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On the Diophantine Equation $3^x + n^y = z^3$

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Abstract

By using elementary methods and Mihailescu's Theorem, the non-existence of integer solutions for the title Diophantine equation is investigated, where n is a positive integer and x, y, z are non-negative integers with some conditions.

Keywords: Diophantine equation, Mihailescu's Theorem, Congruence

1. Introduction

Let m and n be positive integers. In the past ten years, many mathematical researchers have been studying the Diophantine equations of the form $m^x + n^y = z^c$, where $c = 2$, see [1], [2], [3], [4], [5], [6], [7] and [8]. Moreover, Tadee and Siraworakun [9] presented some conditions for the non-existence of positive integer solutions of the Diophantine equation $p^x + (p+2q)^y = z^2$, where p, q and $p + 2q$ are prime numbers.

Meanwhile, the Diophantine equation was investigated, when $c = 3$. For example, Burshtein ([10], [11], [12] and [13]) studied the Diophantine equation $p^x + q^y = z^3$, where p and q are prime numbers. Recently, Tadee [14] found all non-negative integer solutions of the Diophantine equation $8^x + p^y = z^3$, where p is a prime number.

In this paper, we study and find the conditions of the non-existence of integer solutions for the Diophantine equation

$$3^x + n^y = z^3, \tag{1}$$

where n is a positive integer and x, y, z are non-negative integers.

2. Methodology

In this section, we begin by introducing the helpful theorem, which was proved by Mihailescu [15] in 2004:

Theorem 1. *(Mihailescu's Theorem)* [15] The equation $a^x - b^y = 1$ has the only solution $(a, b, x, y) = (3, 2, 2, 3)$, where $a,\ b,\ x$ and y are integers with $\min\{a, b, x, y\} > 1$.

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Corollary 2. The equation $3^x + 1 = z^3$ has no non-negative integer solution.

Proof. Assume that there exist non-negative integers x and z such that $3^x + 1 = z^3$ or $z^3 - 3^x = 1$. It is easy to check that $x > 1$ and $z > 1$. It follows that $\min\{z, 3, 3, x\} > 1$, which is impossible by Theorem 1.

Now, we present some basic properties of congruence, which will be used to prove our results.

Lemma 3. Let a be an integer. Then

(i) $a^3 \equiv 0, 1, 3, 5, 7 \pmod 8$,

(ii) $a^3 \equiv 0, 1, 5, 8, 12 \pmod{13}$.

Proof.

(i) By the division algorithm, there exist integers q and r with $r \in \{0, 1, 2, \dots, 7\}$ such that $a = 8q + r$. We consider the following cases:

Case 1. $r = 0$. Then $a \equiv 0 \pmod 8$ and so $a^3 \equiv 0^3 \equiv 0 \pmod 8$.

Case 2. $r = 1$. Then $a \equiv 1 \pmod 8$ and so $a^3 \equiv 1^3 \equiv 1 \pmod 8$.

Case 3. $r = 2$. Then $a \equiv 2 \pmod 8$ and so $a^3 \equiv 2^3 \equiv 0 \pmod 8$.

Case 4. $r = 3$. Then $a \equiv 3 \pmod 8$ and so $a^3 \equiv 3^3 \equiv 3 \pmod 8$.

Case 5. $r = 4$. Then $a \equiv 4 \pmod 8$ and so $a^3 \equiv 4^3 \equiv 0 \pmod 8$.

Case 6. $r = 5$. Then $a \equiv 5 \pmod 8$ and so $a^3 \equiv 5^3 \equiv 5 \pmod 8$.

Case 7. $r = 6$. Then $a \equiv 6 \pmod 8$ and so $a^3 \equiv 6^3 \equiv 0 \pmod 8$.

Case 8. $r = 7$. Then $a \equiv 7 \pmod 8$ and so $a^3 \equiv 7^3 \equiv 7 \pmod 8$.

(ii) To prove in the same way as (i).

Lemma 4. Let a and b be non-negative integers. Then

(i) if $a \equiv 1, 3 \pmod 8$, then $a^b \equiv 1, 3 \pmod 8$,

(ii) if $a \equiv 1, 3 \pmod{13}$, then $a^b \equiv 1, 3, 9 \pmod{13}$.

Proof.

(i) Case 1. $a \equiv 1 \pmod 8$. Then $a^b \equiv 1 \pmod 8$.

Case 2. $a \equiv 3 \pmod 8$.

If $b = 2k$ for some non-negative integer k, then $a^b \equiv 3^{2k} \equiv 1 \pmod 8$.

If $b = 2k + 1$ for some non-negative integer k, then $a^b \equiv 3^{2k+1} \equiv 3 \pmod 8$.

(ii) Case 1. $a \equiv 1 \pmod{13}$. Then $a^b \equiv 1 \pmod{13}$.

Case 2. $a \equiv 3 \pmod{13}$.

If $b = 3k$ for some non-negative integer k, then $a^b \equiv 3^{3k} \equiv 1 \pmod{13}$.

If $b = 3k + 1$ for some non-negative integer k, then $a^b \equiv 3^{3k+1} \equiv 3 \pmod{13}$.

If $b = 3k + 2$ for some non-negative integer k, then $a^b \equiv 3^{3k+2} \equiv 9 \pmod{13}$.

3. Results

In this section, we investigate some conditions of the non-existence of non-negative integer solutions for the equation (1).

Theorem 5. If $n \equiv 1, 3 \pmod 8$, then the equation (1) has no non-negative integer solution.

Proof. Let $n \equiv 1, 3 \pmod 8$. Assume that the equation (1) has a non-negative integer solution. By Lemma 4(i), we get $3^x \equiv 1, 3 \pmod 8$. Since $n \equiv 1, 3 \pmod 8$ and Lemma 4(i), we obtain $n^y \equiv 1, 3 \pmod 8$. We consider the following cases:

Case 1. $3^x \equiv 1 \pmod 8$ and $n^y \equiv 1 \pmod 8$. Then $3^x + n^y \equiv 2 \pmod 8$.

Case 2. $3^x \equiv 1 \pmod 8$ and $n^y \equiv 3 \pmod 8$. Then $3^x + n^y \equiv 4 \pmod 8$.

Case 3. $3^x \equiv 3 \pmod 8$ and $n^y \equiv 1 \pmod 8$. Then $3^x + n^y \equiv 4 \pmod 8$.

Case 4. $3^x \equiv 3 \pmod 8$ and $n^y \equiv 3 \pmod 8$. Then $3^x + n^y \equiv 6 \pmod 8$.

Thus $3^x + n^y \equiv 2, 4, 6 \pmod 8$. From the equation (1), it implies that $z^3 \equiv 2, 4, 6 \pmod 8$. This is impossible by Lemma 3(i).

Theorem 6. If $n \equiv 5, 7 \pmod 8$ and y is even number, then the equation (1) has no non-negative integer solution.

Proof. Let $n \equiv 5, 7 \pmod 8$ and let y be even number. Assume that the equation (1) has a non-negative integer solution. Since $n \equiv 5, 7 \pmod 8$ and y is even number, we obtain $n^y \equiv 1 \pmod 8$. By Lemma 4(i), we have $3^x \equiv 1, 3 \pmod 8$. Since $n^y \equiv 1 \pmod 8$, we get $3^x + n^y \equiv 2, 4 \pmod 8$. From the equation (1), it implies that $z^3 \equiv 2, 4 \pmod 8$. This is impossible by Lemma 3(i).

Theorem 7. If $n \equiv 1 \pmod{13}$, then the equation (1) has no non-negative integer solution.

Proof. Let $n \equiv 1 \pmod{13}$. Assume that the equation (1) has a non-negative integer solution. Since $n \equiv 1 \pmod{13}$, we have $n^y \equiv 1 \pmod{13}$. By Lemma 4(ii), we get $3^x \equiv 1, 3, 9 \pmod{13}$. Therefore $3^x + n^y \equiv 2, 4, 10 \pmod{13}$. From the equation (1), we obtain $z^3 \equiv 2, 4, 10 \pmod{13}$. This is impossible by Lemma 3(ii).

Theorem 8. If $n \equiv 1, 2 \pmod 3$ and $y \equiv 0 \pmod 3$, then the equation (1) has no non-negative integer solution.

Proof. Let $n \equiv 1, 2 \pmod 3$ and $y \equiv 0 \pmod 3$. Assume that the equation (1) has a non-negative integer solution. Since $y \equiv 0 \pmod 3$, we get $y = 3h$ for some non-negative integer h. From the equation (1), it implies that

$$(z - n^h)(z^2 + z \cdot n^h + n^{2h}) = 3^x. \qquad (2)$$

Then there exists a non-negative integer v such that

$$z - n^h = 3^v \qquad (3)$$

and $$z^2 + z \cdot n^h + n^{2h} = 3^{x-v}. \qquad (4)$$

From the equation (3) and (4), we have

$$3^{2v} + 3^{v+1} \cdot n^h + 3 \cdot n^{2h} = 3^{x-v}. \qquad (5)$$

Assume that $v = 0$. Then $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 3^x$ and so $1 \equiv 0 \pmod 3$, a contradiction. Thus $v > 0$. From the equation (5), we get

$$3^{2v-1} + 3^v \cdot n^h + n^{2h} = 3^{x-v-1}. \qquad (6)$$

It follows that $h > 0$ and so $n \equiv 0 \pmod 3$. This is impossible since $n \equiv 1, 2 \pmod 3$.

Lemma 9. If $n \equiv 0 \pmod{13}$ and the equation (1) has a non-negative integer solution, then $x \equiv 0 \pmod 3$.

Proof. Let $n \equiv 0 \pmod{13}$ and the equation (1) has a non-negative integer solution. By Corollary 2, it implies that $y > 0$. Since $n \equiv 0 \pmod{13}$ and the equation (1), we obtain $3^x \equiv z^3 \pmod{13}$. By Lemma 3(ii), it follows that $3^x \equiv 0, 1, 5, 8, 12 \pmod{13}$. Assume that $x \not\equiv 0 \pmod{3}$. Then $x \equiv 1, 2 \pmod{3}$. We consider the following cases:

Case 1. $x \equiv 1 \pmod{3}$. There exists a non-negative integer k such that $x = 3k + 1$. It implies that $3^x = 3^{3k+1} = 3 \cdot 27^k \equiv 3 \cdot (1)^k \equiv 3 \pmod{13}$. This is impossible since $3^x \equiv 0, 1, 5, 8, 12 \pmod{13}$.

Case 2. $x \equiv 2 \pmod{3}$. There exists a non-negative integer k such that $x = 3k + 2$. It implies that $3^x = 3^{3k+2} = 9 \cdot 27^k \equiv 9 \cdot (1)^k \equiv 9 \pmod{13}$. This is impossible since $3^x \equiv 0, 1, 5, 8, 12 \pmod{13}$.

From Cases 1 and 2, we get $x \equiv 0 \pmod{3}$.

Theorem 10. If $n = 13$, then the equation (1) has no non-negative integer solution.

Proof. Let $n = 13$. Assume that the equation (1) has a non-negative integer solution. By Lemma 9, we have $x \equiv 0 \pmod{3}$. Then $x = 3k$ for some non-negative integer k. From the equation (1), we get

$$(z - 3^k)(z^2 + z \cdot 3^k + 3^{2k}) = 13^y. \tag{7}$$

Then there exists a non-negative integer u such that

$$z - 3^k = 13^u \tag{8}$$

$$\text{and} \quad z^2 + z \cdot 3^k + 3^{2k} = 13^{y-u}. \tag{9}$$

From the equation (8) and (9), we have

$$13^{2u} + 13^u \cdot 3^{k+1} + 3^{2k+1} = 13^{y-u}. \tag{10}$$

Assume that $u > 0$. Then $3^{2k+1} \equiv 0 \pmod{13}$. This is impossible by Lemma 4(ii). Therefore $u = 0$ and so

$$1 + 3^{k+1} + 3^{2k+1} = 13^y. \tag{11}$$

By Lemma 4(ii), we get $3^k \equiv 1, 3, 9 \pmod{13}$. We consider the following cases:

Case 1. $3^k \equiv 1 \pmod{13}$. It follows that $1 + 3^{k+1} + 3^{2k+1} \equiv 1 + 3 \cdot 1 + 3 \cdot 1^2 \equiv 7 \pmod{13}$.

Case 2. $3^k \equiv 3 \pmod{13}$. It follows that $1 + 3^{k+1} + 3^{2k+1} \equiv 1 + 3 \cdot 3 + 3 \cdot 3^2 \equiv 37 \equiv 11 \pmod{13}$.

Case 3. $3^k \equiv 9 \pmod{13}$. It follows that $1 + 3^{k+1} + 3^{2k+1} \equiv 1 + 3 \cdot 9 + 3 \cdot 9^2 \equiv 271 \equiv 11 \pmod{13}$.

By Cases 1, 2 and 3, we have $1 + 3^{k+1} + 3^{2k+1} \equiv 7, 11 \pmod{13}$. From the equation (11), we get $3^y \equiv 7, 11 \pmod{13}$. This is impossible since $13^y \equiv 0, 1 \pmod{13}$.

Theorem 11. If $n = 39$, then the equation (1) has no non-negative integer solution.

Proof. Let $n = 39$. Assume that the equation (1) has a non-negative integer solution. If $x \geq y$, from the equation (1), we have

$$3^y(3^{x-y} + 13^y) = z^3. \tag{12}$$

Since $\gcd(3^y,\ 3^{x-y} + 13^y) = 1$, there exists a positive integer t such that

$$3^{x-y} + 13^{y} = t^{3}. \quad (13)$$

This is impossible by Theorem 10. Thus $x < y$. From the equation (1), we have

$$3^{x}(1 + 3^{y-x} \cdot 13^{y}) = z^{3}. \quad (14)$$

Since gcd $(3^{x}, 1 + 3^{y-x} \cdot 13^{y}) = 1$, there exists a positive integer s such that

$$1 + 3^{y-x} \cdot 13^{y} = s^{3}. \quad (15)$$

Therefore

$$(s-1)(s^{2} + s + 1) = 3^{y-x} \cdot 13^{y}. \quad (16)$$

Then there exist non-negative integers u and v such that

$$s - 1 = 3^{u} \cdot 13^{v} \quad (17)$$

and $$s^{2} + s + 1 = 3^{y-x-u} \cdot 13^{y-v}. \quad (18)$$

From the equation (17) and (18), we have

$$3^{2u} \cdot 13^{2v} + 3^{u+1} \cdot 13^{v} + 3 = 3^{y-x-u} \cdot 13^{y-v}. \quad (19)$$

If $u = 0$, then $13^{2v} \equiv 0 \pmod 3$, a contradiction. Thus $u > 0$ and so

$$3^{2u-1} \cdot 13^{2v} + 3^{u} \cdot 13^{v} + 1 = 3^{y-x-u-1} \cdot 13^{y-v}. \quad (20)$$

Assume that $v = 0$. Therefore $3^{2u-1} + 3^{u} + 1 \equiv 0 \pmod{13}$. This is impossible since $3^{2u-1} + 3^{u} + 1 \equiv 7, 11 \pmod{13}$. Thus $v > 0$. Suppose that $y - v = 0$. From the equation (20), we get $1 \equiv 0 \pmod 3$, a contradiction. Then $y - v > 0$. From the equation (20), we have $1 \equiv 0 \pmod{13}$, which is impossible.

4. Conclusion

In this paper, we showed that the Diophantine equation $3^{x} + n^{y} = z^{3}$ has no non-negative integer solution, where n is a positive integer and x, y, z are non-negative integers with the following conditions: 1) $n \equiv 1, 3 \pmod 8$, 2) $n \equiv 5, 7 \pmod 8$ and y is even, 3) $n \equiv 1 \pmod{13}$, 4) $n \equiv 1, 2 \pmod 3$ and $y \equiv 0 \pmod 3$, 5) $n = 13$ and 6) $n = 39$.

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