

On the Solutions of Two Egyptian Fractional Equations

Suton Tadee^{*}

Department of Mathematics, Faculty of Science and Technology, Thepsatri Rajabhat University

*E-mail: suton.t@lawasri.tru.ac.th

Abstract

In this research, we have studied all positive integer solutions of two Egyptian fractional equations $\frac{1}{a} + \frac{1}{b} = \frac{mn}{m+1}$ and $\frac{1}{a} + \frac{1}{b} = \frac{m}{p^n}$. The results showed that the Egyptian fractional equation $\frac{1}{a} + \frac{1}{b} = \frac{mn}{m+1}$ has twelve positive integer solutions (a, b, m, n) , which are $(2, 3, 5, 1)$, $(2, 4, 3, 1)$, $(1, 2, 3, 2)$, $(2, 6, 2, 1)$, $(1, 3, 2, 2)$, $(3, 6, 1, 1)$, $(1, 2, 1, 3)$, $(3, 3, 2, 1)$, $(1, 1, 2, 3)$, $(4, 4, 1, 1)$, $(2, 2, 1, 2)$ and $(1, 1, 1, 4)$. The Egyptian fractional equation $\frac{1}{a} + \frac{1}{b} = \frac{m}{p^n}$, where p is a prime number and n is a positive integer, has all positive integer solutions (a, b, m) , which are in the form $\left(p^i k, k, \frac{p^{n-i}(p^i+1)}{k}\right)$, where i is an integer with $0 \leq i \leq n$ and k is a positive integer with $k|p^{n-i}(p^i+1)$.

Keywords: Egyptian fractional equation, Diophantine equation, Prime number

* Corresponding author, e-mail: suton.t@lawasri.tru.ac.th

1. Introduction

The Egyptian fractions are rational numbers, which can be represented as the sum of positive unit fractions: $\frac{a}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k}$, where $a, n, x_1, x_2, \dots, x_k$ are positive integers [1]. Meanwhile, we call this equation as the Egyptian fractional equation of length k , which is a Diophantine equation. Many mathematicians have studied and found positive integer solutions of the Egyptian fractional equation of length 2. For example: in 2011, Kishan, Rani and Agarwal [2] showed some positive integer solutions of the Egyptian fractional equations $\frac{2}{n} = \frac{1}{x} + \frac{1}{y}$ and $\frac{3}{n} = \frac{1}{x} + \frac{1}{y}$. In 2013, Rabago and Tagle [3] proved that the Egyptian fractional equation $\frac{q}{p} = \frac{1}{x} + \frac{1}{y}$ has a positive integer solution, where $x \neq y$ and p is a prime number and q is a positive integer, if and only if $p + 1 \equiv 0 \pmod{q}$.

In 2022, Johnson [4] solved the Egyptian fractional equation $\frac{1}{a} + \frac{1}{b} = \frac{q+1}{pq}$, where a, b are positive integers and p, q are distinct prime numbers such that $(q + 1) | (p - 1)$. After that, Prugsapitak ([5], [6]) investigated all positive integer solutions of the Egyptian fractional equation $\frac{1}{a} + \frac{1}{b} = \frac{q-1}{pq}$, where p and q are prime numbers such that $p > q$. In 2023, Tadee [7] found all positive integer solutions of the Egyptian fractional equations $\frac{1}{a} + \frac{1}{b} = \frac{nq+1}{pq}$ and $\frac{1}{a} + \frac{1}{b} = \frac{nq-1}{pq}$, where n is a positive integer and p, q are prime numbers with $p > nq$. Moreover, in 2024, Tadee [8] also found all positive integer solutions of the Egyptian fractional equation $\frac{1}{a} + \frac{1}{b} = \frac{m}{pq}$, where m is a positive integer and p, q are distinct prime numbers.

Inspired by the work mentioned earlier, we will solve two Egyptian fractional equations

$$\frac{1}{a} + \frac{1}{b} = \frac{mn}{m+1} \quad (1)$$

and

$$\frac{1}{a} + \frac{1}{b} = \frac{m}{p^n}, \quad (2)$$

where a, b, m, n are positive integers and p is a prime number.

2. Results

In the beginning of this section, we will find all positive integer solutions of the Egyptian fractional equation (1).

Theorem 1. If $n = 1$, then all positive integer solutions (a, b, m) of the Egyptian fractional equation (1) are $(2, 3, 5)$, $(2, 4, 3)$, $(2, 6, 2)$, $(3, 6, 1)$, $(3, 3, 2)$ and $(4, 4, 1)$.

Proof. Let $n = 1$ and (a, b, m) be a positive integer solution of the equation (1). Without loss of generality, we may assume that $a \leq b$ and so $\frac{1}{b} \leq \frac{1}{a}$. Then $\frac{1}{a} + \frac{1}{b} \leq \frac{2}{a}$. From the equation (1), we have $\frac{m}{m+1} \leq \frac{2}{a}$. It implies that

$$(a - 2)m \leq 2. \quad (3)$$

Since a is a positive integer, we consider the following cases:

Case 1. $a = 1$. Then $\frac{1}{a} + \frac{1}{b} > 1$. From the equation (1), we obtain that $\frac{m}{m+1} > 1$. This is impossible. Therefore, in this case, the equation (1) has no solution.

Case 2. $a = 2$. From the equation (1), we get $m = 1 + \frac{4}{b-2}$. Since m is a positive integer, we have $b - 2 = 1, 2$ and 4 . Then $b = 3, 4$ and 6 . Thus $(a, b, m) = (2, 3, 5), (2, 4, 3)$ and $(2, 6, 2)$, respectively.

Case 3. $a = 3$. From the inequation (3), we get $m = 1$ and 2 . Then $(a, b, m) = (3, 6, 1)$ and $(3, 3, 2)$, respectively.

Case 4. $a = 4$. From the inequation (3), we get $m = 1$. Then $(a, b, m) = (4, 4, 1)$.

Case 5. $a \geq 5$. Then $(a - 2)m \geq 3m$. From the inequation (3), it implies that $2 \geq 3m$. This is impossible. Therefore, in this case, the equation (1) has no solution.

From the above cases, we conclude that if $n = 1$, then the Egyptian fractional equation (1) has only six positive integer solutions (a, b, m) , which are $(2, 3, 5), (2, 4, 3), (2, 6, 2), (3, 6, 1), (3, 3, 2)$ and $(4, 4, 1)$.

Theorem 2. All positive integer solutions (a, b, m, n) of the Egyptian fractional equation (1) are $(2, 3, 5, 1), (2, 4, 3, 1), (1, 2, 3, 2), (2, 6, 2, 1), (1, 3, 2, 2), (3, 6, 1, 1), (1, 2, 1, 3), (3, 3, 2, 1), (1, 1, 2, 3), (4, 4, 1, 1), (2, 2, 1, 2)$, and $(1, 1, 1, 4)$.

Proof. Let a, b, m and n be positive integers such that the equation (1) is true. Without loss of generality, we may assume that $a \leq b$. From the equation (1), we divide both sides by n . Then

$$\frac{1}{an} + \frac{1}{bn} = \frac{m}{m+1} \quad (4)$$

By Theorem 1, we consider the following cases:

Case 1. $(an, bn, m) = (2, 3, 5)$. Then $an = 2, bn = 3$ and $m = 5$. It implies that $n|2$ and $n|3$. Thus $n = 1$. Therefore $(a, b, m, n) = (2, 3, 5, 1)$.

Case 2. $(an, bn, m) = (2, 4, 3)$. Then $an = 2, bn = 4$ and $m = 3$. It implies that $n|2$ and $n|4$. Thus $n = 1$ and 2 . Therefore $(a, b, m, n) = (2, 4, 3, 1)$ and $(1, 2, 3, 2)$, respectively.

Case 3. $(an, bn, m) = (2, 6, 2)$. Then $an = 2, bn = 6$ and $m = 2$. It implies that $n|2$ and $n|6$. Thus $n = 1$ and 2 . Therefore $(a, b, m, n) = (2, 6, 2, 1)$ and $(1, 3, 2, 2)$, respectively.

Case 4. $(an, bn, m) = (3, 6, 1)$. Then $an = 3, bn = 6$ and $m = 1$. It implies that $n|3$ and $n|6$. Thus $n = 1$ and 3 . Therefore $(a, b, m, n) = (3, 6, 1, 1)$ and $(1, 2, 1, 3)$, respectively.

Case 5. $(an, bn, m) = (3, 3, 2)$. Then $an = 3, bn = 3$ and $m = 2$. It implies that $n|3$. Thus $n = 1$ and 3 . Therefore $(a, b, m, n) = (3, 3, 2, 1)$ and $(1, 1, 2, 3)$, respectively.

Case 6. $(an, bn, m) = (4, 4, 1)$. Then $an = 4, bn = 4$ and $m = 1$. It implies that $n|4$. Thus $n = 1, 2$ and 4 . Then $(a, b, m, n) = (4, 4, 1, 1), (2, 2, 1, 2)$ and $(1, 1, 1, 4)$, respectively.

Hence, the Egyptian fractional equation (1) has only twelve positive integer solutions (a, b, m, n) , which are $(2, 3, 5, 1), (2, 4, 3, 1), (1, 2, 3, 2), (2, 6, 2, 1), (1, 3, 2, 2), (3, 6, 1, 1), (1, 2, 1, 3), (3, 3, 2, 1), (1, 1, 2, 3), (4, 4, 1, 1), (2, 2, 1, 2)$ and $(1, 1, 1, 4)$.

Next, we will prove the following theorems, which will be used to find all positive integer solutions of the Egyptian fractional equation (2).

Theorem 3. Let p be a prime number. Then all positive integer solutions (a, b) of the Egyptian fractional equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{p} \quad (5)$$

are $(2p, 2p)$ and $(p(p+1), p+1)$.

Proof. Let a and b be positive integers such that the equation (5) is true. Then $p(a+b) = ab$. Since p is a prime number, we have $p|a$ or $p|b$. Without loss of generality, we may assume that $p|a$. Then there exists a positive integer k such that $a = pk$. Substituting $a = pk$ into the equation $p(a+b) = ab$, we get

$$pk = b(k-1). \quad (6)$$

Therefore $p|b$ or $p|(k-1)$. Next, we consider the following cases:

Case 1. $p|b$. Then there exists a positive integer s such that $b = ps$. From the equation (6), we have $s = 1 + \frac{1}{k-1}$. Therefore $k-1 = 1$ and so $s = 2$. Thus $(a, b) = (2p, 2p)$.

Case 2. $p|(k-1)$. Then there exists a positive integer t such that $k-1 = pt$. It implies that $1 = t(b-p)$ and so $1 = t$. Thus $(a, b) = (p(p+1), p+1)$.

Hence, the Egyptian fractional equation (5) has only two positive integer solutions (a, b) , which are $(2p, 2p)$ and $(p(p+1), p+1)$.

Theorem 4. Let p be a prime number. Then all positive integer solutions (a, b, m) of the Egyptian fractional equation

$$\frac{1}{a} + \frac{1}{b} = \frac{m}{p} \quad (7)$$

are $(2p, 2p, 1), (p, p, 2), (2, 2, p), (1, 1, 2p)$ and $(pk, k, \frac{p+1}{k})$, where k is a positive integer with $k|(p+1)$.

Proof. Let a, b and m be positive integers such that the equation (7) is true. Then

$$\frac{1}{am} + \frac{1}{bm} = \frac{1}{p}. \quad (8)$$

By Theorem 3, we consider the following cases:

Case 1. $(am, bm) = (2p, 2p)$. Then $am = bm = 2p$. Since 2 and p are prime numbers, we have the following subcases:

Subcase 1.1 $m = 1$. Then $(a, b, m) = (2p, 2p, 1)$.

Subcase 1.2 $m = 2$. Then $(a, b, m) = (p, p, 2)$.

Subcase 1.3 $m = p$. Then $(a, b, m) = (2, 2, p)$.

Subcase 1.4 $m = 2p$. Then $(a, b, m) = (1, 1, 2p)$.

Case 2. $(am, bm) = (p(p+1), p+1)$. Therefore $am = p(p+1)$ and $bm = p+1$. Thus $p|a$. Then there exists a positive integer t such that $a = pt$. It implies that $b = t$. Hence $(a, b, m) = (pk, k, \frac{p+1}{k})$, where k is a positive integer with $k|(p+1)$.

Theorem 5. Let p be a prime number and n be a positive integer. Then all positive integer solutions (a, b) of the Egyptian fractional equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{p^n} \quad (9)$$

are $(p^n(p^i + 1), p^{n-i}(p^i + 1))$, where i is an integer with $0 \leq i \leq n$.

Proof. Let a and b be positive integers such that the equation (9) is true. We will prove by induction that $(a, b) = (p^n(p^i + 1), p^{n-i}(p^i + 1))$, where i is an integer with $0 \leq i \leq n$. If $n = 1$, then we have $(a, b) \in \{(2p, 2p), (p(p+1), p+1)\}$, by Theorem 3. Thus, the statement is true for $n = 1$.

Let k be a positive integer. Suppose that all positive integer solutions (a, b) of the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{p^k}. \quad (10)$$

are $(a, b) = (p^k(p^i + 1), p^{k-i}(p^i + 1))$, where i is an integer with $0 \leq i \leq k$. Next, we solve the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{p^{k+1}}. \quad (11)$$

Then

$$p^{k+1}(a + b) = ab. \quad (12)$$

Therefore $p|ab$. Since p is a prime number, we have $p|a$ or $p|b$.

Case 1. $p|a$ and $p|b$. Then there exist two positive integers a_1 and b_1 such that $a = pa_1$ and $b = pb_1$. From the equation (11), we get

$$\frac{1}{a_1} + \frac{1}{b_1} = \frac{1}{p^k}. \quad (13)$$

By assumption, it implies that $(a_1, b_1) = (p^k(p^i + 1), p^{k-i}(p^i + 1))$, where i is an integer with $0 \leq i \leq k$. Therefore $(a, b) = (p^{k+1}(p^i + 1), p^{k+1-i}(p^i + 1))$, where i is an integer with $0 \leq i \leq k$.

Case 2. $p|a$ and $p \nmid b$. Then there exists a positive integer a_1 such that $a = a_0 = pa_1$. From the equation (12), we get

$$p^k(pa_1 + b) = a_1b. \quad (14)$$

Since $p \nmid b$ and p is a prime number, we have $p|a_1$. Then there exists a positive integer a_2 such that $a_1 = pa_2$. From the equation (14), we get

$$p^{k-1}(p^2a_2 + b) = a_2b. \quad (15)$$

By following steps like this, we will get that $a_{j-1} = pa_j$, where i is an integer with $1 \leq j \leq k + 1$. Then $a = p^{k+1}a_{k+1}$. From the equation (12), we have

$$p^{k+1}a_{k+1} + b = a_{k+1}b. \quad (16)$$

Then

$$p^{k+1}a_{k+1} = (a_{k+1} - 1)b. \quad (17)$$

Since $p \nmid b$ and p is a prime number, we have $p|(a_{k+1} - 1)$. Then there exists a positive integer a_{k+2} such that $a_{k+1} - 1 = pa_{k+2}$. From the equation (17), we get

$$p^{k+1}a_{k+2} + p^k = a_{k+2}b. \quad (18)$$

Then

$$p^k = a_{k+2}(b - p^{k+1}). \quad (19)$$

Since $p \nmid b$ and p is a prime number, we have $a_{k+2} = p^k$ and $b = p^{k+1} + 1$. Therefore $a = p^{k+1}(p^{k+1} + 1)$. It follows that $(a, b) = (p^{k+1}(p^{k+1} + 1), p^{k+1} + 1)$.

From two above cases, we obtain that $(a, b) = (p^{k+1}(p^i + 1), p^{k+1-i}(p^i + 1))$, where i is an integer with $0 \leq i \leq k + 1$. The proof is complete.

Theorem 6. Let p be a prime number and n be a positive integer. Then all positive integer solutions (a, b, m) of the Egyptian fractional equation (2) are $(p^i k, k, \frac{p^{n-i}(p^i+1)}{k})$, where i is an integer with $0 \leq i \leq n$ and k is a positive integer with $k|p^{n-i}(p^i + 1)$.

Proof. Let a, b and m be positive integers such that the equation (2) is true. It implies that

$$\frac{1}{am} + \frac{1}{bm} = \frac{1}{p^n}. \quad (20)$$

By Theorem 5, we have $(am, bm) = (p^n(p^i + 1), p^{n-i}(p^i + 1))$, where i is an integer with $0 \leq i \leq n$. Then $am = p^n(p^i + 1)$ and $bm = p^{n-i}(p^i + 1)$. It follows that $a = p^i b$ and $m = \frac{p^{n-i}(p^i+1)}{b}$. Hence $(a, b, m) = (p^i k, k, \frac{p^{n-i}(p^i+1)}{k})$, where i is an integer with $0 \leq i \leq n$ and k is a positive integer with $k|p^{n-i}(p^i + 1)$.

From Theorem 6, we get the following example.

Example 1. The Egyptian fractional equation $\frac{1}{a} + \frac{1}{b} = \frac{m}{4}$ has ten positive integer solutions (a, b, m) . There are $(1, 1, 8)$, $(2, 2, 4)$, $(4, 4, 2)$, $(8, 8, 1)$, $(2, 1, 6)$, $(4, 2, 3)$, $(6, 3, 2)$, $(12, 6, 1)$, $(4, 1, 5)$ and $(20, 5, 1)$.

3. Conclusion

By using elementary methods, two Egyptian fractional equations $\frac{1}{a} + \frac{1}{b} = \frac{mn}{m+1}$ and $\frac{1}{a} + \frac{1}{b} = \frac{m}{p^n}$, where a, b, m, n are positive integers and p is a prime number, are investigated. After studying and researching, we found that two Egyptian fractional equations have not been solved. Moreover, we expect that the methods, which are used in this research, can be applied to find solutions of the Egyptian fractional equations of length greater than 2.

4. Acknowledgments

The author would like to thank the referees for their valuable comments and suggestions. This work was supported by Research and Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

5. References

- [1] Zhao, W., Lu, J. & Wang, L. (2021). On the integral solutions of the Egyptian fraction equation $\frac{a}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. *AIMS Mathematics*, 6(5), 4930 – 4937.
<https://doi.org/10.3934/math.2021289>
- [2] Kishan, H., Rani, M. & Agarwal, S. (2011). The Diophantine equations of second and Higher degree of the form $3xy = n(x + y)$ and $3xyz = n(xy + yz + zx)$ etc. *Asian Journal of Algebra*, 4(1), 31– 37. <https://doi.org/10.3923/aja.2011.31.37>
- [3] Rabago, J.F.T. & Tagle, R.P. (2013). On the area and volume of a certain regular solid and the Diophantine equation $\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. *Notes on Number Theory and Discrete Mathematics*, 19(3), 28 – 32.
- [4] Johnson, J.W. (2022). A Diophantine equation with an elementary solution. *The College Mathematics Journal*, 53(5), 361 – 363.
- [5] Prugsapitak, S. (2023). The Egyptian fraction of the form $\frac{1}{a} + \frac{1}{b} = \frac{q-1}{pq}$. *International Journal of Mathematics and Computer Science*, 18(4), 595 – 597.
- [6] Prugsapitak, S. (2024). Corrigendum to the Egyptian fraction of the form $\frac{1}{a} + \frac{1}{b} = \frac{q-1}{pq}$. *International Journal of Mathematics and Computer Science*, 19(1), 237.
- [7] Tadee, S. (2023). On the Diophantine equations $\frac{1}{a} + \frac{1}{b} = \frac{nq+1}{pq}$ and $\frac{1}{a} + \frac{1}{b} = \frac{nq-1}{pq}$. *Udon Thani Rajabhat University Journal of Sciences and Technology*, 11(3), 51 – 60.
- [8] Tadee, S. (2024). On the Diophantine equation $\frac{1}{a} + \frac{1}{b} = \frac{m}{pq}$. *International Journal of Mathematics and Computer Science*, 19(2), 373 – 375.