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Two Chain-Type Exponential Estimators for the Estimation of the Population Mean

Napattchan Dansawad*

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok
126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok, 10140

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Abstract

This paper attempts to suggest two chain-type exponential estimators to estimate population mean when the information regarding auxiliary variable is complete. To the first-order approximation, the bias and the mean squared error (MSE) of the suggested estimators have been discussed. A few members were also derived from the suggested estimators by allocating the different suitable values of constants. In addition, theoretical and numerical studies were used in order to access the efficiency of the suggested estimators. The results of this study show that the suggested estimators are more efficient under bias and percent of relative efficiencies (PREs) criterion compared to other existing estimators.

Keywords : Exponential Estimator; Auxiliary Variable; Bias; Mean Squared Error; Population Mean

**** Corresponding Author. Tel.: +668 3109 1925, E-mail Address: napattchan.dan@kmutt.ac.th***

1. Introduction

In sample survey, the use of auxiliary information has been played the important role to improve the efficiency of the estimates of population parameters such as population mean $\bar{Y}$, variance $S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$, coefficient of variation $C_y = S_y / \bar{Y}$, and so on of the auxiliary variable y. When the population mean $\bar{X}$ of auxiliary variable x is known, a number of modified versions of ratio and product estimators have been proposed by several researchers. It is to be mentioned that the estimators on some population parameters of the auxiliary variable x may be available to the readers, for examples, H. P. Singh et al. [1], S. Bahl and R. K. Tuteja [2], M. Khoshnevisan et al. [3], H. P. Singh and N. Agnihotri [4], and H. P. Singh and S. K. Pal [5].

Consider finite population $U = \{U_1, U_2, ..., U_N\}$. We select a sample of size n from this population using simple random sampling without replacement (SRSWOR). Let y and x respectively be the study and auxiliary variables. While, y_i and x_i be the observations on the ith unit. Let $\bar{y} = \sum_{i=1}^{n} y_i / n$ and $\bar{x} = \sum_{i=1}^{n} x_i / n$ be the sample means of study variable y and auxiliary variable x respectively. It is desired to estimate the population mean $\bar{Y}$ of the study variable y using information on an auxiliary variable x. We assume that the mean of the auxiliary variable $\bar{X}$ is known.

It is also known that the sample mean $\bar{y}$ of study variable y is an unbiased estimator of the population mean $\bar{Y}$ and its variance under SRSWOR is given by

$$MSE(\bar{y}) = \theta \bar{Y}^2 C_y^2 \qquad (1)$$

where

$$\theta = \left(\frac{1-f}{n}\right), \quad f = \frac{n}{N}.$$

M. Khoshnevisan et al. [3] suggested the following class of estimators for the population mean under SRSWOR that covers the existing ratio and product estimators as follows:

$$\bar{y}_1 = \bar{y}\left(\frac{a\bar{X} + b}{\alpha(a\bar{x} + b) + (1-\alpha)(a\bar{X} + b)}\right)^g \qquad (2)$$

where $a(\neq 0)$ and b are real constants or functions of known parameters of auxiliary variable x. α and g are the suitable choice of scalar that makes the MSE of $\bar{y}_1$ as small as possible.

Later, H. P. Singh and N. Agnihotri [4] have adjusted the estimator of M. Khoshnevisan et al. [3] by removing some real constants or functions of auxiliary variable x in the equation of M. Khoshnevisan et al. [3]. The estimator of H. P. Singh and N. Agnihotri [4] are given as follows:

$$\bar{y}_2 = \bar{y}\left[\frac{a\bar{X} + b}{a\bar{x} + b}\right] \qquad (3)$$

and

$$\bar{y}_3 = \bar{y}\left[\frac{a\bar{x} + b}{a\bar{X} + b}\right] \qquad (4)$$

Nonetheless, H. P. Singh et al. [1] proposed an extension of H. P. Singh and N. Agnihotri [4] by using exponentiation method in SRSWOR. This estimator is given as shown below:

$$\bar{y}_4 = \bar{y}\left[\frac{a\bar{X}+b}{a\bar{x}+b}\right]^{g}\left[\frac{a\bar{x}+b}{a\bar{X}+b}\right]^{1-g}$$

$$= \bar{y}\left[\frac{a\bar{X}+b}{a\bar{x}+b}\right]^{2g-1} \quad (5)$$

Previously, S. Bahl and R. K. Tuteja [2] studied various estimation results for the estimator of a population mean $\bar{Y}$ for an alternative in SRSWOR, by pioneering the ratio and product type of estimators in the form of exponential. The estimators of S. Bahl and R. K. Tuteja [2] are presented as follows:

$$\bar{y}_5 = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \quad (6)$$

and

$$\bar{y}_6 = \bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right) \quad (7)$$

Adopting the estimators of S. Bahl and R. K. Tuteja [2], H. P. Singh and S. K. Pal [5] introduced to replace $\bar{y}$ in (6) with $\bar{y}_5 = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$. The chain ratio-type exponential estimator by H. P. Singh and S. K. Pal [5] is given as follows in (8):

$$\bar{y}_7 = \bar{y}_5\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$$

$$= \bar{y}\exp\left(\frac{2(\bar{X}-\bar{x})}{\bar{X}+\bar{x}}\right) \quad (8)$$

In addition, H. P. Singh and S. K. Pal [5] also replaced $\bar{y}$ in (7) with $\bar{y}_6 = \bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$ and suggested chain product-type exponential estimator as follows:

$$\bar{y}_8 = \bar{y}_6\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$$

$$= \bar{y}\exp\left(\frac{2(\bar{x}-\bar{X})}{\bar{x}+\bar{X}}\right) \quad (9)$$

To the first-order approximation, the biases and MSEs of $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, $\bar{y}_4$, $\bar{y}_5$, $\bar{y}_6$, $\bar{y}_7$ and $\bar{y}_8$ are respectively given by

$$B(\bar{y}_1) = \theta\bar{Y}\tau\alpha C_x^2\left[\frac{g(g+1)}{2}\alpha\tau - gC\right], \quad (10)$$

$$B(\bar{y}_2) = \theta\bar{Y}\tau C_x^2(\tau - C), \quad (11)$$

$$B(\bar{y}_3) = \theta\bar{Y}\tau C C_x^2, \quad (12)$$

$$B(\bar{y}_4) = \theta\bar{Y}\tau(2g-1)(g\tau - C)C_x^2, \quad (13)$$

$$B(\bar{y}_5) = \frac{\theta}{8}\bar{Y}C_x^2(3-4C), \quad (14)$$

$$B(\bar{y}_6) = \frac{\theta}{8}\bar{Y}C_x^2(3+4C), \quad (15)$$

$$B(\bar{y}_7) = \theta\bar{Y}C_x^2(1-C), \quad (16)$$

$$B(\bar{y}_8) = \theta\bar{Y}CC_x^2, \quad (17)$$

$$MSE(\bar{y}_1) = \theta\bar{Y}^2\left[C_y^2 + \alpha\tau g C_x^2(\alpha\tau g - 2C)\right], \quad (18)$$

$$MSE(\bar{y}_2) = \theta\bar{Y}^2\left[C_y^2 + \tau C_x^2(\tau - 2C)\right], \quad (19)$$

$$MSE(\bar{y}_3) = \theta\bar{Y}^2\left[C_y^2 + \tau C_x^2(\tau + 2C)\right], \quad (20)$$

$$MSE(\bar{y}_4) = \theta\bar{Y}^2\left[C_y^2 + (2g-1)\tau C_x^2\left\{(2g-1)\tau - 2C\right\}\right] \quad (21)$$

$$MSE(\bar{y}_5) = \theta\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4}(1-4C)\right], \quad (22)$$

$$MSE(\bar{y}_6) = \theta\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4}(1+4C)\right], \quad (23)$$

$$MSE(\bar{y}_7) = \theta\bar{Y}^2\left[C_y^2 + C_x^2(1-2C)\right], \quad (24)$$

$$MSE(\bar{y}_8) = \theta\bar{Y}^2\left[C_y^2 + C_x^2(1+2C)\right], \quad (25)$$

where

$$C = \rho_{yx}\left(\frac{C_y}{C_x}\right),\ \rho_{yx} = \left(\frac{S_{yx}}{S_y S_x}\right),\ C_x = \left(\frac{S_x}{\bar{X}}\right),$$

$$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}),\ \tau = \frac{a\bar{X}}{a\bar{X}+b},$$

$$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2 .$$

In this paper, the author attempts to develop the estimators of Singh et al. [1] and S. Bahl and R. K. Tuteja [2] as predictor of the mean of the population using the information of auxiliary in sample. Therefore, the suggested estimators with their properties are presented in section 2. Furthermore, we execute the theoretical comparison among existing estimators in the Sections 3. In section 4, the real data sets are used to discuss about the performance of various estimators numerically. Finally, the last section provides conclusion of this paper.

2. The Suggested Estimators with their Properties

By adjusting the estimator of H. P. Singh et al. [1] and S. Bahl and R. K. Tuteja [2], we define two chain-type exponential estimators for the estimation of population mean $\bar{Y}$ as

$$\bar{y}_{N1} = \bar{y}\left[\frac{a\bar{X}+b}{a\bar{x}+b}\right]^{2g-1}\exp\left(\frac{a(\bar{X}-\bar{x})}{a(\bar{X}+\bar{x})+2b}\right) \quad (26)$$

$$\bar{y}_{N2} = \bar{y}\left[\frac{a\bar{X}+b}{a\bar{x}+b}\right]^{2g-1}\exp\left(\frac{a(\bar{x}-\bar{X})}{a(\bar{x}+\bar{X})+2b}\right) \quad (27)$$

For obtaining the expression of bias and MSE of these estimators, we consider the following relative of error terms, as $e_0 = (\bar{y}-\bar{Y})/\bar{Y}$ and $e_1 = (\bar{x}-\bar{X})/\bar{X}$.

Under SRSWOR, we have the following expectations, $E(e_0) = E(e_1) = 0$, $E(e_0^2) = \theta C_y^2$, $E(e_1^2) = \theta C_x^2$, and $E(e_0 e_1) = \theta C C_x^2$.

Rewriting $\bar{y}_{N1}$ and $\bar{y}_{N2}$ at equation (26) and (27) in term of e_i's, we have

$$\bar{y}_{N1} = \bar{Y}(1+e_0)\{1+\tau e_1\}^{-(2g-1)}\exp\left\{\frac{-\tau e_1}{2}\left(1+\frac{\tau e_1}{2}\right)^{-1}\right\} \quad (28)$$

and

$$\bar{y}_{N2} = \bar{Y}(1+e_0)\{1+\tau e_1\}^{-(2g-1)}\exp\left\{\frac{\tau e_1}{2}\left(1+\frac{\tau e_1}{2}\right)^{-1}\right\} \quad (29)$$

Now, we assume that the sample size to be large enough as compared to the population size so as to make $|e_i|$ and $|e_i'|$ very small. Therefore, all the terms of e_i's that have power greater than two in equation (28) and (29), it will be considered ignorable, we have

$$\bar{y}_{N1} \cong \bar{Y}\left[\begin{array}{l} 1+e_0-(2g-1)\tau e_1-(2g-1)\tau e_0 e_1 \\ +g(2g-1)\tau^2 e_1^2 - \frac{\tau e_1}{2} - \frac{\tau e_0 e_1}{2} \\ +(2g-1)\frac{\tau^2 e_1^2}{2} + \frac{3\tau^2 e_1^2}{8} \end{array}\right] \quad (30)$$

and

$$\bar{y}_{N2} \cong \bar{Y}\begin{bmatrix} 1+e_0-(2g-1)\tau e_1-(2g-1)\tau e_0 e_1 \\ +g(2g-1)\tau^2 e_1^2+\frac{\tau e_1}{2}+\frac{\tau e_0 e_1}{2} \\ -(2g-1)\frac{\tau^2 e_1^2}{2}-\frac{\tau^2 e_1^2}{8} \end{bmatrix} \quad (31)$$

Expanding the right-hand sides of equation (30) and (31), then subtracting $\bar{Y}$ on both sides, we get

$$(\bar{y}_{N1}-\bar{Y}) \cong \bar{Y}\begin{pmatrix} e_0-(2g-1)\tau e_1-(2g-1)\tau e_0 e_1 \\ +g(2g-1)\tau^2 e_1^2-\frac{\tau e_1}{2}-\frac{\tau e_0 e_1}{2} \\ +(2g-1)\frac{\tau^2 e_1^2}{2}+\frac{3\tau^2 e_1^2}{8} \end{pmatrix} \quad (32)$$

and

$$(\bar{y}_{N2}-\bar{Y}) \cong \bar{Y}\begin{pmatrix} e_0-(2g-1)\tau e_1-(2g-1)\tau e_0 e_1 \\ +g(2g-1)\tau^2 e_1^2+\frac{\tau e_1}{2}+\frac{\tau e_0 e_1}{2} \\ -(2g-1)\frac{\tau^2 e_1^2}{2}-\frac{\tau^2 e_1^2}{8} \end{pmatrix} \quad (33)$$

Taking expectation on both sides of equation (32) and (33), we have the bias of $\bar{y}_{N1}$ and $\bar{y}_{N2}$ as

$$B(\bar{y}_{N1})=\theta\bar{Y}\tau C_x^2\left[(2g-1)\left\{-C+g\tau+\frac{\tau}{2}\right\}-\frac{C}{2}+\frac{3\tau}{8}\right] \quad (34)$$

and

$$B(\bar{y}_{N2})=\theta\bar{Y}\tau C_x^2\left((2g-1)\left\{-C+g\tau-\frac{\tau}{2}\right\}+\frac{C}{2}-\frac{\tau}{8}\right) \quad (35)$$

By squaring both sides of equation (32) and (33), we get

$$(\bar{y}_{N1}-\bar{Y})^2 \cong \bar{Y}^2\begin{bmatrix} e_0^2+(2g-1)^2\tau^2 e_1^2+\frac{\tau^2 e_1^2}{4} \\ -2(2g-1)\tau e_0 e_1-\tau e_0 e_1 \\ +(2g-1)\tau^2 e_1^2 \end{bmatrix} \quad (36)$$

and

$$(\bar{y}_{N2}-\bar{Y})^2 \cong \bar{Y}^2\begin{bmatrix} e_0^2+(2g-1)^2\tau^2 e_1^2+\frac{\tau^2 e_1^2}{4} \\ -2(2g-1)\tau e_0 e_1+\tau e_0 e_1 \\ -(2g-1)\tau^2 e_1^2 \end{bmatrix} \quad (37)$$

After that, we just taking expectation on both sides of (36) and (37), so the MSE of $\bar{y}_{N1}$ and $\bar{y}_{N2}$ as

$$MSE(\bar{y}_{N1})=\theta\bar{Y}^2\left[C_y^2+\tau C_x^2\left(4g^2\tau-4gC+C-2g\tau+\frac{\tau}{4}\right)\right] \quad (38)$$

and

$$MSE(\bar{y}_{N2})=\theta\bar{Y}^2\left[C_y^2+\tau C_x^2\left(4g^2\tau-4gC+3C-6g\tau+\frac{9\tau}{4}\right)\right] \quad (39)$$

To replace the different choices of the constants $a(a \neq 0)$, b, and g into equations (2) to (9), (26) and (27), and for the convenience of the readers, a few members of various estimators are also shown in **Table 1**:

Table 1. A few members of various estimators

Estimators	Values of constants a	b	g	α
A few members of $\bar{y}_1$				
$\bar{y}_{1(1)}=\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$ The usual ratio estimator	1	0	1	1
$\bar{y}_{1(2)}=\bar{y}\left(\frac{\bar{X}+\rho_{yx}}{\bar{x}+\rho_{yx}}\right)$ H. P. Singh and R. Tailor [6]	1	ρ_{yx}	1	1
$\bar{y}_{1(3)}=\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)$ B. V. S. Sisodia and V. K. Dwivedi [7]	1	C_x	1	1
A few members of $\bar{y}_2$				
$\bar{y}_{2(1)}=\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)=\bar{y}_{1(1)}$	1	0	-	-
$\bar{y}_{2(2)}=\bar{y}\left(\frac{\bar{X}+\rho_{yx}}{\bar{x}+\rho_{yx}}\right)=\bar{y}_{1(2)}$	1	ρ_{yx}	-	-
$\bar{y}_{2(3)}=\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)=\bar{y}_{1(3)}$	1	C_x	-	-
A few members of $\bar{y}_3$				
$\bar{y}_{3(1)}=\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)$ The usual product estimator	1	0	-	-
$\bar{y}_{3(2)}=\bar{y}\left(\frac{\bar{x}+\rho_{yx}}{\bar{X}+\rho_{yx}}\right)$ H. P. Singh and R. Tailor [6]	1	ρ_{yx}	-	-
$\bar{y}_{3(3)}=\bar{y}\left(\frac{\bar{x}+C_x}{\bar{X}+C_x}\right)$ B. N. Pandey and V. Dubey [8]	1	C_x	-	-
A few members of $\bar{y}_4$				
$\bar{y}_{4(1)}=\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)=\bar{y}_{1(1)}=\bar{y}_{2(1)}$	1	0	1	-
$\bar{y}_{4(2)}=\bar{y}\left(\frac{\bar{X}+\rho_{yx}}{\bar{x}+\rho_{yx}}\right)=\bar{y}_{1(2)}=\bar{y}_{2(2)}$	1	ρ_{yx}	1	-
$\bar{y}_{4(3)}=\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)=\bar{y}_{1(3)}=\bar{y}_{2(3)}$	1	C_x	1	-

Table 1. (Cont.)

Estimators	Values of constants a	b	g	α
$\bar{y}_{4(4)}=\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)=\bar{y}_{3(1)}$	1	0	0	-
$\bar{y}_{4(5)}=\bar{y}\left(\frac{\bar{x}+\rho_{yx}}{\bar{X}+\rho_{yx}}\right)=\bar{y}_{3(2)}$	1	ρ_{yx}	0	-
$\bar{y}_{4(6)}=\bar{y}\left(\frac{\bar{x}+C_x}{\bar{X}+C_x}\right)=\bar{y}_{3(3)}$	1	C_x	0	-
A few members of $\bar{y}_{N1}$				
$\bar{y}_{N1(1)}=\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$ H. P. Singh and S. K. Pal [5]	1	0	1	-
$\bar{y}_{N1(2)}=\bar{y}\left(\frac{\bar{X}+\rho_{yx}}{\bar{x}+\rho_{yx}}\right)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\rho_{yx}}\right)$	1	ρ_{yx}	1	-
$\bar{y}_{N1(3)}=\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\rho_{yx}}\right)$ H. P. Singh and S. K. Pal [5]	1	C_x	1	-
$\bar{y}_{N1(4)}=\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	1	0	0	-
$\bar{y}_{N1(5)}=\bar{y}\left(\frac{\bar{x}+\rho_{yx}}{\bar{X}+\rho_{yx}}\right)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\rho_{yx}}\right)$	1	ρ_{yx}	0	-
$\bar{y}_{N1(6)}=\bar{y}\left(\frac{\bar{x}+C_x}{\bar{X}+C_x}\right)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\rho_{yx}}\right)$	1	C_x	0	-
A few members of $\bar{y}_{N2}$				
$\bar{y}_{N2(1)}=\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$	1	0	1	-
$\bar{y}_{N2(2)}=\bar{y}\left(\frac{\bar{X}+\rho_{yx}}{\bar{x}+\rho_{yx}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2\rho_{yx}}\right)$	1	ρ_{yx}	1	-
$\bar{y}_{N2(3)}=\bar{y}\left(\frac{\bar{X}+C_x}{\bar{x}+C_x}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2\rho_{yx}}\right)$	1	C_x	1	-
$\bar{y}_{N2(4)}=\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$	1	0	0	-
$\bar{y}_{N2(5)}=\bar{y}\left(\frac{\bar{x}+\rho_{yx}}{\bar{X}+\rho_{yx}}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2\rho_{yx}}\right)$	1	ρ_{yx}	0	-
$\bar{y}_{N2(6)}=\bar{y}\left(\frac{\bar{x}+C_x}{\bar{X}+C_x}\right)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}+2\rho_{yx}}\right)$	1	C_x	0	-

3. Efficiency Comparison

In this part, the author has compared the efficiency between the suggested estimators and various estimators by using the conditions of absolute biases and MSE values. The details of its are as follows:

3.1 Bias comparison of ratio-type estimators

From (10), (11), (13), and (34), we have

(i) $\left|B(\bar{y}_{1(1)}, \bar{y}_{2(1)}, \bar{y}_{4(1)})\right| < \left|B(\bar{y}_{N1(1)})\right|$

that is, $C < \frac{7}{4}$ (40)

(ii) $\left|B(\bar{y}_{1(2)}, \bar{y}_{2(2)}, \bar{y}_{4(2)})\right| < \left|B(\bar{y}_{N1(2)})\right|$

that is, $C < \frac{7\tau_1}{4}$ (41)

(iii) $\left|B(\bar{y}_{1(3)}, \bar{y}_{2(3)}, \bar{y}_{4(3)})\right| < \left|B(\bar{y}_{N1(3)})\right|$

that is, $C < \frac{7\tau_2}{4}$ (42)

If the conditions (40), (41), and (42) are satisfied, then the suggested estimators ($\bar{y}_{N1(1)}$, $\bar{y}_{N1(2)}$, and $\bar{y}_{N1(3)}$) are less biased respectively to customary the estimators $(\bar{y}_{1(1)}, \bar{y}_{2(1)}, \bar{y}_{4(1)})$, $(\bar{y}_{1(2)}, \bar{y}_{2(2)}, \bar{y}_{4(2)})$, and $(\bar{y}_{1(3)}, \bar{y}_{2(3)}, \bar{y}_{4(3)})$, respectively.

3.2 Bias comparison of product-type estimators

From (12), (13), and (35), we have

(i) $\left|B(\bar{y}_{3(1)}, \bar{y}_{4(4)})\right| < \left|B(\bar{y}_{N2(4)})\right|$

that is, $C > \frac{-3}{4}$ (43)

(ii) $\left|B(\bar{y}_{3(2)}, \bar{y}_{4(5)})\right| < \left|B(\bar{y}_{N2(5)})\right|$

that is, $C > \frac{-3\tau_1}{4}$ (44)

iii) $\left|B(\bar{y}_{3(3)}, \bar{y}_{4(6)})\right| < \left|B(\bar{y}_{N2(6)})\right|$

that is, $C > \frac{-3\tau_2}{4}$ (45)

If the conditions (43), (44), and (45) are satisfied, then the suggested estimators ($\bar{y}_{N2(4)}$, $\bar{y}_{N2(5)}$, and $\bar{y}_{N2(6)}$) are less biased respectively to customary the estimators $(\bar{y}_{3(1)}, \bar{y}_{4(4)})$, $(\bar{y}_{3(2)}, \bar{y}_{4(5)})$, and $(\bar{y}_{3(3)}, \bar{y}_{4(6)})$, respectively.

3.3 MSE comparison of ratio-type estimators

From (18), (19), (21), and (38), we have

(i) $MSE(\bar{y}_{1(1)}, \bar{y}_{2(1)}, \bar{y}_{4(1)}) < MSE(\bar{y}_{N1(1)})$

if $C < \frac{5}{4}$ (46)

(ii) $MSE(\bar{y}_{1(2)}, \bar{y}_{2(2)}, \bar{y}_{4(2)}) < MSE(\bar{y}_{N1(2)})$

if $C < \frac{5\tau_1}{4}$ (47)

(iii) $MSE(\bar{y}_{1(3)}, \bar{y}_{2(3)}, \bar{y}_{4(3)}) < MSE(\bar{y}_{N1(3)})$

if $C < \frac{5\tau_2}{4}$ (48)

It is follows that if the conditions (46), (47), and (48) are satisfied, then the suggested estimators ($\bar{y}_{N1(1)}$, $\bar{y}_{N1(2)}$, and $\bar{y}_{N1(3)}$) are better than the estimators $(\bar{y}_{1(1)}, \bar{y}_{2(1)}, \bar{y}_{4(1)})$, $(\bar{y}_{1(2)}, \bar{y}_{2(2)}, \bar{y}_{4(2)})$, and $(\bar{y}_{1(3)}, \bar{y}_{2(3)}, \bar{y}_{4(3)})$, respectively.

3.4 MSE comparison of product-type estimators

From (20), (21), and (39), we have

(i) $MSE(\bar{y}_{3(1)},\ \bar{y}_{4(4)}) < MSE(\bar{y}_{N2(4)})$

$$\text{if } C < \frac{-1}{4} \quad (49)$$

(ii) $MSE(\bar{y}_{3(2)},\ \bar{y}_{4(5)}) < MSE(\bar{y}_{N2(5)})$

$$\text{if } C < \frac{-\tau_1}{4} \quad (50)$$

iii) $MSE(\bar{y}_{3(3)},\ \bar{y}_{4(6)}) < MSE(\bar{y}_{N2(6)})$

$$\text{if } C < \frac{-\tau_2}{4} \quad (51)$$

where $\tau_1 = \dfrac{\bar{X}}{\bar{X} + \rho_{yx}}$, $\tau_2 = \dfrac{\bar{X}}{\bar{X} + C_x}$.

It is follows that if the conditions (49), (50), and (51) are satisfied, then the suggested estimators ($\bar{y}_{N2(4)}$, $\bar{y}_{N2(5)}$, and $\bar{y}_{N2(6)}$) are better than the estimators $(\bar{y}_{3(1)}, \bar{y}_{4(4)})$, $(\bar{y}_{3(2)}, \bar{y}_{4(5)})$, and $(\bar{y}_{3(3)}, \bar{y}_{4(6)})$, respectively.

4. Numerical Study

In section 4 , to judge the merits of the suggested estimators $\bar{y}_{N1}$ and $\bar{y}_{N2}$ over the various estimators, the author has considered three natural population data sets. The description of the population data sets is given in **Table 2**.

Table 2. The population data sets

Term	Population I	Population II	Population III
	M. N. Murthy [9]	M. Khoshnevisan *et al.* [3]	W. G. Cochran [10]
N	80	20	10
n	20	8	4
$\bar{Y}$	11.264	19.550	5.920
$\bar{X}$	51.826	18.800	3.590
C_y	0.7500	0.3552	0.144
C_x	0.354	0.394	0.128
ρ_{yx}	0.941	-0.920	1.680

To examine the biasedness of various estimators of population mean $\bar{Y}$ we have computed the following absolute bias quantities:

$$B_{(\bar{y}_{1(1)}, \bar{y}_{2(1)}, \bar{y}_{4(1)})} = \left| \frac{B(\bar{y}_{1(1)}, \bar{y}_{2(1)}, \bar{y}_{4(1)})}{\theta \bar{Y} C_x^2} \right| = |1 - C|,$$

$$B_{(\bar{y}_{1(2)}, \bar{y}_{2(2)}, \bar{y}_{4(2)})} = \left| \frac{B(\bar{y}_{1(2)}, \bar{y}_{2(2)}, \bar{y}_{4(2)})}{\theta \bar{Y} C_x^2} \right| = |\tau_1(\tau_1 - C)|,$$

$$B_{(\bar{y}_{1(3)}, \bar{y}_{2(3)}, \bar{y}_{4(3)})} = \left| \frac{B(\bar{y}_{1(3)}, \bar{y}_{2(3)}, \bar{y}_{4(3)})}{\theta \bar{Y} C_x^2} \right| = |\tau_2(\tau_2 - C)|,$$

$$B_{(\bar{y}_{3(1)}, \bar{y}_{4(4)})} = \left| \frac{B(\bar{y}_{3(1)}, \bar{y}_{4(4)})}{\theta \bar{Y} C_x^2} \right| = |C|,$$

$$B_{(\bar{y}_{3(2)}, \bar{y}_{4(5)})} = \left| \frac{B(\bar{y}_{3(2)}, \bar{y}_{4(5)})}{\theta \bar{Y} C_x^2} \right| = |\tau_1 C|,$$

$$B_{(\bar{y}_{3(3)}, \bar{y}_{4(6)})} = \left| \frac{B(\bar{y}_{3(3)}, \bar{y}_{4(6)})}{\theta \bar{Y} C_x^2} \right| = |\tau_2 C|,$$

$$B_{(\bar{y}_{N1(1)})} = \left| \frac{B(\bar{y}_{N1(1)})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{3}{8}(5 - 4C) \right|,$$

$$B_{(\bar{y}_{N1(2)})} = \left| \frac{B(\bar{y}_{N1(2)})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{3\tau_1}{8}(5\tau_1 - 4C) \right|,$$

$$B_{(\bar{y}_{N1(3)})} = \left| \frac{B(\bar{y}_{N1(3)})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{3\tau_2}{8}(5\tau_2 - 4C) \right|,$$

$$B_{(\bar{y}_{N1(4)})} = \left| \frac{B(\bar{y}_{N1(4)})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{1}{8}(4C - 1) \right|,$$

$$B_{(\bar{y}_{N1(5)})} = \left| \frac{B(\bar{y}_{N1(5)})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{\tau_1}{8}(4C - \tau_1) \right|,$$

$$B_{(\bar{y}_{N1(6)})} = \left| \frac{B(\bar{y}_{N1(6)})}{\theta \bar{Y} C_x^2} \right| = \left| \frac{\tau_2}{8}(4C - \tau_2) \right|,$$

$$B_{(\bar{y}_{N2(1)})} = \left|\frac{B(\bar{y}_{N2(1)})}{\theta \bar{Y} C_x^2}\right| = \left|\frac{1}{8}(3-4C)\right|,$$

$$B_{(\bar{y}_{N2(2)})} = \left|\frac{B(\bar{y}_{N2(2)})}{\theta \bar{Y} C_x^2}\right| = \left|\frac{\tau_1}{8}(3\tau_1-4C)\right|,$$

$$B_{(\bar{y}_{N2(3)})} = \left|\frac{B(\bar{y}_{N2(3)})}{\theta \bar{Y} C_x^2}\right| = \left|\frac{\tau_2}{8}(3\tau_2-4C)\right|,$$

$$B_{(\bar{y}_{N2(4)})} = \left|\frac{B(\bar{y}_{N2(4)})}{\theta \bar{Y} C_x^2}\right| = \left|\frac{3}{8}(4C+1)\right|,$$

$$B_{(\bar{y}_{N2(5)})} = \left|\frac{B(\bar{y}_{N2(5)})}{\theta \bar{Y} C_x^2}\right| = \left|\frac{3\tau_1}{8}(4C+\tau_1)\right|,$$

$$B_{(\bar{y}_{N2(6)})} = \left|\frac{B(\bar{y}_{N2(6)})}{\theta \bar{Y} C_x^2}\right| = \left|\frac{3\tau_2}{8}(4C+\tau_2)\right|,$$

and finding are shown in **Table 3**.

Table 3. Bias quantities of different estimators

Estimators	Population		
	I	II	III
$\bar{y}_{1(1)}$, $\bar{y}_{2(1)}$, $\bar{y}_{4(1)}$	0.9936	1.8288	0.8900
$\bar{y}_{1(2)}$, $\bar{y}_{2(2)}$, $\bar{y}_{4(2)}$	0.9934	1.9770	0.8234
$\bar{y}_{1(3)}$, $\bar{y}_{2(3)}$, $\bar{y}_{4(3)}$	0.9936	1.7711	0.8926
$\bar{y}_{3(1)}$, $\bar{y}_{4(4)}$	1.9936	0.6288	1.8900
$\bar{y}_{3(2)}$, $\bar{y}_{4(5)}$	1.9581	0.6014	1.2875
$\bar{y}_{3(3)}$, $\bar{y}_{4(6)}$	1.9801	0.6117	1.8249
$\bar{y}_{N1(1)}$	0.5355	1.7718	0.4600
$\bar{y}_{N1(2)}$	0.5284	1.7658	0.4181
$\bar{y}_{N1(3)}$	**0.5105**	1.5811	**0.4013**
$\bar{y}_{N1(4)}$	0.8718	0.5394	0.8200
$\bar{y}_{N1(5)}$	0.8585	0.5739	0.5857
$\bar{y}_{N1(6)}$	0.8667	0.5258	0.7959
$\bar{y}_{N2(1)}$	0.6218	0.6394	0.5700
$\bar{y}_{N2(2)}$	0.6173	0.6211	0.4697
$\bar{y}_{N2(3)}$	0.6201	0.6461	0.5628
$\bar{y}_{N2(4)}$	1.3655	0.6182	1.2100
$\bar{y}_{N2(5)}$	1.2989	**0.5117**	1.1053
$\bar{y}_{N2(6)}$	1.3401	0.5774	1.0870

From the **Table 3**, it is observed that the values of bias quantities of some few members of the suggested estimators $\bar{y}_{N1}$ and $\bar{y}_{N2}$ are less in comparison of the other estimators. For population data sets I and III, in the case of positive correlation coefficient, we perceived that the values of bias quantity $\bar{y}_{N1(3)}$ is lowest with respect to all bias quantities of other estimators. For population II, in the case of negative correlation coefficients, one found that the $\bar{y}_{N2(5)}$ have lowest value of bias quantity with comparison to other estimators.

In comparing the efficiency of the suggested estimator over other estimators in this present study, we have used the percent relative efficiencies (PREs) as criteria for comparison with respect to unbiased the estimator $\bar{y}$ which details are presented in the following **Table 4**.

Table 4. The PREs of different estimators with respect to $\bar{y}$

Estimators	Population		
	I	II	III
$\bar{y}$	100.0000	100.0000	100.0000
$\bar{y}_{1(1)}$, $\bar{y}_{2(1)}$, $\bar{y}_{4(1)}$	298.9715	23.3928	108.5382
$\bar{y}_{1(2)}$, $\bar{y}_{2(2)}$, $\bar{y}_{4(2)}$	288.7267	22.5057	104.7123
$\bar{y}_{1(3)}$, $\bar{y}_{2(3)}$, $\bar{y}_{4(3)}$	331.5316	23.7668	151.0022
$\bar{y}_{3(1)}$, $\bar{y}_{4(4)}$	47.3689	527.0649	20.9346
$\bar{y}_{3(2)}$, $\bar{y}_{4(5)}$	47.8177	675.5013	27.9892
$\bar{y}_{3(3)}$, $\bar{y}_{4(6)}$	47.5387	484.5549	21.5203
$\bar{y}_{N1(1)}$	592.3890	14.6275	154.4982
$\bar{y}_{N1(2)}$	571.9153	13.7235	162.1239
$\bar{y}_{N1(3)}$	**640.5185**	15.0133	**163.0460**
$\bar{y}_{N1(4)}$	520.6022	348.6796	173.6499
$\bar{y}_{N1(5)}$	301.5186	374.8604	136.4437
$\bar{y}_{N1(6)}$	531.0142	338.6975	178.0580
$\bar{y}_{N2(1)}$	420.3491	391.2801	132.5411
$\bar{y}_{N2(2)}$	355.6011	369.0207	123.7205

Table 4. (Cont.)

Estimators	Population		
	I	II	III
$\bar{y}_{N2(3)}$	416.3933	401.7911	139.5793
$\bar{y}_{N2(4)}$	168.9643	780.3622	113.3635
$\bar{y}_{N2(5)}$	181.6991	**793.6975**	133.5651
$\bar{y}_{N2(6)}$	173.6042	790.9843	126.4133

It is observed from **Table 4** that:

(i) For population data sets I and III, when the correlation between study and auxiliary variables is positive, the estimator $\bar{y}_{N1(3)}$ is superior to the other estimators. Because it gives the largest value of PRE as compared to other estimators within the same populations.
(ii) Also, when comparing the efficiency between the estimators of population data sets II in the case of negative correlation, it has been seen that the PRE of the estimator $\bar{y}_{N2(5)}$ is superior among all of the other estimators. Therefore, the estimator $\bar{y}_{N2(5)}$ appears to be the best in the sense of having largest percent relative efficiency as well as least bias in Population II.

5. Conclusion

In this paper, two chain-type exponential estimators for the estimation of population mean using the information of auxiliary in sample following H. P. Singh et al. [1] and S. Bahl and R. K. Tuteja [2] has been suggested. It has been found that some known estimators of the population mean such as the usual ratio, usual product estimators, and estimators, according to H. P. Singh and S. K. Pal [5], H. P. Singh and R. Tailor [6], B. V. S. Sisodia and V. K. Dwivedi [7], and B. N. Pandey and V. Dubey [8], are members of suggested estimators. The expressions for bias and MSE of the suggested estimators have been obtained. It has been also found from the results of theoretical and numerical studies that the suggested estimators are less bias and more efficient than other existing estimators. Therefore, we introduced the use of the suggested estimators in practice. However, this conclusion cannot be extrapolated due to limited numerical study.

6. References

[1] H. P. Singh, R. S. Solanki and A. K. Singh, “A generalized ratio-cum-product estimator for estimating the finite population mean in survey sampling,” *Communications in Statistics - Theory and Methods*, vol. 45, no. 1, pp. 158-172, Jan. 2016.

[2] S. Bahl and R. K. Tuteja, “Ratio and product type exponential estimators,” *Journal of Information and Optimization Sciences*, vol. 12, no. 1, pp. 159-164, May 1991.

[3] M. Khoshnevisan, R. Singh, P. Chauhan, N. Sawan and F. Smarandache, “A general family of estimators for estimating population mean using known value of some population parameter(s),” *Far East Journal of Theoretical Statistics*, vol. 22, pp. 181-191, Jul. 2007.

[4] H. P. Singh and N. Agnihotri, “A general procedure of estimating population mean using auxiliary information in sample surveys,” *Statistics in Transition New Series*, vol. 9, no. 1, pp. 71-87, Jun. 2008.

[5] H. P. Singh and S. K. Pal, “A new chain ratio-ratio-type exponential estimator using auxiliary information in sample surveys,” *International Journal of Mathematics And its*

Applications, vol. 3, no. 4-B, pp. 37-46, Sep. 2015.

[6] H. P. Singh and R. Tailor, “Use of known correlation coefficient in estimating the finite population mean,” *Statistics in Transition*, vol. 6, no. 1, pp. 555-560, Jan. 2003.

[7] B. V. S. Sisodia and V. K. Dwivedi, “A modified ratio estimator using coefficient of variation of auxiliary variable,” *Journal Indian Society Agricultural Statistics*, vol. 32, no. 2, pp. 13-18, Jan. 1981.

[8] B. N. Pandey and V. Dubey, “Modified product estimator using coefficient of variation of auxiliary variate,” *Assam Statistical Review*, vol. 2, no. 2, pp. 64-66, Jan. 1988.

[9] M. N. Murthy, *Sampling Theory and Methods*, 2nd ed. Calcutta: Statistical Publishing Society, 1967.

[10] W. G. Cochran, *Sampling Techniques*, 3rd ed. New York: John Wiley & Sons, 1977.