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Differential Operator and Laplace Transform for Linear Differential Equations

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Abstract

The solutions of differential equations, there are many different ways to find values. In this research, Laplace transform for time and space differential operators are used to find solutions to equations. Which works very well and more quickly.

Keywords: Differential operator, Laplace transform, Linear Differential Equation

1. Introduction

Differential equations can be exploited and solved. In science Engineering, Physics and Economics the problem can be solved in a variety of ways. It depends on the nature of the work that needs to be solved. In order to be efficient and to get the most accurate values by solving the problem of differential equations, there are various methods that can be used to solve the problem, for example, using the Laplace transform. There are also a variety of conversions that have gained some attention, such as the Laplace transform. Elzaki transform Sumudu transform, etc. In addition to converting there is also a solution using differential operators, integrations, matrix operations. That can be used to solve problems. Many researchers have used this method to help solve problems and find solutions of differential equations, for example, Sadikali Latif Shaikh has proven to determine the properties of Sadik transformations and to determine the relationship of the transformations. In addition to the conversion, the problem is also solved by using the differential operator. Wenfeng Chen uses the differential operator method to solve linear differential equations with constant coefficients.

Therefore, this research, I am interested differential operator and transform to solve problems of linear differential equations.

2. Basic definition

Finding solutions for differential equations, there are several methods as follows:

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2.1 Differential Operator

There is a way to find the value is to make it in the form of a new symbol. Which represents the derivative with respect to the independent variable x as follows:

from $a_n \dfrac{d^{(n)}y}{dx^{(n)}} + a_{n-1}\dfrac{d^{(n-1)}y}{dx^{(n-1)}} + ... + a_1 \dfrac{dy}{dx} + a_0 y = f(x)$ when $a_n \neq 0$

or $a_n D^n y + a_{n-1} D^{n-1} y + ... + a_1 Dy + a_0 y = f(x)$ can be written instead $L(D)y = f(x)$

let $D = \dfrac{d}{dx}$ so $Dy = \dfrac{dy}{dx}$ call D the differential operator for first order differential operator. Second order differential operator $D^2 = \dfrac{d^2}{dx^2}$ so $D^2 y = \dfrac{d^2 y}{dx^2}$

2.2 Laplace transform

The Laplace transform is an integral transform, this form

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s).$$

In general $F(s)$ will exist for $s > \alpha$ where α is some constant. L is called the Laplace transform operator.

Theorem: [Laplace transform of derivatives] Suppose f is of exponential order, and that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Then

$$L\{ f'(t)\} = sF(s) - f(0)$$

Applying the theorem multiple times yields:

$$L\{ f''(t)\} = s^2 F(s) - sf(0) - f'(0),$$

$$L\{ f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0),$$

Therefor $$L\left\{ f^n(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - s^0 f^{n-1}(0)$$

3. The solution

In this section author Laplace transform method.in time and differential operator in space

Example 3.1 Consider the one dimensional heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ and initial temperature is $u(x,0) = \sin \pi x$ The analytical solution is $u(x,t) = e^{-\pi^2 t} \sin \pi x$

Solution

From $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

then $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} = D^2 u(x,t)$

taking Laplace transform on both sides gives

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{D^2 u(x,t)\right\}$$

$$sL\{u(x,t)\} - u(x,0) = L\{D^2u(x,t)\}$$
$$(s-D^2)L\{u(x,t)\} = \sin \pi x$$
$$L\{u(x,t)\} = \frac{1}{s-D^2}(\sin \pi x)$$
$$u(x,t) = e^{-\pi^2 t}\sin \pi x$$

Example 3.2 Consider one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and the initial conditions are $u(x,0) = \sin x$ and $u'(x,0) = 0$. The analytical solution is $u(x,t) = \sin x \cos t$

Solution

from $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

then $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} = D^2u(x,t)$

taking Laplace transform on both sides gives

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = L\{D^2u(x,t)\}$$
$$s^2L\{u(x,t)\} - su(x,0) - u'(x,0) = L\{D^2u(x,t)\}$$
$$(s^2-D^2)L\{u(x,t)\} = s(\sin x)$$
$$L\{u(x,t)\} = \frac{s}{s^2-D^2}(\sin x)$$
$$u(x,t) = \sin x \cos t$$

Example 3.3 Consider the partial equation $u_{xx} + 2u_{tt} + 3u_x = 0$ and the initial conditions are $u(x,0) = 0$ and $u_t(x,0) = e^{-3x}$. The analytical solution is $u(x,t) = te^{-3x}$

Solution

from $u_{xx} + 2u_{tt} + 3u_x = 0$

then $D^2u + 2u_{tt} + 3Du = 0$

taking Laplace transform on both sides gives

$$L\{D^2u\} + L\{2u_{tt}\} + L\{3Du\} = 0$$
$$L\{D^2u(x,t)\} + 2s^2L\{u(x,t)\} - 2su(x,0) - 2u'(x,0) + 3L\{Du(x,t)\} = 0$$
$$(D^2 + 2s^2 + 3D)L\{u(x,t)\} = 2e^{-3x}$$
$$L\{u(x,t)\} = \frac{1}{D^2+2s^2+3D}(2e^{-3x})$$
$$u(x,t) = te^{-3x}$$

Example 3.4 Consider the partial equation $u_{xx} - u_{tt} + u_t + 9u = \sin 3x$ and the initial conditions are $u(x,0) = 0$ and $u_t(x,0) = \sin 3x$.

Solution

from $u_{xx} - u_{tt} + u_t + 9u = \sin 3x$

then $D^2 u - u_{tt} + u_t + 9u = \sin 3x$

taking Laplace transform on both sides gives

$$L\left\{D^2 u\right\} - L\left\{u_{tt}\right\} + L\left\{u_t\right\} + L\left\{9u\right\} = L\left\{\sin 3x\right\}$$

$$L\{D^2 u(x,t)\} - s^2 L\{u(x,t)\} + su(x,0) + u'(x,0) + sL\left\{u(x,t)\right\} - u(x,0) + 9L\left\{u(x,t)\right\} = \frac{\sin 3x}{s}$$

$$(D^2 - s^2 + s + 9)L\{u(x,t)\} = \frac{\sin 3x}{s} - \sin 3x$$

$$L\left\{u(x,t)\right\} = \frac{1}{D^2 - s^2 + s + 9}\left[\frac{\sin 3x}{s} + \sin 3x\right]$$

$$u(x,t) = t \sin 3x$$

4. Conclusion

This paper shown the results for Laplace transform method in time and differential operator in space. The study found that this method is efficient, simple and can be applied to application.

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