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On Finding Integer Solutions to Homogeneous Ternary Quadratic Diophantine Equation $x^2+(2k+1)y^2=(k+1)^2z^2$

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Abstract

The theoretical importance of polynomial equations of second degree in three unknowns with integral coefficients is great as they are closely connected with many problems of number theory. Specifically, the second degree polynomial equations with three unknowns in connection with geometrical figures occupy a pivotal role in the region of mathematics. The successful completion of exhibiting all integers satisfying the requirements set forth in the problem add to further progress of Number Theory as they offer good applications in the field of Graph theory, Modular theory, Coding and Cryptography, Engineering, Music and so on. Integers have repeatedly played a crucial role in the evolution of the Natural Sciences. The theory of integers provides answers to real world problems. **Objectives:** The objective of this research paper is to obtain varieties of integer solutions to homogeneous ternary quadratic Diophantine equation represented by $x^2+(2k+1)y^2=(k+1)^2z^2$. Geometrically, the considered polynomial equations of degree two with three unknowns represents cone. **Methods:** Various choices of integer solutions are secure from beginning to end employing linear modifications and used to simplify expressions. Patterns of solutions in integers are obtained by reducing the given polynomial equation to the equation which is solvable through employing suitable transformations and applying the factorization method. **Findings:** Six distinct transformations are applied to obtain choices of integral solutions for the considered second degree equation having three unknowns. **Novelty:** The equation in title has been reduced to either solvable ternary quadratic equation or a system of simultaneous equations through employing suitable transformations. Many lattice points satisfying the given cone are obtained through analytical process by means of substitution strategy and method of factorization.

Keywords: Quadratic equation, Homogeneous quadratic, Ternary quadratic, Integer solutions

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1. Introduction

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety [1, 2]. In particular, the ternary quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. These types of equations are significant since they geometrically represent three dimensional figures namely cones. These solutions play a vital role in different area of mathematics & science and help us in understanding the significance of number patterns. The theoretical importance of quadratic equations in three unknowns with integral coefficients is great as they are closely connected with many problems of number theory. Specifically, the second degree homogeneous equations with three unknowns occupy a pivotal role in the region of mathematics. It is well-known that the quadratic Diophantine equations, homogeneous or non-homogeneous, have aroused the interest of many mathematicians. In particular, one may refer [3-10] for quadratic equations with three unknowns representing different geometrical figures. The above problems motivated us to search for the distinct integer solutions to ternary homogeneous quadratic equations.

In this paper, a typical ternary quadratic Diophantine equation representing cone given by $x^2+(2k+1)y^2=(k+1)^2z^2$ is studied for determining its integer solutions successfully through elementary algebra, namely, substitution strategy and factorization method. **Methodology**

The homogeneous second degree equation in three unknowns to be solved is

$$x^2+(2k+1)y^2=(k+1)^2z^2 \tag{1}$$

To start with, (1) is satisfied by

$$x=4k^3+6k^2+3k, y=2k+1, z=4k^2+2k+1.$$

However, there are many more integer solutions and the process of obtaining various solution patterns is illustrated below:

Process 1

Taking

$$x=(k+1)X, y=(k+1)Y \tag{2}$$

in (1), it leads to the ternary quadratic equation

$$X^2+(2k+1)Y^2=z^2 \tag{3}$$

which is satisfied by

$$Y=2pq, X=(2k+1)p^2-q^2 \tag{4}$$

and

$$z=(2k+1)p^2+q^2 \tag{5}$$

Using (4) in (2), we have

$$x = (k+1)\,[(2k+1)\,p^2 - q^2\,]\,,$$
$$y = 2(k+1)\,p\,q. \qquad (6)$$

Thus, (5) & (6) represent the integer solutions to (1).

A few numerical examples are given in Table 1.

Table 1 Numerical Examples:

k	p	q	x	y	z
2	1	3	-12	18	14
1	2	4	-8	32	28
3	3	2	236	48	67
4	4	5	595	200	169

Process 2

Consider (3) as the system of double equations as shown below

$$z + X = Y^2$$
$$z - X = 2k+1$$

Solving the above pair of equations, we have

$$Y = 2s+1\,,\, X = 2s^2 + 2s - k \qquad (7)$$

and

$$z = 2s^2 + 2s + k + 1 \qquad (8)$$

From (7) and (2), we get

$$x = (k+1)\;(2s^2 + 2s - k)\;,$$
$$y = (k+1)(2s+1). \qquad (9)$$

Thus, (8) & (9) satisfy (1).

Note 1

It is to be noted that, one may write (3) as the pair of equations as follows:

$$z + X = (2k+1)\,Y^2$$
$$z - X = 1$$

In this case, the solutions to (1) are obtained as

$$x = (k+1)\,[k(2s+1)^2 + 2s^2 + 2s]\,,$$
$$y = (k+1)\;(2s+1)\,,$$
$$z = [k(2s+1)^2 + 2s^2 + 2s + 1].$$

Process 3

The substitution of the transformations

$$x=k(k+1)X\,,z=(k+1)P+(2k+1)\beta\,,y=(k+1)P+(k+1)^2\beta \tag{10}$$

in (1) leads to the ternary quadratic equation

$$P^2=X^2+(2k+1)\beta^2 \tag{11}$$

which is satisfied by

$$\beta=2pq\,,X=(2k+1)p^2-q^2\,,P=(2k+1)p^2+q^2 \tag{12}$$

In view of (10), the integer solutions to (1) are given by

$$\begin{aligned} x&=k(k+1)[(2k+1)p^2-q^2]\,,\\ y&=(k+1)[(2k+1)p^2+q^2]+2pq\,(k+1)^2\,,\\ z&=(k+1)[(2k+1)p^2+q^2]+2pq\,(2k+1). \end{aligned} \tag{13}$$

Note 2

Apart from (10), one may consider the transformations as

$$x=k(k+1)X\,,z=(k+1)P-(2k+1)\beta\,,y=(k+1)P-(k+1)^2\beta\;.$$

For this choice, the corresponding integer solutions to (1) are given by

$$\begin{aligned} x&=k(k+1)[(2k+1)p^2-q^2]\,,\\ y&=(k+1)[(2k+1)p^2+q^2]-2pq\,(k+1)^2\,,\\ z&=(k+1)[(2k+1)p^2+q^2]-2pq\,(2k+1). \end{aligned}$$

Process 4

Assume

$$z=a^2+(2k+1)b^2 \tag{14}$$

Consider

$$(k+1)^2=(k+i\sqrt{2k+1})\,(k-i\sqrt{2k+1}) \tag{15}$$

Using (14) & (15) in (1) and employing the factorization technique, we write

$$x+i\sqrt{2k+1}\,y=(k+i\sqrt{2k+1})\,(a+i\sqrt{2k+1}\,b)^2$$

On equating the real and imaginary parts in the above equation, we have

$$\begin{aligned} x&=k[a^2-(2k+1)b^2]-2(2k+1)ab\,,\\ y&=2kab+[a^2-(2k+1)b^2]\,. \end{aligned} \tag{16}$$

Observe that (14) & (16) satisfy (1).

Process 5

Write (1) as

$$x^2 + (2k+1)\, y^2 = (k+1)^2\, z^2 * 1 \qquad (17)$$

Express the integer 1 on the R.H.S. of (17) as

$$1 = \frac{(k + i\sqrt{2k+1})\,(k - i\sqrt{2k+1})}{(k+1)^2} \qquad (18)$$

Assume

$$z = (k+1)^2 [a^2 + (2k+1) b^2] \qquad (19)$$

Substituting (15), (18) & (19) in (17) and following the procedure as in Process 4, we get

$$x = (k+1)\{(k^2 - 2k - 1)\,[a^2 - (2k+1)b^2] - 4k(2k+1)\,ab\}\,,$$
$$y = (k+1)\{2k[a^2 - (2k+1)b^2] + 2(k^2 - 2k - 1)\,ab\}\,. \qquad (20)$$

Thus, (19) & (20) satisfy (1).

Remark 1:

It is worth to observe that, the representation of integer 1 given by (18) may have other choices. A few examples are given below.

Example-1:

$$1 = \frac{(9k + 4 + i3\sqrt{2k+1})\,(9k + 4 - i3\sqrt{2k+1})}{(9k+5)^2}$$

Example-2:

$$1 = \frac{(25k + 12 + i5\sqrt{2k+1})\,(25k + 12 - i5\sqrt{2k+1})}{(25k+13)^2}$$

Example-3:

$$1 = \frac{(49k + 24 + i7\sqrt{2k+1})\,(49k + 24 + i7\sqrt{2k+1})}{(49k+25)^2}$$

Example-4:

$$1 = \frac{[(2k+1)p^2 - q^2 + i\,2pq\sqrt{2k+1}]\,[(2k+1)p^2 - q^2 - i\,2pq\sqrt{2k+1}]}{[(2k+1)p^2 + q^2]^2}$$

Example-5:

$$1 = \frac{[p^2 - (2k+1)q^2 + i\,2pq\sqrt{2k+1}]\,[p^2 - (2k+1)q^2 - i\,2pq\sqrt{2k+1}]}{[p^2 + (2k+1)q^2]^2}$$

Process 6

It is to be observed that, choosing the values of k to be

$$k = 2s^2 + 2s$$

in (1) and employing the transformations

$$x = (2s+1)(2s^2+2s+1)\,X\;,\;y = (2s^2+2s+1)\,Y\,,\;z = (2s+1)\,w \tag{21}$$

in (1), it reduces to the Pythagorean equation given by

$$X^2 + Y^2 = w^2 \tag{22}$$

Considering the most cited solutions of (22) and utilizing (21), the corresponding integer 0solutions to (1) are obtained.

2. Conclusion:

In this paper, the homogeneous ternary quadratic equation representing homogeneous cone given by $x^2 + (2k+1)y^2 = (k+1)^2 z^2$ is studied for obtaining its integer solutions through substitution technique and factorization method. One may search for other forms of quadratic equations with multiple variables to determine their integer solutions.

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