mean Squared Error (MSE) and Total Mean Squared Error (TMSE) than the traditional methods.

A restricted feasible SUR estimator was suggested for coefficients of SUR model to solve the problem of multicollinearity by Erdugan and Akdeniz [10]. This restricted feasible estimator was then compared with Feasible Generalised Least Squares (FGLS) and the estimator proposed by Revanker [12] in terms of MSE. The study revealed that their suggested estimator has smaller MSE.

All the aforementioned works are classical method of estimation and they do not take into account of uncertainties related to the SUR model and parameter values. However, Bayesian approach has an ability of incorporating prior information into the parameters of the model. Bayesian technique is known to have a principled way of combining prior information with data using a decision framework.

There is no work that has used Bayesian method of estimation in solving the problem of correlated regressors in SUR model.In this paper, the problem of multicollinearity in a two-equation SUR model is considered using Bayesian method to examine if there is a gain in efficiency in the method. Hence, the performance of the Bayesian, SUR and OLS estimators at different sample sizes and varying degree of collinearity is examined with the aid of simulation study.

The rest of the study is organized as follows. Section 2 deals with methods of solving multicollinearity in SUR model. Monte Carlo experiment was conducted in Section 3 to determine the performances of the estimators. Section 4 presents and discusses the results while Section 5 concludes the paper.

2. Methodology

In this section, we present a SUR model and also introduce Bayesian method of estimation for SUR model with correlated regressors.

2.1 Seemingly Unrelated Regression (SUR) Model

Consider the following SUR model:

$$y_{ip} = x'_{ip}\beta_i + \varepsilon_{ip} \tag{1}$$

$$i = 1, \dots, \text{m}, \quad p = 1, \dots, \text{P}$$

In this model, i represents the equation number and p is the time period. y_{ip} is the dependent variable in equation i, x_{ip} is a k-dimensional vector of regressors in the i-th equation and β_i is a ki-dimensional coefficient vector of regression for the i–th equation while ε_{ip} is the disturbance term.

The i-equations can be compactly expressed as:

$$y_i = x_i\beta_i + \varepsilon_i \tag{2}$$

If we further stack the observations together, equation (2) becomes:

$$y = X\beta + \varepsilon \tag{3}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_p \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & X_p \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}$$

Equation (3) is a familiar linear regression model.

The name of the model implies the equations of the model are independent of one another but have a cotemporaneous covariance that is correlated through their error terms Zellner [13].

Therefore, the assumption of the variance-covariance matrix is given as:

$$\text{E}(\varepsilon) = 0$$

$$\begin{aligned} \text{E}(\varepsilon\varepsilon') = \Sigma \otimes I_T &= \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \cdots & \sigma_{1M}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \ddots & \sigma_{2M}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{M1}^2 & \sigma_{M2}^2 & \vdots & \sigma_{MM}^2 \end{pmatrix} \otimes I_T \\ &= \begin{pmatrix} \sigma_{11}^2 I_T & \sigma_{12}^2 I_T & \cdots & \sigma_{1M}^2 I_T \\ \sigma_{21}^2 I_T & \sigma_{22}^2 I_T & \ddots & \sigma_{2M}^2 I_T \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{M1}^2 I_T & \sigma_{M2}^2 I_T & \vdots & \sigma_{MM}^2 I_T \end{pmatrix} \\ &= \begin{pmatrix} \text{H}^{-1} & \text{H}^{-1} & \cdots & \text{H}^{-1} \\ \text{H}^{-1} & \text{H}^{-1} & \ddots & \text{H}^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ \text{H}^{-1} & \text{H}^{-1} & \vdots & \text{H}^{-1} \end{pmatrix} \\ &= \Psi \end{aligned}$$

where ⊗ is the kronecker product operator

Ψ is ($M\ T$ X $M\ T$) positive definite variance covariance matrix.

With the above assumption, it can been seen that ε is N(0, Ψ).

2.2 Bayesian method

The likelihood function of equation (3) is given by:

$$L\ (\mathrm{y} \mid \beta, \mathrm{H}) \propto \prod_{i=1}^{m} \frac{|H|^{nm/2}}{(2\pi)^{nm/2}} exp[-\tfrac{1}{2}(\mathrm{y} - x\ \beta)' \mathrm{H}^{-1}\ (\mathrm{y} - x\ \beta)]$$

$$L\ (\mathrm{y} \mid \beta, \mathrm{H}) \propto \prod_{i=1}^{m} \frac{|H|^{nm/2}}{(2\pi)^{nm/2}} exp[\tfrac{1}{2}\ \mathrm{tr}\ \{\mathrm{P}\Psi^{-1}\}] \qquad (4)$$

Where “tr” denotes the trace of matrix, | Ψ | is the value of determinant of Ψ and P is m X m matrix given by:

$$\mathrm{P} = (r_{ij}) = (y_j - x_{ij}\beta_j)'\ (y_j - x_{ij}\beta_j) \qquad (5)$$

Different forms of priors were used in literature in the estimation of SUR model. These include the use of recursive extended natural conjugate prior by Richard and Steel [14]. Normal-Wishart conjugate prior was known to have analytical results for SUR model. However, it was found to be too restrictive because the prior covariances between coefficients of each of equations are always proportional to the same matrix.

Here, independent Normal-Wishart prior will be used because of the nature of block diagonal structure of matrix Ψ which will allow the matrix inversion to be partly done analytically.Hence, the prior is given as:

$$\mathrm{P}(\beta, H) = \mathrm{P}\ (\beta)\ \mathrm{P}\ (H) \qquad (6)$$

where

$$\mathrm{P}(\beta) = f_N\ (\beta|\beta^0, Q^0)$$

Also written as:

$$= \frac{1}{(2\pi)^{k/2}}\ |Q^0|^{-\frac{1}{2}} \exp\ \{-\ \tfrac{1}{2}\ (\beta - \beta^0)'(Q^0)^{-1}\ (\beta - \beta^0)\} \qquad (7)$$

and

$$\mathrm{P}\ (H) = f_W\ (H\ |v^0, H^0)$$

This can also be written as

$$= \frac{1}{C_W}\ |\ H|^{\frac{v^0 - N - 1}{2}}\ |\Psi|^{-\frac{v^0}{2}} \exp[-\ \tfrac{1}{2}\ \mathrm{tr}\ (\Psi^{-1}H)] \qquad (8)$$

where

$$C_W = 2^{\frac{v^0 N}{2}} \pi^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} \Gamma(\frac{v^0 + 1 - i}{C_W})$$

while

f_N and f_W are respectively normal and wishart densities.

If equation (4) is combined with equations (7) and (8), we ignore the terms that does not depends on parameters β and H then have:

$$P(\beta, H| y) = \{\exp [\frac{1}{2}\{\text{tr } P\Psi^{-1} + (\beta - \beta^0)'(Q^0)^{-1} (\beta - \beta^0) + \text{tr } (\Psi^{-1}H)\}]\}| H|^{\frac{v^0-n-1+nm}{2}} \quad (9)$$

It is observed that (9) does not looks like a well known distribution, we therefore obtain the posterior conditional for β by treating equation (9) as a function of β keeping H fixed which is given as:

$$P(\beta| y, H) = \exp \{\frac{1}{2}(\beta - \beta^0)'(Q^0)^{-1} (\beta - \beta^0)\} \quad (10)$$

Equation (10) is a kernel of multivariate normal distribution and can also be written as:

$$\beta| y, H \sim N (\beta^*, Q^*) \quad (11)$$

Also, the posterior for H conditional on β is:

$$H| y, \beta \sim W (v^*, H^*) \quad (12)$$

where

$$Q^* = ((Q^0)^{-1} + \sum_{i=1}^{n} x_i' H x_i)^{-1}$$

$$v^* = n + v^0$$

$$H^* = [(H^0)^{-1} + \sum_{i=1}^{n} (y_i - x_i\beta)(y_i - x_i\beta)']$$

The Bayesian estimator for β is:

$$\beta^* = ((Q^0)^{-1}\beta^0 + \sum_{i=1}^{n} x_i' H y_i)^{-1} \quad (13)$$

N.B: The symbol "0" under the parameters are the priors while symbol represented by " * " over the parameters are the posterior parameters.

3. Monte Carlo experiment

In this Section, we set up a Monte Carlo experiment to determine the most efficient estimator among the estimators considered in SUR model. Therefore, we consider a two-equation SUR model and are given as:

$$y_1 = 0.4 + 1.7x_{11} + 2.5\, x_{12} + \varepsilon_1 \quad (14)$$

$$y_2 = 2.0 + 0.8x_{21} + 3.0\, x_{22} + \varepsilon_2 \quad (15)$$

The main aim of this work is to compare the performance of different estimators when regressors of SUR model are correlated. To achieve the aim, different degrees of collinearity (0.8, 0.85, 0.90, 0.95, 0.99 and 0.999) will be considered and the regressors of one of the equations in the model are assumed to be correlated.

The simulation algorithms are as follows:

i. Generate the regressors respectively from MVN(0, Σ)
ii. Set the initial values for parameter β
iii. Generate the disturbance term ε from MVN (0, $\Sigma \otimes I_T$).

iv. The regressors, initial values and disturbance terms obtained in (i), (ii) and (iii) will enabled the generation of simulated response observations.

Gibbs sampler method will be used based on the prior presented in sub-section 2.2, since the posterior conditionals will involve inverting the matrix Ψ which is computationally challenging. The Gibbs sampler will make use of 10000 replications with 1000 replications as burn-in-replications while 9000 replications will be retained.

In conducting the Monte Carlo experiment, there are factors that will be considered which will be used to compare the performances of the estimators. The estimators are Bayesian, SUR and OLS. The factors to be considered are number of observations and collinearity among the regressors in the first equation of the model. The design of Monte Carlo experiment is summarized in Table 1.

Table 1: Factors and design of Monte Carlo experiment.

Factor	**Design**
Number of observations	n = 20, 70, 100, 200, 500 and 1000.
Correlation among the regressors	ρ_x= 0.8, 0.85, 0.90, 0.95, 0.99 and 0.999.
True value of parameters	β = 0.4, 1.7, 2.5, 2.0, 0.8 and 3.0.
Correlation among the equations	ρ_Ψ = 0.6
Number of equations	m = 2

Prior specification

For prior elicitation, the following specifications will be used

Prior degree of freedom is given by: v^0 = 6,

The SUR coefficient is set as: $\beta^0 = 0_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The prior covariance is also given as:

$$Q^0 = \begin{pmatrix} 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 \end{pmatrix}$$

$$\mathrm{H}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4. Results and discussion

In this section results of Monte Carlo simulation experiment will be discussed. The results are presented in the appendix as shown in Tables 2-7. Tables 2-7 give the estimates of MSE for Bayesian, SUR and OLS for the entire sample sizes based on their levels of collinearity while visual examination of all the results was also given in Figures 1-6 in appendix. The best estimator among all the estimators is the one with minimum MSE. Increase in levels of collinearity does not affect the MSE of all the estimators considered, this was also demonstrated in the work of Manssson et al. (2010) while increase in sample sizes does not also have effect on the performance of all the estimators. Based on the results, it is noticeable in all cases that the MSEs of Bayesian are smaller than other estimators (SUR and OLS). However, SUR estimator outperformed OLS estimator. The results are also presented by plots using a line graph in terms of MSE for all the estimators based on levels of collinearity. It is also observed that Bayesian estimator shows a better performance than the classical estimator (SUR and OLS) for all the levels of collinearity and sample sizes.

5. Conclusion

Few works have suggested solutions for the problem of multicollinearity in seemingly unrelated regression model, but all these works are classical method of estimation. However, when relevant prior information about the behaviour of situation is available and also known to an investigator, such information may be useful in the estimation of seemingly unrelated regression model.

This work provided a Bayesian approach for estimating the parameters of a seemingly unrelated regression model when the regressors are correlated. The performance of the Bayesian method was compared with the classical SUR estimator and OLS through Monte Carlo simulations. The design of the Monte Carlo study involved a system of two-equation SUR model with six correlated regressors. The number of observations and the degree of collinearity were also varied. The performance of these estimators was assessed by mean squared error criterion over 10000 replications. The results showed that the Bayesian estimator produced the least mean squared errors than both SUR and OLS estimators for all the sample sizes and correlation levels considered while increase in both levels of correlation and sample sizes do not improve the performance of the estimators considered.

Hence, the Bayesian approach is more efficient than both SUR and OLS estimators when faced with the problem of multicollinearity. This study suggests that the Bayesian approach can be used when the available data is characterized by strong multicollinearity in seemingly unrelated regression model.

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Appendix

Table 2: Result of MSE when n = 20.

ρ_x	OLS	SUR	BAYESIAN
0.8	0.6235	0.5693	0.00737
0.85	0.9561	0.8456	0.0095
0.90	1.3844	1.1812	0.00928
0.95	1.4255	1.2177	0.00917
0.99	1.0633	0.9317	0.00937
0.999	1.2597	1.0753	0.00963

Table 3: Result of MSE when n = 70.

ρ_x	OLS	SUR	BAYESIAN
0.8	0.4588	0.4485	0.00797
0.85	0.7965	0.7626	0.00833
0.90	1.2376	1.2023	0.00841
0.95	1.1262	1.1262	0.00822
0.99	0.9710	0.9571	0.00814
0.999	1.0836	1.0373	0.00774

Table 4: Result of MSE when n = 100.

ρ_x	OLS	SUR	BAYESIAN
0.8	1.1220	1.1110	0.00811
0.85	0.7485	0.7261	0.00783
0.90	0.9775	0.9544	0.00792
0.95	1.1217	1.1093	0.00792
0.99	0.8198	0.8037	0.00747
0.999	1.0313	1.0450	0.00731

Table 5: Result of MSE when n = 200.

ρ_x	OLS	SUR	BAYESIAN
0.8	1.0670	1.0706	0.00472
0.85	0.9343	0.9378	0.00052
0.90	0.8585	0.8502	0.00049
0.95	0.9606	0.9563	0.00439
0.99	1.1506	1.1338	0.00566
0.999	1.0053	0.9980	0.00568

Table 6: Result of MSE when n = 500.

ρ_x	OLS	SUR	BAYESIAN
0.8	0.9944	0.9888	0.00302
0.85	1.0368	1.0348	0.00015
0.90	0.9480	0.9466	0.00012
0.95	0.9769	0.9724	0.00756
0.99	1.0317	1.0380	0.00012
0.999	1.0169	1.0143	0.00014

Table 7: Result of MSE when n = 1000.

ρ_x	OLS	SUR	BAYESIAN
0.8	1.0357	1.0353	0.00458
0.85	0.9944	0.9914	0.00758
0.90	1.0863	1.0855	0.00334
0.95	0.9466	0.9457	0.00130
0.99	0.9297	0.9271	0.00297
0.999	1.0058	1.0042	0.00220

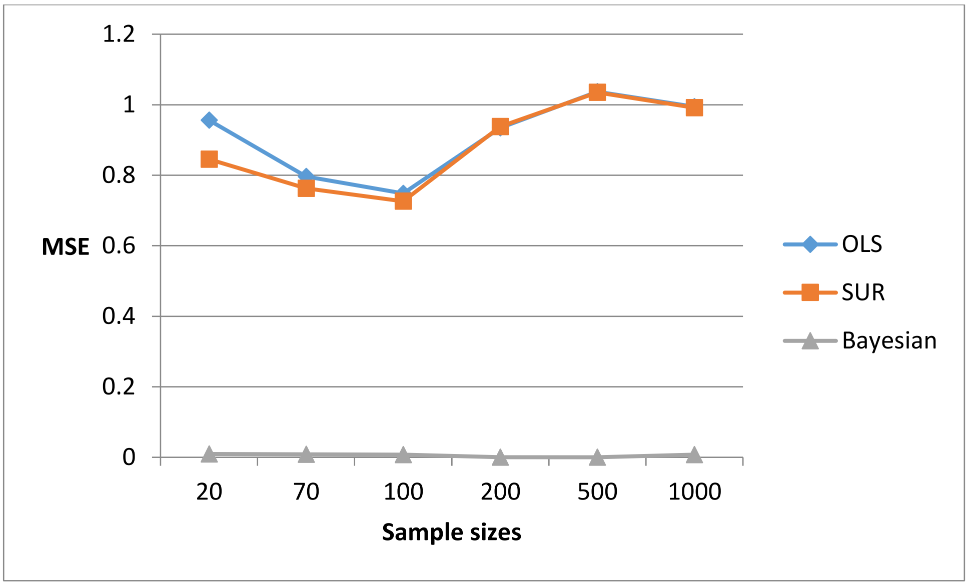

Figure 1. Plot of MSE when the degree of collinearity is 0.8.

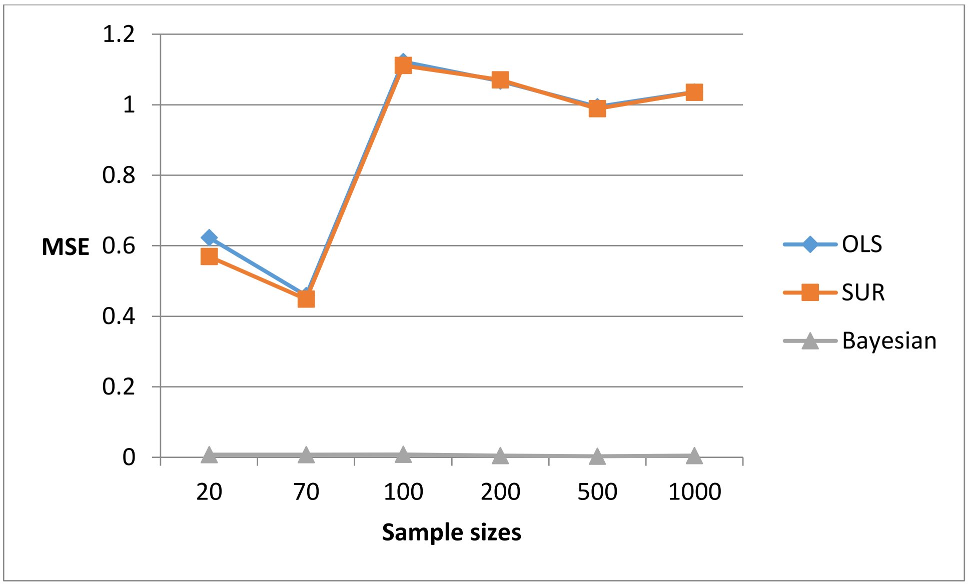

Figure 2. Plot of MSE when the degree of collinearity is 0.85.

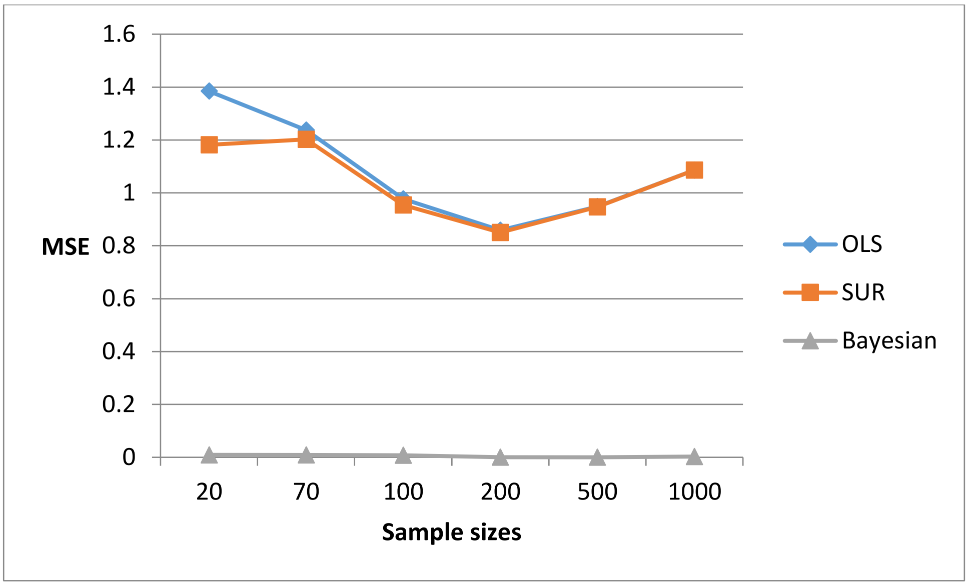

Figure 3. Plot of MSE when the degree of collinearity is 0.90.

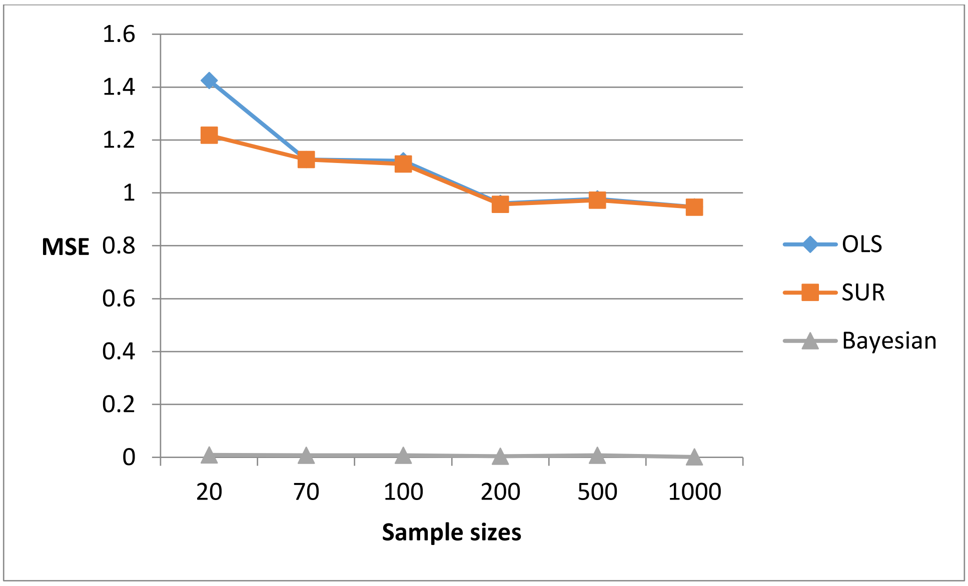

Figure 4. Plot of MSE when the degree of collinearity is 0.95.

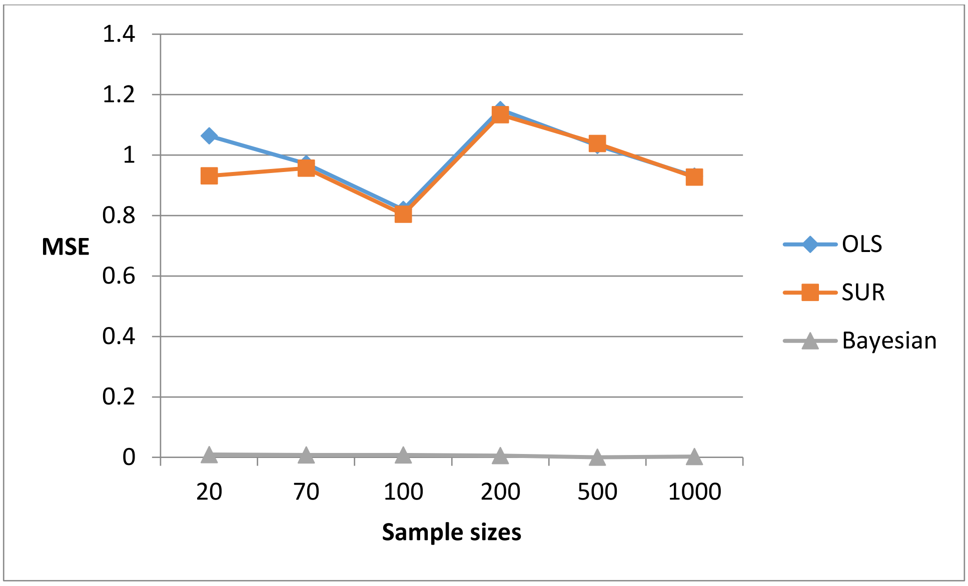

Figure 5. Plot of MSE when the degree of collinearity is 0.99.

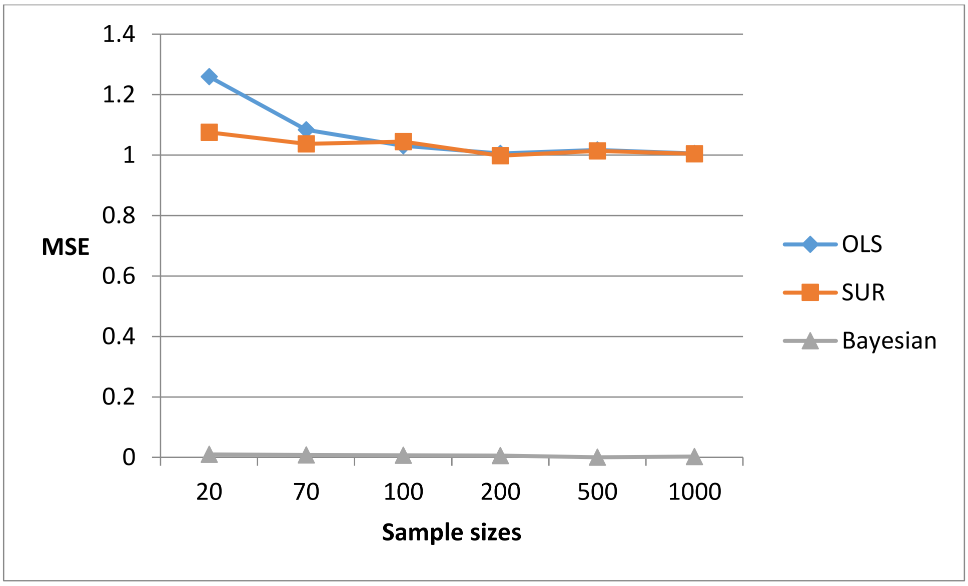

Figure 6. Plot of MSE when the degree of collinearity is 0.999.